BACHELOR THESIS

C^*-Algebras

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under the supervision of

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by

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Eidesstattliche Erklärung

I hereby declare in lieu of an oath that I have completed the present Bachelor's thesis independently and without illegitimate assistance from third parties. I have used no other than the specified sources and aids.

Vienna, April 24, 2023

signature of the author

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1 Introduction

In this Bachelor-Thesis we present some basics of C^*-algebra theory leading to the Gelfand-Naimark Theorem. This theorem shows that every C^*-algebra can be identified as a particular C^*-subalgebra of all bounded linear operators on some Hilbert space. The theorem uses the Gelfand-Naimark-Segal construction which provides an important tool in C^*-algebra theory. The thesis is based on chapter 4 of 'Fundamentals of the Theory of Operator Algebras' by Richard V. Kadison and John R. Ringrose [1].

We start in Chapter 2 by defining C^*-algebras and certain special elements like selfadjoint elements of a C^*-algebra. Then we discuss some examples and generalise some facts about operators on a Hilbert space to C^*-algebras. One of these is the functional calculus which provides an important tool later.

In Chapter 3, we consider positive elements of a C^*-algebra. These behave like positive semidefinite operators on a Hilbert space. We see that we can decompose every selfadjoint element into a positive and negative part. Since we have a notion of positive elements, we can use this to define a partial order structure on the selfadjoint elements of a C^*-algebra.

Chapter 4 is about linear functionals on a C^*-algebra. Just like positive elements, we can consider positive linear functionals. If a positive linear functional has norm 1, we call it a state. Extremal points of the set of states are called pure states. This is in some sense the smallest subset of linear functionals we need to reconstruct all of them.

In Chapter 5, we want to represent a C^*-algebra on a Hilbert space. With the Gelfand-Naimark-Segal construction, we can produce a Hilbert space and representation on that Hilbert space from a state of a C^*-algebra. Using the sum of Hilbert spaces and the pure states, we prove the Gelfand-Naimark Theorem which tells us that every C^*-algebra is $^*-$isomorphic to a C^*-subalgebra of operators on a Hilbert space.

The reader is expected to have basic knowledge in functional analysis similar to the material presented in the course 'Funktionalanalysis 1' at the Technical University of Vienna. Any reference to functional analysis means that the result can be found in the lecture notes written by Martin Blümlinger, Michael Kaltenbäck and Harald Woracek [2].

2 Foundations

We use the same notation as in the lecture notes of 'Funktionalanalysis 1'[2]. Therefore, if $\mathcal{A}$ is an algebra, then $\text{Inv}(\mathcal{A})$ are all invertible elements of $\mathcal{A}$. For $A \in \mathcal{A}$ we consider the spectrum $\sigma(A)$ and the residue $\rho(A)$. The unit in every unitary algebra is referred to as I regardless of the surrounding space. By context, it is clear which unit is meant. For complex numbers z, we denote the complex conjugate as $\overline{z}$.

2.1 C^*-Algebras

2.1 Definition. Let $\mathcal{A}$ be a complex Banach algebra. $\mathcal{A}$ is a C^*-*algebra* if it has a conjugate linear Involution * which satisfies the following rules:

- $\forall A, B \in \mathcal{A}, s, t \in \mathbb{C} : (sA + tB)^* = \overline{s}A^* + \overline{t}B^*$
- $\forall A \in \mathcal{A} : (A^*)^* = A$
- $\forall A, B \in \mathcal{A} : (AB)^* = B^*A^*$
- $\forall A \in \mathcal{A} : \|A^*A\| = \|A\|^2$

Often we consider special elements of a C^*-algebra.

2.2 Definition. Let $\mathcal{A}$ be a C^*-algebra. $A \in \mathcal{A}$ is called

- *normal* if $AA^* = A^*A$,
- *selfadjoint* if $A = A^*$, and
- *unitary* if $AA^* = A^*A = I$.

2.3 Proposition. *Let $\mathcal{A}$ be a C^*-algebra.*

(i) For all $A \in \mathcal{A}$, we have $\|A\| = \|A^*\|$.

(ii) $I^* = I$.

(iii) For $A \in \mathcal{A}$, we find selfadjoint elements $H, K \in \mathcal{A}$ such that $A = H + iK$.

(iv) $A \in \mathcal{A}$ is invertible if and only if A^* is invertible.

(v) $\sigma(A^*) = \{\overline{a} : a \in \sigma(A)\}$.

Proof.

(i) Let $A \in \mathcal{A}$. From Definition 2.1, we get

$$\|A\|^2 = \|A^*A\| \leq \|A^*\|\|A\| \implies \|A^*\| \geq \|A\|.$$

We can use the same inequality with A^* and get $\|A^*\| \leq \|A\|$. Therefore, $\|A^*\| = \|A\|$.

(ii)

$$I = (I^*)^* = (I^*I)^* = I^*(I^*)^* = I^*.$$

(iii) We define $H := \frac{1}{2}(A + A^*), K := \frac{1}{2}i(A^* - A)$. Obviously they satisfy the conditions from (iii).

(iv) If A is invertible with inverse A^{-1}, we get

$$A^*(A^{-1})^* = (A^{-1}A)^* = I^* = I = (AA^{-1})^* = (A^{-1})^*A^*.$$

Therefore, A^* is invertible with $(A^*)^{-1} = (A^{-1})^*$. If A^* is invertible, then we already know that $A = (A^*)^*$ is invertible.

(v) We know from (iv)

$$aI - A \in \text{Inv}(\mathcal{A}) \iff \overline{a}I - A^* \in \text{Inv}(\mathcal{A}).$$

By negating both sides, we get $a \in \sigma(A) \iff \overline{a} \in \sigma(A^*)$.

■

Proposition 2.3 shows that in a C^*-algebra * is isometric (and since it is conjugate linear therefore continuous). Furthermore, I is selfadjoint und unitary. Every element in $\mathcal{A}$ is a linear combination of two selfadjoint elements. We call them real and imaginary part. From (v) we also get that the spectral radii of A and A^* are the same.

2.4 Definition. Let $\mathcal{A}, \mathcal{B}$ be two C^*-algebras.
$\Phi : \mathcal{A} \to \mathcal{B}$ is called a C^*-homomorphism if

- $\forall A, B \in \mathcal{A}, s, t \in \mathbb{C} : \Phi(sA + tB) = s\Phi(A) + t\Phi(B)$,
- $\forall A, B \in \mathcal{A} : \Phi(AB) = \Phi(A)\Phi(B)$,
- $\Phi(I) = I$ and
- $\forall A \in \mathcal{A} : \Phi(A^*) = \Phi(A)^*$.

Φ is a C^*-*isomorphism* if Φ is a bijective C^*-homomorphism.

We specify certain subsets of a C^*-algebra.

2.5 Definition. Let $\mathcal{A}$ be a C^*-algebra. A subset $\mathcal{B} \subseteq \mathcal{A}$ is called

- *selfadjoint* if $\mathcal{B}$ is closed under *,
- a $^*-$*subalgebra* if $\mathcal{B}$ is selfadjoint and a subalgebra, and
- a C^*-*subalgebra* if $\mathcal{B}$ is a closed $^*-$subalgebra and contains the unit I.

If $\mathcal{B} \subseteq \mathcal{A}$ is a C^*-subalgebra of a C^*-algebra $\mathcal{A}$, then $\mathcal{B}$ is also a C^*-algebra.

2.6 Example. The most elementary example of a C^*-algebra is the set of all linear and bounded operators $\mathcal{B}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$, where T^* is the Hilbert adjoint of an operator $T \in \mathcal{B}(\mathcal{H})$. Every closed subalgebra of $\mathcal{B}(\mathcal{H})$ with I is also a C^*-algebra. We will show with the GNS-construction that for every C^*-algebra we find a C^*-isomorphism onto a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

2.7 Example. Other important examples are sets of complex valued functions where the operation * is defined by $f^*(x) = \overline{f}(x) = \overline{f(x)}$ which is pointwise complex conjugation. For a compact Hausdorff space X, the set $C(X)$ of all continuous functions from X to $\mathbb{C}$ is a C^*-algebra. Other examples include all bounded functions $\ell_\infty(M) \subseteq \mathbb{C}^M$ on an arbitrary set M, as well as $L_\infty(\Omega)$ which are all essentially bounded functions on a measure space Ω which are C^*-algebras.

2.2 Functional calculus

In this section we prove generalisations of some standard facts about operators on a Hilbert space to C^*-algebras.

2.8 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and $A \in \mathcal{A}$.*

(i) If A is normal, then $r(A) = \|A\|$.

(ii) If A is selfadjoint, then $\sigma(A) \subseteq \mathbb{R}$ and contains one or both of $\|A\|, -\|A\|$.

(iii) If $U \in \mathcal{A}$ is unitary, then $\|U\| = 1$ and $\sigma(U)$ is part of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Proof.

(i) For normal $A \in \mathcal{A}$, we have

$$\begin{aligned}\left\|A^2\right\|^2 &= \left\|(A^2)^* A^2\right\| = \|A^*(A^*A)A\| \\ &= \|A^*(AA^*)A\| = \|(A^*A)^*(A^*A)\| \\ &= \|A^*A\|^2 = (\|A\|^2)^2.\end{aligned}$$

Therefore, $\left\|A^2\right\| = \|A\|^2$ and by induction we get $\left\|A^{2^k}\right\| = \|A\|^{2^k}$ for all $k \in \mathbb{N}$. Then,

$$r(A) = \lim_{n\to\infty} \left\|A^k\right\|^{\frac{1}{k}} = \lim_{k\to\infty} \left\|A^{2^k}\right\|^{\frac{1}{2^k}} = \|A\|.$$

(ii) Let A be selfadjoint, then $\sigma(A)$ is compact and if $\sigma(A) \subseteq \mathbb{R}$, then it contains a scalar with absolute value $r(A) = \|A\|$ which are only $\|A\|$ or $-\|A\|$. Therefore, we only need to show $\sigma(A) \subseteq \mathbb{R}$. Let $a+ib \in \sigma(A)$. We define $B_n := A - aI + inbI$ for $n \in \mathbb{N}$. Then we get with the spectral mapping theorem for polynomials

$$i(n+1)b = a + ib - a + inb \in \sigma(B_n).$$

Then,

$$\begin{aligned}(n^2+2n+1)b^2 = |i(n+1)b|^2 &\leq (r(B_n))^2 \leq \|B_n\|^2 = \|B_n^* B_n\| \\ &= \|(A - aI - inbI)(A - aI + inbI)\| = \left\|(A-aI)^2 + n^2b^2I\right\| \\ &\leq \|A - aI\|^2 + n^2b^2.\end{aligned}$$

For $n \in \mathbb{N}$, we get the inequality $(2n+1)b^2 \leq \|A - aI\|^2$ which means b must be zero. Therefore, $\sigma(A) \subseteq \mathbb{R}$.

(iii) For unitary U, we get

$$\|U\|^2 = \|U^*U\| = \|I\| = 1.$$

If $z \in \sigma(U)$, then $z^{-1} \in \sigma(U^{-1}) = \sigma(U^*)$. We get

$$|z| \leq \|U\| = 1 \wedge |z|^{-1} \leq \|U^*\| = 1.$$

Therefore, $|z| = 1$.

■

Later we will need the following corollary for normal elements.

2.9 Corollary. *Let $A \in \mathcal{A}$ be normal and $\mathcal{A}$ a C^*-algebra. If there exists $k \in \mathbb{N}, k \geq 1$ such that $A^k = 0$, then $A = 0$.*

Proof. We get $A^n = 0$ for all $n \geq k$ and therefore $\|A\| = r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = 0$. ■

Next, we discuss the functional calculus. From functional analysis, we know the spectral theorem for selfadjoint operators A on a Hilbert space $\mathcal{H}$. There we proved that we find a C^*-homomorphism from $C(\sigma(A))$ to $\mathcal{B}(\mathcal{H})$ which maps the identity on $\sigma(A)$ to A. The argument also works for general C^*-algebras and is the foundation for many properties of selfadjoint elements in a C^*-algebra.

2.10 Theorem. *Let $\mathcal{A}$ be a C^*-algebra. For selfadjoint $A \in \mathcal{A}$, there exists an injective C^*-homomorphism $\Phi : C(\sigma(A)) \to \mathcal{A}, f \mapsto \Phi(f)$ with $\Phi(\iota) = A$ where ι is the identity on $\sigma(A)$, which means that $\iota : \sigma(A) \to \sigma(A), a \mapsto a$.*

Proof. For polynomials $p(x) \in \mathbb{C}[x]$, we can consider $p(A)$ in the C^*-algebra. Then we know

$$\begin{aligned}\forall p(x), q(x) \in \mathbb{C}[x], s, t \in \mathbb{C} : & s \cdot p(A) + t \cdot q(A) = (s \cdot p(x) + t \cdot q(x))(A) \\ & p(A)q(A) = (p(x) \cdot q(x))(A) \\ & p(A)^* = \overline{p}(A).\end{aligned}$$

Here $\overline{p}(x)$ is the polynomial with complex conjugated coefficients. Since $\sigma(A) \subseteq \mathbb{R}$, we get $\overline{p}(a) = \overline{p(a)}$ for all $a \in \sigma(A)$. Therefore, all polynomial functions on $\sigma(A)$ are a separating subalgebra of $C(\sigma(A))$ which contains the unit and is closed under complex conjugation. Using the Stone-Weierstrass theorem, we know that they are everywhere dense in $C(\sigma(A))$. Since $p(A)$ is normal, we can use Proposition 2.8 (i). Together with the spectral mapping theorem for polynomials we get

$$\|p(A)\| = r(p(A)) = \max\{|z| : z \in \sigma(p(A))\} = \max\{|p(a)| : a \in \sigma(A)\}.$$

From this, we get that for every polynomial function $p(x)$ on $\sigma(A)$ the expression $p(A)$ is welldefined and that $p(x) \mapsto p(A)$ is isometric and injective. Since all polynomial functions on $\sigma(A)$ are everywhere dense in $C(\sigma(A))$, we can find an unique extension $\Phi : C(\sigma(A)) \to \mathcal{A}$ which is also injective. It satisfies $\Phi(p(x)) = p(A)$ and therefore also $\Phi(\iota) = \iota(A) = A$. With continuity, all calculation rules are extended from the polynomials to Φ. Then Φ is a C^*-homomorphism with $\Phi(\iota) = A$. ■

It is clear that Φ with these properties is unique. Therefore, we write $f(A)$ instead of $\Phi(f)$ for $f \in C(\sigma(A))$. From the proof, we know that $f(A)$ is the limit of a sequence of polynomial evaluations $p_n(A)$ whence p_n converge uniformly to f on $\sigma(A)$. Since $p_n(A)$ is normal, the limit $f(A)$ is also normal.

2.3 Further results

With the functional calculus, we can show the following corollary.

2.11 Corollary. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{B} \subseteq \mathcal{A}$ a C^*-subralgebra and $A \in \mathcal{B}$. Then $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$. The spectrum is therefore independent of the surrounding C^*-algebra.*

Proof. If $aI - A$ is invertible in $\mathcal{B}$, then $aI - A$ is also invertible in $\mathcal{A}$. Therefore, we get $\sigma_{\mathcal{A}}(A) \subseteq \sigma_{\mathcal{B}}(A)$. For the converse implication, it is sufficient to show that if $A \in \mathcal{B}$ has an inverse A^{-1} in $\mathcal{A}$ then $A^{-1} \in \mathcal{B}$. First, we show this for selfadjoint elements.
Let $A \in \mathcal{B}$ be selfadjoint and assume that there exists $A^{-1} \in \mathcal{A}$. Since then $0 \notin \sigma_{\mathcal{A}}(A)$, we know that $f(a) = \frac{1}{a}$ is a continuous function on $\sigma(A)$. We can therefore consider $f(A)$. Since $f(a)a = 1$ for all $a \in \sigma(A)$, we get $f(A)A = I$ and analogously $Af(A) = I$. Thus, $f(A) = A^{-1}$. Choose a sequence of polynomials $p_n(x)$ which converge uniformly to f on $\sigma(A)$. Then $p_n(A) \to f(A)$, and since $\mathcal{B}$ is a closed subalgebra $A^{-1} = f(A) \in \mathcal{B}$.
Now we need to show the same thing for arbitrary $A \in \mathcal{B}$. Let $A \in \mathcal{B}$ and $A^{-1} \in \mathcal{A}$. Then $A^* \in \mathcal{B}$ is also invertible in $\mathcal{A}$ and furthermore $A^*A \in \mathcal{B}$ is also invertible. $C = A^*A \in \mathcal{B}$ is selfadjoint and has an inverse $C^{-1} \in \mathcal{A}$. From before $C^{-1} \in \mathcal{B}$. Then

$$C^{-1}A^*A = I \implies A^{-1} = C^{-1}A^* \in \mathcal{B}.$$

That shows $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$. ■

In the course of this corollary, we have also shown that for a selfadjoint invertible element A we can find polynomials p_n such that $p_n(A)$ converges to A^{-1}. We can even show that the p_n can be chosen in a way that the constant term of every p_n is zero. For that, we consider a bounded interval that contains $\sigma(A)$ und 0 and a continuous function f on I such that $f(a) = \frac{1}{a}$ for all $a \in \sigma(A)$ and $f(0) = 0$. Then we can find polynomials $q_n(x)$ which converge to f uniformly on I. Then $q_n(0) \to f(0) = 0$. Therefore, we can define $p_n(x) := q_n(x) - q_n(0)$ and p_n converges to f uniformly on I in particular on $\sigma(A)$. Then $p_n(A)$ tend to A^{-1} and the constant terms of the p_n are zero.

2.12 Proposition. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ a C^*-homomorphism and $A \in \mathcal{A}$.*

(i) $\sigma(\varphi(A)) \subseteq \sigma(A)$ and $\|\varphi(A)\| \leq \|A\|$. In particular φ is continuous.

(ii) For $f \in C(\sigma(A))$, there is $\varphi(f(A)) = f(\varphi(A))$.

(iii) If φ is an injective C^*-homomorphism, then $\sigma(\varphi(A)) = \sigma(A)$ and $\|\varphi(A)\| = \|A\|$. Furthermore, $\varphi(\mathcal{A})$ is a C^*-subalgebra.

Proof.

(i) First, we show $\sigma(\varphi(A)) \subseteq \sigma(A)$. Let $a \notin \sigma(A)$, then $aI - A \in Inv(\mathcal{A})$. Therefore, $aI - \varphi(A) = \varphi(aI - A) \in Inv(\mathcal{B})$ and $a \notin \sigma(\varphi(A))$. This proves $\sigma(\varphi(A)) \subseteq \sigma(A)$.
Since A^*A is selfadjoint, $\varphi(A^*A)$ is selfadjoint as well and both are normal. Then we can use Proposition 2.8 and get

$$\|A^*A\| = r(A^*A), \|\varphi(A^*A)\| = r(\varphi(A^*A)).$$

We also know $\sigma(\varphi(A^*A)) \subseteq \sigma(A^*A)$ which means $r(\varphi(A^*A)) \leq r(A^*A)$. All in all, we get

$$\|\varphi(A)\|^2 = \|(\varphi(A))^*\varphi(A)\| = \|\varphi(A^*A)\| = r(\varphi(A^*A)) \leq r(A^*A) = \|A^*A\| = \|A\|^2.$$

Then $\|\varphi(A)\| \leq \|A\|$.

(ii) Let $f \in C(\sigma(A))$ and p_n be a sequence of polynomials which converge to f uniformly on $\sigma(A)$. From (i) we know that p_n also converge to f uniformly on $\sigma(\varphi(A)) \subseteq \sigma(A)$. Then $p_n(\varphi(A)) \to f(\varphi(A))$ and $\varphi(p_n(A)) \to \varphi(f(A))$ since φ is continuous. Since φ is a homomorphism, $p_n(\varphi(A)) = \varphi(p_n(A))$ and by taking the limit, we get the equality $f(\varphi(A)) = \varphi(f(A))$.

(iii) Let φ be an injective C^*-homomorphism and $B \in \mathcal{A}$ selfadjoint. Suppose $\sigma(\varphi(B)) \subsetneq \sigma(B)$. Then there exists a continuous function $f \in C(\sigma(B))$ such that f is not constant 0 but $f|_{\sigma(\varphi(B))} \equiv 0$. We also get with (ii)

$$f(B) \neq 0 \wedge \varphi(f(B)) = f(\varphi(B)) = 0.$$

That is a contradiction to φ being injective. Thus, $\sigma(\varphi(B)) = \sigma(B)$.
With $B = A^*A$, we get analogously to (i) $r(\varphi(B)) = r(B)$ and $\|\varphi(A)\| = \|A\|$. Therefore, φ is isometric. $\varphi(\mathcal{A})$ is the image of an isometry and therefore closed. The other properties of a C^*-subalgebra follow since φ is a C^*-homomorphism. Because of Corollary 2.11 and since φ is a bijective homomorphism from $\mathcal{A}$ to $\varphi(\mathcal{A})$, we get $\sigma_{\mathcal{B}}(\varphi(A)) = \sigma_{\varphi(\mathcal{A})}(\varphi(A)) = \sigma_{\mathcal{A}}(A)$. ■

2.13 Proposition. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ a C^*-homomorphism. Then $\varphi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$.*

Proof. It is clear that $\varphi(\mathcal{A})$ is a selfadjoint subalgebra of $\mathcal{B}$ with $I \in \varphi(\mathcal{A})$. Therefore, we only need to show that $\varphi(\mathcal{A})$ is closed. Now let A_n be a sequence in $\mathcal{A}$ with $\varphi(A_n) \to B$ and $B \in \mathcal{B}$. We have to show $B \in \varphi(\mathcal{A})$. For the real parts, we have

$$\varphi(\frac{1}{2}(A_n + A_n^*)) = \frac{1}{2}(\varphi(A_n) + \varphi(A_n)^*) \to \frac{1}{2}(B + B^*).$$

Analogously the imaginary parts of $\varphi(A_n)$ converge to the imaginary part of B. Obviously it is sufficient to show that the real and imaginary part of B are in $\varphi(\mathcal{A})$ to get that B is in $\varphi(\mathcal{A})$. By considering real and imaginary part, we can suppose that A_n and B are selfadjoint without loss of generality. Furthermore, by considering a subsequence of A_n, we can assume that $\|\varphi(A_{n+1}) - \varphi(A_n)\| < 2^{-n}$.
Now for $n \in \mathbb{N}$, we consider the function

$$f_n : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} -2^{-n} & x < -2^{-n} \\ x & -2^{-n} \le x \le 2^{-n} \\ 2^{-n} & x > 2^{-n}. \end{cases}$$

Since $r(\varphi(A_{n+1}) - \varphi(A_n)) < 2^{-n}$, we get that f_n is the identity on $\sigma(\varphi(A_{n+1}) - \varphi(A_n))$. With Proposition 2.12 (ii), we have

$$\varphi(f_n(A_{n+1} - A_n)) = f_n(\varphi(A_{n+1}) - \varphi(A_n)) = \varphi(A_{n+1}) - \varphi(A_n)$$

Furthermore, $\|f_n\|_\infty \le 2^{-n}$ and therefore $\|f_n(A_{n+1} - A_n)\| \le \|f_n\|_\infty \le 2^{-n}$. Then the series $A_1 + \sum_{n=1}^{\infty} f_n(A_{n+1} - A_n)$ is absolutely convergent and also convergent because of the completeness of $\mathcal{A}$. Let $A \in \mathcal{A}$ be the limit of the series. Then

$$\begin{aligned} \varphi(A) &= \lim_{m\to\infty} \left(\varphi(A_1) + \sum_{n=1}^{m-1} \varphi(f_n(A_{n+1} - A_n)) \right) \\ &= \lim_{m\to\infty} \left(\varphi(A_1) + \sum_{n=1}^{m-1} \varphi(A_{n+1}) - \varphi(A_n) \right) \\ &= \lim_{m\to\infty} \varphi(A_m) = B \end{aligned}$$

This means $B \in \varphi(\mathcal{A})$ and $\varphi(\mathcal{A})$ is closed. ■

Using the functional calculus, we can show a spectral mapping theorem for continuous functions.

2.14 Theorem. *Let $\mathcal{A}$ be a C^*-algebra, $A \in \mathcal{A}$ selfadjoint and $f \in C(\sigma(A))$. Then*

$$\sigma(f(A)) = \{f(a) : a \in \sigma(A)\}.$$

Proof. The function $g \mapsto g(A)$ is an injective C^*-homomorphism from $C(\sigma(A))$ to $\mathcal{A}$. Then we derive from Proposition 2.12 (iii) that

$$\sigma_{\mathcal{A}}(f(A)) = \sigma_{C(\sigma(A))}(f) = \{f(a) : a \in \sigma(A)\}.$$

■

3 Order structure

3.1 Positive elements

We extend the concept of positive operators or functions to general C^*-algebras.

3.1 Definition. Let $\mathcal{A}$ be a C^*-algebra and $A \in \mathcal{A}$ selfadjoint. We call A *positive*, if $\sigma(A) \subseteq \mathbb{R}_0^+$, and denote set of all positive elements in $\mathcal{A}$ as $\mathcal{A}^+$.

For a C^*-algebra $\mathcal{A}$ and a C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$, we already know that the spectrum of $B \in \mathcal{B}$ is independent of the surrounding C^*-algebra (Corollary 2.11). Therefore, the property whether $B \in \mathcal{B}$ is positive is also independent of the surrounding C^*-algebra. Thus, we have $\mathcal{B}^+ = \mathcal{B} \cap \mathcal{A}^+$. For another C^*-algebra $\mathcal{C}$ and a C^*-homomorphism Φ from $\mathcal{A}$ to $\mathcal{C}$, we know that for every $A \in \mathcal{A}$ the spectrum $\sigma(\varphi(A)) \subseteq \sigma(A)$ (Proposition 2.12 (i)). Therefore, if $A \in \mathcal{A}$ is positive, then $\varphi(A)$ is positive in $\mathcal{C}$.

3.2 Example. If $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$, then A is positive semidefinite if $\langle Ax, x\rangle \geq 0$ for every $x \in \mathcal{H}$. In functional analysis, we proved that this is equivalent to $\sigma(A) \subseteq \mathbb{R}_0^+$. Therefore, A is positive in the sense of C^*-algebras if and only if A is positive semidefinite in the sense of operators on a Hilbert space.

3.3 Lemma. *Let $\mathcal{A}$ be a C^*-algebra, $A \in \mathcal{A}$ selfadjoint, $a \in \mathbb{R}$ and $a \geq \|A\|$. Then $A \in \mathcal{A}^+$ if and only if $\|A - aI\| \leq a$.*

Proof. We have

$$\|A - aI\| = r(A - aI) = \sup_{t\in\sigma(A)} |t - a| = \sup_{t\in\sigma(A)} (a - t).$$

and hence $\|A - aI\| \leq a$ if and only if $\sigma(A) \subseteq \mathbb{R}_0^+$ which is what we wanted to show. ■

3.4 Proposition. *Let $\mathcal{A}$ be a C^*-Algebra.*

(i) *$\mathcal{A}^+$ is closed.*

(ii) *For $A \in \mathcal{A}^+$ and $a \in \mathbb{R}_0^+$, we have $aA \in \mathcal{A}^+$.*

(iii) *If $A, B \in \mathcal{A}^+$, then $A + B \in \mathcal{A}^+$.*

(iv) *If $A \in \mathcal{A}^+$ and $-A \in \mathcal{A}^+$, then $A = 0$.*

Proof.

(i) Because of Lemma 3.3, we can characterise $\mathcal{A}^+$ as

$$\mathcal{A}^+ = \{A \in \mathcal{A} : A = A^* \wedge \|A - \|A\|I\| \leq \|A\|\}.$$

The norm and * are continuous which shows that $\mathcal{A}^+$ is closed.

(ii) For $A \in \mathcal{A}^+, a \in \mathbb{R}_0^+$, we get
$$\sigma(aA) = \{at : t \in \sigma(A)\} \subseteq \mathbb{R}_0^+.$$

(iii) Using Lemma 3.3 for $A, B \in \mathcal{A}^+$, we get
$$\|A - \|A\|I\| \leq \|A\| \wedge \|B - \|B\|I\| \leq \|B\|.$$
Therefore,
$$\|A + B - (\|A\| + \|B\|)I\| \leq \|A - \|A\|I\| + \|B - \|B\|I\| \leq \|A\| + \|B\|.$$
By Lemma 3.3 with $a = \|A\| + \|B\| \geq \|A + B\|$, we have $A + B \in \mathcal{A}^+$.

(iv) Let $A, -A \in \mathcal{A}^+$. By Definition 3.1, A is selfadjoint and $\sigma(A) \subseteq \mathbb{R}_0^+ \cap \mathbb{R}_0^- = \{0\}$. Using Proposition 2.8 (i) we have $\|A\| = r(A) = 0$.

■

3.2 Decomposition

3.5 Proposition. *Let $\mathcal{A}$ be a C^*−algebra and $A \in \mathcal{A}$ selfadjoint.*

(i) For $f \in C(\sigma(A))$, we have $f(A) \in \mathcal{A}^+$ if and only if $f(a) \geq 0$ for all $a \in \sigma(A)$.

(ii) $\|A\|I \pm A \in \mathcal{A}^+$.

(iii) We can find $A^+, A^- \in \mathcal{A}^+$ with $A = A^+ - A^-$ and $A^+A^- = A^-A^+ = 0$. A^+, A^- are uniquely defined by these properties and $\|A\| = \max(\|A^+\|, \|A^-\|)$.

Proof.

(i) Using Theorem 2.14, we know $\sigma(f(A)) = \{f(a) : a \in \sigma(A)\}$. If $f(A) \in \mathcal{A}^+$, there is immediately $f(a) \geq 0$ for all $a \in \sigma(A)$. Inversely, if $f(a) \geq 0$ for all $a \in \sigma(A)$, then $f(A)$ is selfadjoint because $f(A)^* = \overline{f}(A) = f(A)$. From knowing the spectrum $\sigma(f(A)) = \{f(a) : a \in \sigma(A)\}$, we can follow that $f(A)$ is positive.

(ii) We define $f : \sigma(A) \to \mathbb{R}, a \mapsto \|A\| \pm a$. Then $f \in C(\sigma(A))$ and $f(a) \geq 0$ for all $a \in \sigma(A)$. Therefore, $\|A\|I \pm A = f(A) \in \mathcal{A}^+$ from (i).

(iii) We define the continuous functions u, u^+, u^- from $\mathbb{R}$ to $\mathbb{R}$ by
$$u(t) = t, u^+(t) = \max\{t, 0\}, u^-(t) = \max\{-t, 0\}.$$
Then
$$u = u^+ - u^-, u^+u^- = u^-u^+ = 0. \tag{3.2.1}$$
We define $A^+ := u^+(A), A^- := u^-(A)$, then (3.2.1) implies
$$A = u(A) = u^+(A) - u^-(A) = A^+ - A^-, A^+A^- = A^-A^+ = 0.$$

Since $u^+(a) \geq 0, u^-(a) \geq 0$ for all $a \in \sigma(A)$, we have $A^+, A^- \in \mathcal{A}^+$ by (i). They satisfy the expected conditions. Furthermore, we get

$$\|u\|_\infty = \max\{\|u^+\|_\infty, \|u^-\|_\infty\}$$

and then $\|A\| = \max\{\|A^+\|, \|A^-\|\}$.
We only need to show that they are unique. Let $B, C \in \mathcal{A}^+$ with $A = B - C$ and $BC = CB = 0$. Then by induction, we get

$$\forall n \geq 1 : A^n = B^n + (-C)^n.$$

This means that $p(A) = p(B) + p(-C)$ for all polynomials with constant term 0. Since $u^+(0) = 0$, we find a sequence of polynomials p_n which converge to u^+ uniformly on $\sigma(A) \cup \sigma(B) \cup \sigma(-C)$ and the constant term of each p_n is 0. Then

$$u^+(A) = \lim_{n\to\infty} p_n(A), = \lim_{n\to\infty} (p_n(B) + p_n(-C)) = u^+(B) + u^+(-C).$$

Since $\sigma(B) \subseteq \mathbb{R}_0^+$ and $\sigma(-C) \subseteq \mathbb{R}_0^-$, we have

$$\forall b \in \sigma(B) : u^+(b) = b \implies u^+(B) = B,$$
$$\forall c \in \sigma(-C) : u^+(c) = 0 \implies u^+(-C) = 0.$$

Then $A^+ = u^+(A) = u^+(B) + u^+(-C) = B$ and therefore $C = B - A = A^+ - A = A^-$. We have shown that A^+, A^- are unique.

■

Proposition 3.5 (iii) shows that for every selfadjoint element we find an unique decomposition into a positive and a negative part.

3.6 Corollary. *Let $\mathcal{A}$ be a C^*-algebra. Then every element $A \in \mathcal{A}$ is the linear combination of 4 elements in $\mathcal{A}^+$.*

Proof. Every element $A \in \mathcal{A}$ has a decomposition into real and imaginary part, which are the difference of two elements in $\mathcal{A}^+$ because of Proposition 3.5 (iii). ■

3.7 Lemma. *Let $\mathcal{A}$ be a C^*-algebra. For $A \in \mathcal{A}$ with $-A^*A \in \mathcal{A}^+$, we have $A = 0$.*

Proof. First we prove that for every $A, B \in \mathcal{A}$ we have $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$. This is equivalent to $\rho(AB) \backslash \{0\} = \rho(BA) \backslash \{0\}$ and because of the symmetry it is sufficient to show that one is a subset of the other. Therefore, let $\lambda \neq 0$ and $\lambda \notin \sigma(AB)$ which means that $AB - \lambda I$ is invertible. Then $I - (\lambda^{-1}A)B$ is invertible. Let $C = \lambda^{-1}A$

$$\begin{aligned}(I - BC)(B(I - CB)^{-1}C + I) &= B(I - CB)^{-1}C + I - BCB(I - CB)^{-1}C - BC \\ &= B\left((I - CB)^{-1} - CB(I - CB)^{-1}\right)C + I - BC \\ &= B(I - CB)(I - CB)^{-1}C + I - BC \\ &= BC - BC + I = I\end{aligned}$$

This shows that $B(I-CB)^{-1}C+I$ is the right inverse of $I-BC$. A similar computation shows that it is the left inverse, too. Thus $I-BC = I-B(\lambda^{-1}A)$ is invertible. This shows that $BA-\lambda I$ is invertible and $\lambda \notin \sigma(BA)\cup\{0\}$. Therefore, $\sigma(AB)\cup\{0\} = \sigma(BA)\cup\{0\}$.
Now we can prove the lemma. Let $A \in \mathcal{A}$ with $-A^*A \in \mathcal{A}^+$. We split A into real part H and imaginary part K. This means $A = H+iK$ with H,K selfadjoint. Since $\sigma(H^2) = \{h^2 : h \in \sigma(H)\} \subseteq \mathbb{R}_0^+$, we have that H^2 and similarly K^2 are positive. From before we have $\sigma(-AA^*) \subseteq \sigma(-A^*A)\cup\{0\} \subseteq \mathbb{R}_0^+$ and since AA^* is also selfadjoint $-AA^*$ is positive. Then we get

$$\begin{aligned} & A^*A + AA^* = (H-iK)(H+iK) + (H+iK)(H-iK) = 2H^2 + 2K^2 \\ \implies & A^*A = 2H^2 + 2K^2 + (-AA^*) \end{aligned}$$

Thus A^*A is the sum of three positive elements and therefore positive as well. Then A^*A and $-A^*A$ are positive which means that $A^*A = 0$ using Proposition 3.4 (iv). Then also $\|A\|^2 = \|A^*A\| = 0$ and $A = 0$. ∎

3.8 Theorem. *Let $\mathcal{A}$ be a C^*-Algebra and $A \in \mathcal{A}$. Then the following statements are equivalent:*

(i) $A \in \mathcal{A}^+$

(ii) There exists $H \in \mathcal{A}^+$ with $A = H^2$.

(iii) There exists $B \in \mathcal{A}$ with $A = B^*B$.

If the conditions (i)-(iii) are satisfied, the element H in (ii) is unique.

Proof. *(i)* $\implies$ *(ii)*: $f(t) = t^{\frac{1}{2}}$ is for $t \in \sigma(A) \subseteq \mathbb{R}_0^+$ well defined and continuous. We define $H := f(A)$. Since $f(a) \geq 0$ for all $a \in \sigma(A)$, we get that the element H is positive. Since $f(t)^2 = t$, we get $H^2 = A$.
(ii) $\implies$ *(iii)*: We set $B = H$ and with $H^* = H$ we get $B^*B = H^2 = A$.
(iii) $\implies$ *(i)*: We have $A = B^*B$ with $B \in \mathcal{A}$. Obviously A is selfadjoint therefore by Proposition 3.5 (iii) we get $A = A^+ - A^-$ with the properties from there. Let $C := BA^-$ then we have

$$C^*C = A^-B^*BA^- = A^-(A^+ - A^-)A^- = -(A^-)^3.$$

A^- is in $\mathcal{A}^+$ therefore $-C^*C = (A^-)^3 \in \mathcal{A}^+$ as well. From Lemma 3.7, we know $C = 0$, whereas $(A^-)^3 = 0$ follows. With Corollary 2.9, we get $A^- = 0$ and $A = A^+ \in \mathcal{A}^+$.
Now we only need to show that H is unique. Let $A \in \mathcal{A}^+$, $f(t) = t^{\frac{1}{2}}$ on $\sigma(A)$, $H = f(A)$ and $K \in \mathcal{A}^+$ with $K^2 = A$. We want to show $H = K$. Let (p_n) be a sequence of polynomials which converge uniformly on $\sigma(A)$ to f. We define $q_n(x) := p_n(x^2)$. Since we know that $\sigma(A) = \sigma(K^2) = \{x^2 : x \in \sigma(K)\}$, the limits are

$$\lim_{n\to\infty} q_n(x) = \lim_{n\to\infty} p_n(x^2) = f(x^2) = x$$

uniformly for $x \in \sigma(K)$. Then

$$K = \lim_{n\to\infty} q_n(K) = \lim_{n\to\infty} p_n(K^2) = \lim_{n\to\infty} p_n(A) = f(A) = H.$$

Therefore, we proved that H is unique. ∎

We call the element H from statement (ii) of Theorem 3.8 the *positive square root of* A. We also reference it by $A^{\frac{1}{2}}$. Analogously, we can define $f_\alpha(a) = a^\alpha$ for $\alpha > 0$ and get $A^\alpha := f_\alpha(A)$. Then we have $f_\alpha(a)f_\beta(a) = f_{\alpha+\beta}(a)$ and $f_1(a) = a$ for all $a \in \sigma(A)$. From this we get $A^\alpha A^\beta = A^{\alpha+\beta}, A^1 = A$. If A is invertible, then we can also consider f_α and $A^\alpha = f_\alpha(A)$ for $\alpha \leq 0$. Then we get $A^0 = I$ and the inverse of A as $A^{-1} = f_{-1}(A)$. We see that A^α has the elementary meaning for integer values of α.

3.9 Corollary. *Let $\mathcal{A}$ be a C^*-algebra, $A \in \mathcal{A}^+$ and $B \in \mathcal{A}$. Then $B^*AB \in \mathcal{A}^+$.*

Proof. Since $B^*AB = (A^{\frac{1}{2}}B)^*A^{\frac{1}{2}}B$, we get B^*AB by Theorem 3.8. ∎

3.3 Order structure

Now we can define a partial order for the selfadjoint elements of a C^*-algebra.

3.10 Definition. Let $\mathcal{A}$ be a C^*-algebra. We define $\mathcal{A}_h$ as the set of all selfadjoint elements of $\mathcal{A}$. Furthermore, we define the relation $\leq$ by

$$\forall A, B \in \mathcal{A}_h : A \leq B :\Longleftrightarrow\ B - A \in \mathcal{A}^+.$$

3.11 Proposition. *Let $\mathcal{A}$ be a C^*-algebra.*

(i) *$\mathcal{A}_h$ is closed.*

(ii) *$\leq$ is a partial order.*

(iii) $\mathcal{A}^+ = \{A \in \mathcal{A}_h : A \geq 0\}$.

(iv) *For all $A \in \mathcal{A}_h$, there is $-\|A\|I \leq A \leq \|A\|I$.*

(v) $\|A\| = \inf\{a \geq 0 : -aI \leq A \leq aI\}$.

(vi) *For $A, B, A', B' \in \mathcal{A}_h$ with $A \leq B$ and $A' \leq B'$, it follows that $A + B \leq A' + B'$.*

(vii) *For $A, B \in \mathcal{A}_h, a \geq 0$ with $A \leq B$ the inequality $aA \leq aB$ also holds.*

(viii) *For $A, B \in \mathcal{A}_h, a \leq 0$ with $A \leq B$, the inequality changes directions to $aB \leq aA$.*

(ix) *For $(A_n), (B_n)$ sequences of elements in $\mathcal{A}_h$ and $A_n \to A \in \mathcal{A}_h, B_n \to B \in \mathcal{A}_h$ with $A_n \leq B_n$ for all $n \in \mathbb{N}$, there is also $A \leq B$.*

(x) *For $A, B \in \mathcal{A}_h$ and $C \in \mathcal{A}$ with $A \leq B$, there is $C^*AC \leq C^*BC$.*

(xi) *$A \in \mathcal{A}^+$ is invertible if and only if there exists $a > 0$ such that $A \geq aI$.*

Proof.

(i) This follows immediately from the fact that * is continuous.

(ii) $\leq$ is transitive: Let $A, B, C \in \mathcal{A}_h$ with $A \leq B, B \leq C$. Then $B - A, C - B \in \mathcal{A}^+$. From Proposition 3.4 (iii), we get $C - A = (C - B) + (B - A) \in \mathcal{A}^+$ and $A \leq C$.
$\leq$ is reflective: $\sigma(0) = \{0\} \subseteq \mathbb{R}_0^+$. Therefore, for all $A \in \mathcal{A}_h$, we get $A - A = 0 \in \mathcal{A}^+$ and $A \leq A$.
$\leq$ is antisymmetric: Let $A, B \in \mathcal{A}_h$ with $A \leq B, B \leq A$. Then of course $B - A$ and $-(B - A) = A - B$ are in $\mathcal{A}^+$. From Proposition 3.4 (iv), we get $B - A = 0$ and $A = B$.
Therefore, $\leq$ is a partial order on $\mathcal{A}_h$.

(iii) For $A \in \mathcal{A}_h$ we have

$$A \in \mathcal{A}^+ \iff A - 0 \in \mathcal{A}^+ \iff A \geq 0.$$

(iv) This follows immediately from Proposition 3.5 (ii) $\|A\|I \pm A \in \mathcal{A}^+$ for all $A \in \mathcal{A}_h$.

(v) We have

$$\|A\| = \inf\{a \geq 0 : \forall t \in \sigma(A) : -a \leq t \leq a\}.$$

With Proposition 3.5 (i), we can see that we can use functional calculus here and get

$$\|A\| = \inf\{a \geq 0 : -aI \leq A \leq aI\}.$$

(vi) Let $A, B, A', B' \in \mathcal{A}_h$ with $A \leq B, A' \leq B'$. Now we have

$$(B + B') - (A + A') = (B - A) + (B' - A') \in \mathcal{A}^+$$

from Proposition 3.4 (iii). Therefore, $A + A' \leq B + B'$.

(vii) For $A, B \in \mathcal{A}_h, a \geq 0$ with $A \leq B$, we can use Proposition 3.4 (ii) where we know that $aB - aA = a(B - A) \in \mathcal{A}^+$ which implies $aA \leq aB$.

(viii) Let $A, B \in \mathcal{A}_h, a \leq 0$ with $A \leq B$. Now we use Proposition 3.4 (ii) with $-a$ to get $aA - aB = (-a)(B - A) \in \mathcal{A}^+$ which means $aB \leq aA$.

(ix) This follows from the fact that $\mathcal{A}^+$ is closed.

(x) For $A, B \in \mathcal{A}_h$ and $C \in \mathcal{A}$ with $A \leq B$, we can use Corollary 3.9 to know that $C^*BC - C^*AC = C^*(B - A)C \in \mathcal{A}^+$. This means $C^*AC \leq C^*BC$.

(xi) Let $A \in \mathcal{A}^+$. Then A is invertible if and only if $0 \notin \sigma(A)$ which is equivalent to $\sigma(A) \subseteq [a, \infty)$ for some $a > 0$ since the spectrum is compact. However, $\sigma(A) \subseteq [a, \infty)$ is equivalent to the statement that the function $t \mapsto t - a$ is positive on $\sigma(A)$. By Proposition 3.5 (i), we know that this happens if and only if $A - aI \geq 0$. This proves the equivalence.

■

We see that this partial order behaves much in the same way as the order of real numbers. We can add inequalities and multiply with a positive integer. If we multiply with a negative integer, the sign changes direction and we can take limits.

4 Linear functionals

4.1 Hermitian linear functionals

Now we consider linear functionals on selfadjoint subspaces $\mathcal{M}$ of a C^*-algebra $\mathcal{A}$ which contain the unit I. We refer to the set of all selfadjoint elements of $\mathcal{M}$ as $\mathcal{M}_h$. The set of all positive elements of $\mathcal{M}$ is called $\mathcal{M}^+$ which satisfies $\mathcal{M}^+ = \mathcal{M} \cap \mathcal{A}^+$. If $\mathcal{B} \subseteq \mathcal{A}$ is a C^*-subalgebra and $\mathcal{M} \subseteq \mathcal{B}$, then $\mathcal{M} \cap \mathcal{B}^+ = \mathcal{M} \cap \mathcal{B} \cap \mathcal{A}^+ = \mathcal{M} \cap \mathcal{A}^+$ which shows that $\mathcal{M}^+$ is independent of the surrounding C^*-algebra. $\mathcal{M}$ contains real and imaginary part of every $A \in \mathcal{M}$. Selfadjoint elements $A \in \mathcal{M}$ are the difference of two positive elements $\|A\|I \pm A \in \mathcal{M}^+$ which shows that $\mathcal{M}$ is the linear span of $\mathcal{M}^+$.

4.1 Definition. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a selfadjoint subspace of $\mathcal{A}$. For every linear functional ρ on $\mathcal{M}$, we can define $\rho^* : \mathcal{M} \to \mathbb{C}, A \mapsto \overline{\rho(A^*)}$ which is also a linear functional. Then we call ρ *hermitian* if $\rho = \rho^*$.

4.2 Proposition. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ and ρ a linear functional on $\mathcal{M}$.*

(i) ρ is hermitian if and only if $\rho(H)$ is real for every selfadjoint $H \in \mathcal{H}$.

(ii) ρ can be expressed uniquely in the form $\rho = \rho_1 + i\rho_2$ with hermitian linear functionals ρ_1, ρ_2 on $\mathcal{M}$.

(iii) If ρ is bounded and hermitian, then $\|\rho\| = \sup\{\rho(H) : H = H^*, H \in \mathcal{M}\}$.

Proof.

(i) Let $A \in \mathcal{M}$ and $A = H + iK$ with selfadjoint $H, K \in \mathcal{M}_h$. Then

$$\rho(A) = \rho(H + iK) = \rho(H) + i\rho(K),$$
$$\rho^*(A) = \rho^*(H) + i\rho^*(K) = \overline{\rho(H)} + i\overline{\rho(K)}.$$

We see that $\rho(A) = \rho^*(A)$ for every $A \in \mathcal{M}$ if and only if $\rho(H) = \overline{\rho(H)}$ for every $H \in \mathcal{M}_h$ which is exactly when $\rho(H)$ is real for every selfadjoint $H \in \mathcal{M}$.

(ii) We only need to define $\rho_1 = \frac{1}{2}(\rho + \rho^*), \rho_2 = \frac{1}{2}i(\rho^* - \rho)$.

(iii) Suppose ρ is bounded and hermitian. We note that $\sup\{\rho(H) : H = H^*, H \in \mathcal{M}\}$ is well defined because $\rho(H)$ is real and bounded by $\|\rho\|$ for every selfadjoint $H \in \mathcal{M}$. Therefore, $\sup\{\rho(H) : H = H^*, H \in \mathcal{M}\} \leq \|\rho\|$. For every $\varepsilon > 0$, we can find $A \in \mathcal{M}$ such that $\|A\| \leq 1$ and $|\rho(A)| > \|\rho\| - \varepsilon$. Now we choose $a \in \mathbb{C}$ such that $a\rho(A) = |\rho(A)|$ and get

$$\|\rho\| - \varepsilon < |\rho(A)| = \rho(aA) = \overline{\rho(aA)} = \rho((aA)^*)$$

since ρ is hermitian. We define $H_0 := \frac{1}{2}(aA + (aA)^*) \in \mathcal{M}_h$ as the real part of aA and get $\rho(H_0) = \frac{1}{2}(\rho(aA) + \rho((aA)^*)) = |\rho(A)| > \|\rho\| - \varepsilon$. This shows that $\|\rho\| \leq \sup\{\rho(H) : H = H^*, H \in \mathcal{M}\}$ which proves that they are equal.

■

4.3 Definition. Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ and ρ a linear functional on $\mathcal{M}$. If $\rho(A) \geq 0$ for every $A \in \mathcal{M}^+$ then ρ is called *positive*.

4.4 Proposition. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ and ρ a positive linear functional on $\mathcal{M}$. Then ρ is hermitian.*

Proof. For every selfadjoint $A \in \mathcal{M}$, the elements $\|A\|I \pm A$ are positive. Therefore, we know that $\rho(\|A\|I \pm A) \geq 0$ and $\rho(A) = \frac{1}{2}(\rho(\|A\|I + A) - \rho(\|A\|I - A))$ is real. Then Proposition 4.2 (i) implies that ρ is hermitian. ■

4.5 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and ρ a positive linear functional on $\mathcal{A}$. Then $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}, (A, B) \mapsto \langle A, B \rangle := \rho(B^*A)$ defines a positive semi-definite hermitian sesquilinear form and*

$$\forall A, B \in \mathcal{A} : |\rho(B^*A)|^2 \leq \rho(A^*A)\rho(B^*B). \tag{4.1.1}$$

Proof. We see immediately that $\langle \cdot, \cdot \rangle$ is sesquilinear. For every $A, B \in \mathcal{A}$,

$$\langle A, B \rangle = \rho(B^*A) = \rho((A^*B)^*) = \overline{\rho(A^*B)} = \overline{\langle B, A \rangle}$$

because ρ is hermitian which shows that $\langle \cdot, \cdot \rangle$ is symmetric. Now for every $A \in \mathcal{A}$, the element A^*A is positive and, since ρ is positive, we get $\langle A, A \rangle = \rho(A^*A) \geq 0$. Therefore, $\langle \cdot, \cdot \rangle$ is a positive semi-definite hermitian sesquilinear form. Then it satisfies the Cauchy-Schwarz inequality which yields

$$|\rho(B^*A)|^2 = |\langle A, B \rangle|^2 \leq \langle A, A \rangle \langle B, B \rangle = \rho(A^*A)\rho(B^*B)$$

for all $A, B \in \mathcal{A}$. ■

This proposition shows that for every positive functional ρ on a C^*-algebra $\mathcal{A}$ we can define a corresponding positive semi-definite hermitian sesquilinear form on $\mathcal{A}$. Then we refer to (4.1.1) as the *Cauchy-Schwarz inequality* for ρ.

4.6 Theorem. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}$ a selfadjoint subspace of $\mathcal{M}$ containing I. A linear functional ρ on $\mathcal{M}$ is positive if and only if ρ is bounded and $\rho(I) = \|\rho\|$.*

Proof. First let ρ be a positive linear functional on $\mathcal{M}$ and $A \in \mathcal{M}$. Then we choose $a \in \mathbb{C}$ with $|a| = 1$ such that $a\rho(A) = |\rho(A)|$ and define $H = \frac{1}{2}(aA + (aA)^*)$ the real part of aA. We see $\|H\| \leq \|A\|$ and

$$H \leq \|H\|I \leq \|A\|I \implies \|A\|\rho(I) - \rho(H) = \rho(\|A\|I - H) \geq 0.$$

Since ρ is hermitian, we get

$$|\rho(A)| = \rho(aA) = \overline{\rho(aA)} = \rho((aA)^*)$$

which shows

$$|\rho(A)| = \rho\left(\frac{1}{2}(aA + (aA)^*)\right) = \rho(H) \leq \|A\|\rho(I).$$

This proves that ρ is bounded and $\rho(I) \geq \|\rho\|$. Since $\rho(I) = |\rho(I)| \leq \|\rho\|$, we have $\|\rho\| = \rho(I)$.
For the other implication, suppose ρ is a bounded linear functional with $\rho(I) = \|\rho\|$. Without loss of generality, we can suppose $\|\rho\| = 1$. Let $A \in \mathcal{M}^+$ and $\rho(A) = a + ib$ with real a, b. We want to show $a \geq 0$ and $b = 0$. Since $\sigma(A) \subseteq \mathbb{R}_0^+$, we get for $0 < s \leq \frac{1}{\|A\|}$

$$\sigma(I - sA) = \{1 - st : t \in \sigma(A)\} \subseteq [0, 1].$$

Then $\|I - sA\| = r(I - sA) \leq 1$. Therefore,

$$1 - sa \leq |1 - s(a + ib)| = |\rho(I - sA)| \leq 1.$$

This shows $a \geq 0$. We define $B_n := A - aI + inbI$ for every $n \in \mathbb{N}$ and get

$$\begin{aligned}\|B_n\|^2 &= \|B_n^* B_n\| = \|(A - aI - inbI)(A - aI + inbI)\| \\ &= \|(A - aI)^2 + n^2 b^2 I\| \\ &\leq \|A - aI\|^2 + n^2 b^2.\end{aligned}$$

Then with $\rho(B_n) = i(n + 1)b$, we have

$$(n^2 + 2n + 1)b^2 = |\rho(B_n)|^2 \leq \|B_n\|^2 \leq \|A - aI\|^2 + n^2 b^2.$$

We see $b = 0$ and therefore ρ is positive. ■

4.2 States

4.7 Definition. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ containing I. A linear functional ρ on $\mathcal{M}$ is called a *state* of $\mathcal{M}$ if ρ is positive and $\rho(I) = 1$. We denote the set of all states of $\mathcal{M}$ as $\mathcal{S}(\mathcal{M})$. We call $\mathcal{S}(\mathcal{M})$ the *state space* of $\mathcal{M}$.

4.8 Example. Given a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$, we define a linear functional on $\mathcal{B}(\mathcal{H})$ with $\omega_x : \mathcal{B}(\mathcal{H}) \to \mathbb{C}, T \mapsto \langle Tx, x\rangle$. This linear functional is positive because for every $T \in \mathcal{B}(\mathcal{H})^+$ we have $\omega_x(T) = \langle Tx, x\rangle \geq 0$. We see $\omega_x(I) = \|x\|^2$ and therefore ω_x is a state if $\|x\| = 1$. For a C^*-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ which contains I, we call $\omega_x|_{\mathcal{M}}$ a *vector state* if $\|x\| = 1$.

The next Proposition shows that the state space with the weak* topology is a compact Hausdorff space.

4.9 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ containing I. Then $\mathcal{S}(\mathcal{M})$ is part of $\{\rho \in \mathcal{M}' : \|\rho\| = 1\}$ where $\mathcal{M}'$ is the Banach dual space of $\mathcal{M}$. Furthermore, $\mathcal{S}(\mathcal{M})$ is convex and weak*-compact.*

Proof. Theorem 4.6 shows that, for every state $\rho \in \mathcal{S}(\mathcal{M})$, we know that $\rho \in \mathcal{M}'$ and $\|\rho\| = 1$. We also get the characterisation

$$\mathcal{S}(\mathcal{M}) = \{\rho \in \mathcal{M}' : \rho(I) = 1 \land \forall A \in \mathcal{M}^+ : \rho(A) \geq 0\}.$$

From this, we immediately see that $\mathcal{S}(\mathcal{M})$ is convex and weak*-closed. Since the unit ball in $\mathcal{M}'$ is weak*-compact by the Banach-Alaoglu Theorem, we know that $\mathcal{S}(\mathcal{M})$ is also weak*-compact. ■

4.10 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ containing I. For every $A \in \mathcal{M}$ and $a \in \sigma(A)$, we find a state ρ of $\mathcal{M}$ with $\rho(A) = a$.*

Proof. Let $A \in \mathcal{M}$ and $a \in \sigma(A)$. We define

$$\rho_0 : \{bA + cI : b, c \in \mathbb{C}\} \to \mathbb{C}, bA + cI \mapsto ba + c$$

Then ρ_0 is obviously linear and since $ba + c \in \sigma(bA + cI)$ for $b, c \in \mathbb{C}$, we can see that $|\rho_0(bA+cI)| \leq r(bA+cI) \leq \|bA+cI\|$. Therefore, ρ_0 is bounded and since also $\rho_0(I) = 1$, we know $\|\rho_0\| = 1$. Using the Hahn-Banach Theorem, we can extend ρ_0 from the subspace $\{bA + cI : b, c \in \mathbb{C}\}$ of $\mathcal{M}$ to a bounded linear functional ρ on $\mathcal{M}$. Then ρ satisfies $\|\rho\| = \|\rho_0\| = 1 = \rho_0(I) = \rho(I)$ and $\rho(A) = \rho_0(A) = a$. From Theorem 4.6, we get that ρ is a state of $\mathcal{M}$ with $\rho(A) = a$. ■

4.11 Theorem. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ containing I and $A \in \mathcal{M}$.*

(i) $A = 0$ if $\rho(A) = 0$ for every state ρ of $\mathcal{M}$.

(ii) A is selfadjoint if $\rho(A)$ is real for every state ρ of $\mathcal{M}$.

(iii) $A \in \mathcal{M}^+$ if $\rho(A) \geq 0$ for every state ρ of $\mathcal{M}$.

(iv) If A is normal, we find a state ρ of $\mathcal{M}$ with $|\rho(A)| = \|A\|$.

Proof.

(i) First, we assume A is selfadjoint and $\rho(A) = 0$ for every state ρ of $\mathcal{M}$. Using Proposition 4.10, we get $\sigma(A) = \{0\}$ and $\|A\| = r(A) = 0$.
Now, for the general case let $A \in \mathcal{M}$ with $\rho(A) = 0$ for every state ρ of $\mathcal{M}$. Let $A = H + iK$ with the selfadjoint real and imaginary part H and K. Then we get $\rho(A) = \rho(H) + i\rho(K) = 0$ with $\rho(H), \rho(K)$ real for every state ρ of $\mathcal{M}$. Therefore, we see $\rho(H) = \rho(K) = 0$ for every state ρ of $\mathcal{M}$. From before we know $H = K = 0$ and $A = H + iK = 0$.

(ii) Suppose $\rho(A)$ is real for every state ρ of $\mathcal{M}$. Then for every state ρ of $\mathcal{M}$

$$\rho(A - A^*) = \rho(A) - \overline{\rho(A)} = 0.$$

From (i) we get $A - A^* = 0$ and A is selfadjoint.

(iii) If $\rho(A) \geq 0$ for every state ρ of $\mathcal{M}$, we already know that A is selfadjoint. By Proposition 4.10, we know that $\sigma(A) \subseteq \mathbb{R}_0^+$ and therefore A is positive.

(iv) If A is normal, we know $r(A) = \|A\|$ which means we find $a \in \sigma(A)$ with $|a| = \|A\|$. Using Proposition 4.10, we find a state ρ such that $|\rho(A)| = |a| = \|A\|$.

■

4.3 Decomposition

Just like we found a decomposition for selfadjoint elements of a C^*-algebra into positive and negative part, we can find a decomposition for bounded hermitian linear functionals into positive and negative part. To prove this, we need the following two lemmas first.

4.12 Lemma. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ which contains I and $\mathcal{S}_0$ a set of states of $\mathcal{M}$. If for every selfadjoint $H \in \mathcal{M}$*

$$\|H\| = \sup\{|\rho(H)| : \rho \in \mathcal{S}_0\}$$

then the set $\overline{co(\mathcal{S}_0 \cup -\mathcal{S}_0)}$ is the set of all bounded hermitian linear functionals on $\mathcal{M}$ with norm less than or equal to 1 where $\overline{co(\mathcal{S}_0 \cup -\mathcal{S}_0)}$ is the weak-closed convex hull of $\mathcal{S}_0 \cup -\mathcal{S}_0$.*

Proof. The set of all bounded hermitian linear functionals on $\mathcal{M}$ with norm smaller or equal than 1 is convex and weak*-closed and since it contains $\mathcal{S}_0 \cup -\mathcal{S}_0$ we know that it contains $\overline{\text{co}(\mathcal{S}_0 \cup -\mathcal{S}_0)}$. Now we only need to show that every bounded hermitian linear functional on $\mathcal{M}$ with norm smaller or equal than 1 is an element of $\overline{\text{co}(\mathcal{S}_0 \cup -\mathcal{S}_0)}$. Suppose there is a bounded hermitian linear functional ρ_0 with $\|\rho_0\| \leq 1$ that is not element of $\overline{\text{co}(\mathcal{S}_0 \cup -\mathcal{S}_0)}$. By the Hahn-Banach Theorem, we find a continuous (in respect to the weak* topology) linear functional on $\mathcal{M}'$ which separates ρ_0 and $\overline{\text{co}(\mathcal{S}_0 \cup -\mathcal{S}_0)}$. By definition of the weak* topology, all continuous linear functionals on $\mathcal{M}'$ can be represented by an element in $\mathcal{M}$. Therefore, we find $A \in \mathcal{M}$ and $a \in \mathbb{R}$ such that

$$\text{Re}\rho_0(A) > a \land \forall \rho \in \overline{\text{co}(\mathcal{S}_0 \cup -\mathcal{S}_0)} : \text{Re}\rho(A) \leq a.$$

Let H be the real part of A, then we get for a hermitian linear functional ρ on $\mathcal{M}$

$$\rho(H) = \frac{1}{2}(\rho(A) + \rho(A^*)) = \frac{1}{2}(\rho(A) + \overline{\rho(A)}) = \text{Re}\rho(A).$$

Therefore, we have $\rho_0(H) > a$ and $\rho(H) \leq a$ for every $\rho \in \overline{\text{co}(\mathcal{S}_0 \cup -\mathcal{S}_0)}$. In particular, we have $|\rho(H)| \leq a$ for every $\rho \in \mathcal{S}_0$. Then we see

$$a < \rho_0(H) \leq \|H\| = \sup\{|\rho(H)| : \rho \in \mathcal{S}_0\} \leq a.$$

This is a contradiction. ■

4.13 Lemma. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ containing I. Then*

$$\overline{co(\mathcal{S}(\mathcal{M}) \cup -\mathcal{S}(\mathcal{M}))} = \{a\sigma - b\tau : \sigma, \tau \in \mathcal{S}(\mathcal{M}), a, b \in \mathbb{R}_0^+, a + b = 1\}$$

Proof. Let $\mathcal{S}_0$ be the right-hand set. It is obvious that $\mathcal{S}_0 \subseteq \overline{\mathrm{co}(\mathcal{S}(\mathcal{M}) \cup -\mathcal{S}(\mathcal{M}))}$. We want to show that they are equal. First, we show that $\mathcal{S}_0$ is convex. Let $a\sigma - b\tau, a'\sigma' - b'\sigma' \in \mathcal{S}_0$ and $a + b = a' + b' = c + d = 1$ with $a, b, a', b', c, d \in \mathbb{R}_0^+$ and $\sigma, \sigma', \tau, \tau' \in \mathcal{S}(\mathcal{M})$. We have to differentiate three cases:

1. case $(a > 0 \vee a' > 0) \wedge (b > 0 \vee b' > 0)$: Let $\lambda := ac + a'd > 0, \mu := bc + b'd > 0$. Then $\lambda + \mu = 1$. We define $\rho := \frac{1}{\lambda}(ac\sigma + a'd\sigma')$ and $\nu := \frac{1}{\mu}(bc\tau + b'd\tau')$. Then we know since $\mathcal{S}(\mathcal{M})$ is convex that ρ and ν are states of $\mathcal{M}$. Therefore,
$$c(a\sigma - b\tau) + d(a'\sigma' - b'\tau') = \lambda\rho - \mu\nu \in \mathcal{S}_0.$$
2. case $(a = 0 \wedge a' = 0)$: Now we see that $b = b' = 1$ and because $\mathcal{S}(\mathcal{M})$ is convex $c\tau + d\tau' \in \mathcal{S}(\mathcal{M})$. Then
$$c(a\sigma - b\tau) + d(a'\sigma' - b'\tau') = 0 - 1 \cdot (c\tau + d\tau') \in \mathcal{S}_0.$$
3. case $(b = 0 \wedge b' = 0)$: Now we see that $a = a' = 1$ and $c\sigma + d\sigma' \in \mathcal{S}(\mathcal{M})$. Then
$$c(a\sigma - b\tau) + d(a'\sigma' - b'\tau') = 1 \cdot (c\sigma + d\sigma') - 0 \in \mathcal{S}_0.$$

Therefore, $\mathcal{S}_0$ is convex. Since $\mathcal{S}_0$ is the range of the continuous function
$$F : \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M}) \times [0, 1] \to \mathcal{M}', (\sigma, \tau, a) \mapsto a\sigma - (1 - a)\tau$$
and $\mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M}) \times [0, 1]$ is weak*-compact (Proposition 4.9), we know that $\mathcal{S}_0$ is also weak*-compact. Since $\mathcal{S}_0$ is convex, weak*-closed and contains $\mathcal{S}(\mathcal{M}) \cup -\mathcal{S}(\mathcal{M})$, we know $\mathcal{S}_0 = \overline{\mathrm{co}(\mathcal{S}(\mathcal{M}) \cup -\mathcal{S}(\mathcal{M}))}$. ■

Now we can prove the decomposition of bounded hermitian linear functionals.

4.14 Theorem. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a selfadjoint subspace of $\mathcal{A}$ which contains I and ρ a bounded hermitian linear functional on $\mathcal{M}$. Then we find two positive linear functionals ρ^+, ρ^- on $\mathcal{M}$ which satisfy $\rho = \rho^+ - \rho^-$ and $\|\rho\| = \|\rho^+\| + \|\rho^-\|$. In the case that $\mathcal{M}$ is the whole C^*-algebra $\mathcal{A}$, then ρ^+, ρ^- are uniquely determined by these conditions.*

Proof. We assume without loss of generality $\|\rho\| = 1$. For every selfadjoint $A \in \mathcal{M}$, Theorem 4.11 (iv) tells us that
$$\|A\| = \sup\{|\tau(A)| : \tau \in \mathcal{S}(\mathcal{M})\}$$
since $|\tau(A)| \leq \|A\|$ for every state τ of $\mathcal{M}$. Therefore, the state space $\mathcal{S}(\mathcal{M})$ satisfies the conditions of Lemma 4.12 and we know $\rho \in \overline{\mathrm{co}(\mathcal{S}(\mathcal{M}) \cup -\mathcal{S}(\mathcal{M}))}$. Using Lemma 4.13, we know $\rho = a\sigma - b\tau$ with $\sigma, \tau \in \mathcal{S}(\mathcal{M})$, $a, b \in \mathbb{R}_0^+$ and $a + b = 1$. Then we define $\rho^+ = a\sigma, \rho^- = b\tau$. Then we have $\rho = \rho^+ - \rho^-$ and
$$\|\rho\| = 1 = a + b = \|\rho^+\| + \|\rho^-\|.$$

Now we need to show that for $\mathcal{M} = \mathcal{A}$ the decomposition is unique. Therefore, we assume from now on $\mathcal{M} = \mathcal{A}$ and let μ, ν be positive linear functionals on $\mathcal{A}$ which satisfy $\rho = \mu - \nu$ and $\|\rho\| = 1 = \|\mu\| + \|\nu\|$ just like ρ^+, ρ^-. Let $\varepsilon > 0$. Then we find a selfadjoint $H \in \mathcal{A}$ with $\|H\| \leq 1$ and $\rho(H) > \|\rho\| - \frac{1}{2}\varepsilon^2$. We define $K := \frac{1}{2}(I - H)$. Since $-I \leq H \leq I$, we know $0 \leq K \leq I$ and therefore $K, I - K \in \mathcal{A}^+$. Then

$$\mu(I) + \nu(I) = \|\mu\| + \|\nu\| = \|\rho\| < \rho(H) + \frac{1}{2}\varepsilon^2 = \mu(H) - \nu(H) + \frac{1}{2}\varepsilon^2$$
$$\Longrightarrow \mu(I - H) + \nu(I + H) < \frac{1}{2}\varepsilon^2 \implies \mu(K) + \nu(I - K) < \frac{1}{4}\varepsilon^2$$
$$\Longrightarrow 0 \leq \mu(K) < \frac{1}{4}\varepsilon^2 \wedge 0 \leq \nu(I - K) < \frac{1}{4}\varepsilon^2$$

because μ, ν are positive linear functionals. Now let $A \in \mathcal{A}$ and with the Cauchy-Schwartz inequality for μ and ν we get

$$|\mu(KA)|^2 = |\mu(K^{\frac{1}{2}}K^{\frac{1}{2}}A)|^2 \leq \mu(K)\mu(A^*KA) \leq \frac{1}{4}\varepsilon^2\|K^{\frac{1}{2}}A\|^2 \leq \frac{1}{4}\varepsilon^2\|A\|^2,$$
$$|\nu((I - K)A)| \leq \nu(I - K)\nu(A^*(I - K)A) \leq \frac{1}{4}\varepsilon^2\|A\|^2.$$

We can use the same argument for ρ^+, ρ^- and get

$$|\mu(KA)| \leq \frac{\varepsilon}{2}\|A\|, \qquad |\rho^+(KA)| \leq \frac{\varepsilon}{2}\|A\|$$
$$|\nu((I - K)A)| \leq \frac{\varepsilon}{2}\|A\|, \qquad |\rho^-((I - K)A)| \leq \frac{\varepsilon}{2}\|A\|.$$

From $\rho = \rho^+ - \rho^- = \mu - \nu$, we get $\mu - \rho^+ = \nu - \rho^-$. Then we have

$$\begin{aligned}\mu(A) - \rho^+(A) &= \nu(A) - \rho^-(A) + \mu(KA) - \rho^+(KA) - \nu(KA) + \rho^-(KA)\\ &= \mu(KA) - \rho^+(KA) + \nu((I - K)A) - \rho^-((I - K)A).\end{aligned}$$

Therefore, $|\mu(A) - \rho^+(A)| < 2\varepsilon\|A\|$ for every $\varepsilon > 0$. Then we know $\mu = \rho^+$ and $\nu = \rho^-$. ■

4.15 Corollary. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a selfadjoint subspace of $\mathcal{A}$ which contains I and ρ a bounded linear functional on $\mathcal{M}$. Then ρ is the linear combination of four states of $\mathcal{M}$.*

Proof. By Proposition 4.2 (ii), we find σ, τ bounded hermitian linear functionals on $\mathcal{M}$ such that $\rho = \sigma + i\tau$ with . Theorem 4.14 shows that we can find $\sigma^+, \sigma^-, \tau^+, \tau^-$ positive linear functionals on $\mathcal{M}$ such that $\sigma = \sigma^+ - \sigma^-, \tau = \tau^+ - \tau^-$ which implies $\rho = \sigma^+ - \sigma^- + i\tau^+ - i\tau^-$. Every positive linear functional on $\mathcal{M}$ is a scalar multiple of a state of $\mathcal{M}$ which proves that every bounded linear functional on $\mathcal{M}$ is a linear combination of four states of $\mathcal{M}$. ■

4.4 Pure states

4.16 Definition. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ which contains I. We call ρ a *pure state* of $\mathcal{M}$ if ρ is an extreme point of $\mathcal{S}(\mathcal{M})$. We define $\mathcal{P}(\mathcal{M})$ as the set of all pure states of $\mathcal{M}$. Then the weak* closure $\overline{\mathcal{P}(\mathcal{M})}$ is called the *pure state space.*

From the Krein-Milman Theorem, we know that $\mathcal{S}(\mathcal{M})$ is weak*-closed convex hull of $\mathcal{P}(\mathcal{M})$.

4.17 Theorem. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ which contains I and $A \in \mathcal{M}$.*

(i) $A = 0$ if $\rho(A) = 0$ for every pure state ρ of $\mathcal{M}$.

(ii) A is selfadjoint if $\rho(A)$ is real for every pure state ρ of $\mathcal{M}$.

(iii) A is positive if $\rho(A) \geq 0$ for every pure state ρ of $\mathcal{M}$.

(iv) If A is normal, there exists a pure state ρ_0 of $\mathcal{M}$ with $|\rho_0(A)| = \|A\|$

Proof. All the properties of (i)-(iii) are extended to weak* limits of convex combinations of pure states. Since we know $\mathcal{S}(\mathcal{M}) = \overline{\text{co}(\mathcal{P}(\mathcal{M}))}$, the statements (i)-(iii) are proven by the corresponding statements in Theorem 4.11.
For (iv) let A be normal. Using Theorem 4.11, we know that there exists a state τ such that $\tau(A) = c$ and $|c| = \|A\|$. Now let $a \in \mathbb{C}$ be such that $\tau(aA) = ac = \|A\|$ and $|a| = 1$. We consider the set $\mathcal{S}_0 := \{\rho \in \mathcal{S}(\mathcal{M}) : \text{Re}\rho(aA) = \|A\|\}$. We see that $\mathcal{S}_0$ is convex, compact and nonempty since $\tau \in \mathcal{S}_0$. From the Krein-Milman theorem, we know that there exists an extreme point ρ_0 of $\mathcal{S}_0$. We prove that ρ_0 is a pure state. Let $0 < t < 1, \rho_1, \rho_2 \in \mathcal{S}(\mathcal{M})$ with $(1-t)\rho_1 + t\rho_2 = \rho_0$. We know $\text{Re}\rho_1(aA) \leq |\rho_1(aA)| \leq \|A\|$ and similarly $\text{Re}\rho_2(aA) \leq \|A\|$. We also know

$$(1-t)\text{Re}\rho_1(aA) + t\text{Re}\rho_2(aA) = \text{Re}(1-t)\rho_1(aA) + t\rho_2(aA) = \text{Re}\rho_0(aA) = \|A\|.$$

Then we see $\text{Re}\rho_1(aA) = \text{Re}\rho_2(aA) = \|A\|$ and $\rho_1, \rho_2 \in \mathcal{S}_0$. Since ρ_0 is an extreme point of $\mathcal{S}_0$, we know $\rho_1 = \rho_2 = \rho_0$ and ρ_0 is a pure state. Furthermore, we get

$$\|A\| = \text{Re}\rho_0(aA) \leq |\rho_0(A)| \leq \|A\|.$$

Therefore, $|\rho_0(A)| = \|A\|$. ■

4.18 Theorem. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a selfadjoint subspace of $\mathcal{A}$ which contains I and $\mathcal{S}_0$ a subset of the state space $\mathcal{S}(\mathcal{M})$. Then these four conditions are equivalent:*

(i) A is positive if $\rho(A) \geq 0$ for every $\rho \in \mathcal{S}_0$.

(ii) For every selfadjoint $H \in \mathcal{M}$, there is $\|H\| = \sup\{|\rho(H)| : \rho \in \mathcal{S}_0\}$.

(iii) $\overline{co(\mathcal{S}_0)} = \mathcal{S}(\mathcal{M})$.

(iv) $\mathcal{P}(\mathcal{M}) \subseteq \overline{\mathcal{S}_0}$.

Proof. (i) $\Longrightarrow$ (ii): Let $H \in \mathcal{M}$ be selfadjoint. We define $a = \sup\{|\rho(H)| : \rho \in \mathcal{S}_0\}$. Then we know $a \leq \|H\|$ and

$$\forall \rho \in \mathcal{S}_0 : \rho(aI \pm H) = a \pm \rho(H) \geq 0.$$

From (i), we know $aI \pm H \in \mathcal{M}^+$ and $-aI \leq H \leq aI$ which implies $\|H\| \leq a$ because of Proposition 3.11 (v). Therefore, $\|H\| = \sup\{|\rho(H)| : \rho \in \mathcal{S}_0\}$.
(ii) $\Longrightarrow$ (iii): We define $\mathcal{S}_1 = \overline{\text{co}(\mathcal{S}_0)}$. Now we know $\mathcal{S}_1 \subseteq \mathcal{S}(\mathcal{M})$. Therefore, from (ii) we also get $\|H\| = \sup\{|\rho(H)| : \rho \in \mathcal{S}_1\}$. Lemma 4.12 tells us that $\overline{\text{co}(\mathcal{S}_1 \cup -\mathcal{S}_1)}$ is the set of all hermitian linear functionals which are bounded by 1. Of course $\mathcal{S}(\mathcal{M})$ is a part of this. We define

$$\mathcal{S}_2 := \{a\sigma - b\tau : \sigma, \tau \in \mathcal{S}_1, a, b \in \mathbb{R}_0^+, a + b = 1\}.$$

Just like in Lemma 4.13, we get that $\mathcal{S}_2$ is convex and weak*-compact because $\mathcal{S}_1$ is convex and weak*-compact. Since it also contains $\mathcal{S}_1 \cup -\mathcal{S}_1$, we know that $\mathcal{S}_2 = \overline{\text{co}(\mathcal{S}_1 \cup -\mathcal{S}_1)}$. Therefore, for every state ρ of $\mathcal{M}$, we find $\sigma, \tau \in \mathcal{S}_1$,$a, b \in \mathbb{R}_0^+$ with $\rho = a\sigma - b\tau$ and $a + b = 1$. Then

$$1 = \rho(I) = a\sigma(I) - b\tau(I) = a - b = 1 - 2b.$$

This shows $b = 0$ and $\rho = \sigma \in \mathcal{S}_1$. This shows $\mathcal{S}(\mathcal{M}) = \mathcal{S}_1$.
(iii) $\Longrightarrow$ (iv): From functional analysis, we know since $\overline{\text{co}(\mathcal{S}_0)} = \mathcal{S}(\mathcal{M})$ is compact that all extremal points of $\mathcal{S}(\mathcal{M})$ are contained in $\overline{\mathcal{S}_0}$. This means $\mathcal{P}(\mathcal{M}) \subseteq \overline{\mathcal{S}_0}$.
(iv) $\Longrightarrow$ (i): If $\rho(A) \geq 0$ for every ρ in $\mathcal{S}_0$, then it is also true for $\rho \in \overline{\mathcal{S}_0}$. Since the pure states are contained in $\overline{\mathcal{S}_0}$, Theorem 4.17 (iii) implies that A is positive. ∎

4.19 Corollary. *Let $\mathcal{H}$ be a Hilbert space and $H \in \mathcal{B}(\mathcal{H})$ a selfadjoint operator. Then*

$$\|H\| = \sup\{|\langle Hx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

For a selfadjoint subspace $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ which contains I the set $\mathcal{S}_0$ of all vector states of $\mathcal{M}$ satisfies $\mathcal{P}(\mathcal{M}) \subseteq \overline{\mathcal{S}_0}$ and $\mathcal{S}(\mathcal{M}) = \overline{co(\mathcal{S}_0)}$.

Proof. If there is $A \in \mathcal{M}$ with $\rho(A) \geq 0$ for every $\rho \in \mathcal{S}_0$, we have $\langle Ax, x\rangle \geq 0$ for every $x \in \mathcal{H}$. Then A is positive by definition. Therefore, we immediately get from Theorem 4.18 that $\mathcal{S}(\mathcal{M}) = \overline{\text{co}(\mathcal{S}_0)}$ and $\mathcal{P}(\mathcal{M}) \subseteq \overline{\mathcal{S}_0}$. If we set $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we get

$$\|H\| = \sup\{|\langle Hx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$$

for every selfadjoint $H \in \mathcal{B}(\mathcal{H})$. ∎

5 GNS construction

5.1 Representations

5.1 Definition. If $\mathcal{A}$ is a C^*-algebra and $\mathcal{H}$ is a Hilbert space, then a C^*-homomorphism φ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is called a *representation* of $\mathcal{A}$ on $\mathcal{H}$. If further φ is injective, then it is called a *faithful representation* of $\mathcal{A}$ on $\mathcal{H}$.

The objective in this chapter is to show that every C^*-algebra has a faithful representation on some Hilbert space.

5.2 Definition. Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{H}$ be a Hilbert space and φ be a representation of $\mathcal{A}$ on $\mathcal{H}$. If $\varphi(\mathcal{A})x = \{\varphi(A)x : A \in \mathcal{A}\}$ is everywhere dense in $\mathcal{H}$ for $x \in \mathcal{H}$, we call φ a *cyclic* representation and x a *cyclic vector* (or *generating vector*) for φ.

We will see that there is a connection between states of a C^*-algebra $\mathcal{A}$ and cyclic representations.

5.3 Example. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.
The first and most simple example of a representation is the inclusion mapping of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H})$. The inclusion mapping is obviously an injective C^*-homomorphism and therefore a faithful representation of $\mathcal{A}$ on $\mathcal{H}$.
If $\mathcal{K} \subseteq \mathcal{H}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{K}$ is invariant under every operator of $\mathcal{A}$, then we can define the *compression* $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{K}), A \mapsto A|_{\mathcal{K}}$. We know

$$\forall A \in \mathcal{A} \forall x, y \in \mathcal{K} : \langle Ax, y\rangle = \langle x, A^*y\rangle.$$

This shows $(A|_{\mathcal{K}})^* = A^*|_{\mathcal{K}}$ for every $A \in \mathcal{A}$. The other properties of a C^*-homomorphism are apparent for φ. Thus, we know that the compression φ is a representation of $\mathcal{A}$ on $\mathcal{K}$.

5.4 Example. Let again $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. We can use the construction from the last example to construct a cyclic representation. We consider $\mathcal{A}x := \{Ax : A \in \mathcal{A}\}$ for $x \in \mathcal{H}$. Then $\mathcal{A}x$ is invariant under each operator in $\mathcal{A}$ wherefore the same is true for its closure $\overline{\mathcal{A}x}$. We have seen before that the compression $\varphi : \mathcal{A} \to \mathcal{B}(\overline{\mathcal{A}x}), A \mapsto A|_{\overline{\mathcal{A}x}}$ is a representation. Since $\mathcal{A}x$ is everywhere dense in $\overline{\mathcal{A}x}$, we know that φ is a cyclic representation with cyclic vector x.

5.5 Example. For other types of C^*-algebras we can also find representations. In Chapter 1 we have discussed that L^∞ is a C^*-algebra for a measure space $(\Omega, \mathscr{A}, \mu)$. For every $f \in L^\infty$ we can define the multiplication operator $M_f : L^2 \to L^2, g \mapsto fg$. This is well-defined since f is essentially bounded. M_f is linear and bounded by $\|f\|_\infty$. The function $\varphi : L^\infty \to \mathcal{B}(L^2), f \mapsto M_f$ is linear and multiplicative. For $f \in L^\infty, g, h \in L^2$, we have

$$\langle M_f(g), h\rangle = \langle fg, h\rangle = \int_\Omega fg\overline{h} \, \mathrm{d}\mu = \int_\Omega g\overline{(\overline{f}h)} \, \mathrm{d}\mu = \langle g, \overline{f}h\rangle = \langle g, M_{\overline{f}}(h)\rangle.$$

Since $f^* = \overline{f}$ in L^∞, we see that φ is a representation. It is faithful since $fg = 0$ for every $g \in L^2$ implies that $f = 0$ almost everywhere. Similarly, if X is a compact interval of $\mathbb{R}$, then $C(X)$ is a C^*-subalgebra of L^∞ with the Lebesgue measure on X. Thus restricting φ to $C(X)$ that is $\varphi|_{C(X)} : C(X) \to \mathcal{B}(L^2), f \mapsto M_f$ yields a faithful representation as well.

5.2 The Gelfand-Naimark-Segal construction

5.6 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and ρ a state of $\mathcal{A}$. Then*

$$\mathcal{L}_\rho := \{A \in \mathcal{A} : \rho(A^*A) = 0\}$$

is a closed left ideal in $\mathcal{A}$ and there is $\rho(B^*A) = 0$ if $A \in \mathcal{L}_\rho$ and $B \in \mathcal{A}$. Furthermore, we define

$$\langle A + \mathcal{L}_\rho, B + \mathcal{L}_\rho \rangle := \rho(B^*A)$$

where $\langle \cdot, \cdot, \rangle$ is an inner product on the quotient linear space $\mathcal{A}/\mathcal{L}_\rho$.

Proof. ρ is positive and using Proposition 4.5 we know that

$$\langle A, B \rangle_0 := \rho(B^*A)$$

defines a positive semi-definite hermitian sesquilinear form on $\mathcal{A}$. Then $\mathcal{L}_\rho = \{A \in \mathcal{A} : \rho(A^*A) = 0\}$ is a linear subspace of $\mathcal{A}$. For $A, B \in \mathcal{L}_\rho$ and $s, t \in \mathbb{C}$, we have

$$\langle sA + tB, sA + tB \rangle_0 = s\overline{s}\langle A, A \rangle_0 + s\overline{t}\langle A, B \rangle_0 + t\overline{s}\overline{\langle A, B \rangle_0} + t\overline{t}\langle B, B \rangle_0 = 2\mathrm{Re}\ s\overline{t}\langle A, B \rangle_0.$$

Proposition 4.5 shows that $|\langle A, B \rangle_0|^2 = |\rho(B^*A)|^2 \leq \rho(A^*A)\rho(B^*B) = 0$ and therefore $\langle sA + tB, sA + tB \rangle_0 = 0$. Then $sA + tB \in \mathcal{L}_\rho$ and $\mathcal{L}_\rho$ is indeed a linear subspace of $\mathcal{A}$. If $A \in \mathcal{L}_\rho$ and $B \in \mathcal{A}$, we also have $|\langle A, B \rangle_0|^2 = |\rho(B^*A)|^2 \leq \rho(A^*A)\rho(B^*B) = 0$ which shows $\langle A, B \rangle_0 = \rho(B^*A) = 0$. This means that

$$\langle A + \mathcal{L}_\rho, B + \mathcal{L}_\rho \rangle := \langle A, B \rangle_0 = \rho(B^*A)$$

is well-defined for $A, B \in \mathcal{A}$ and an inner product on the quotient linear space $\mathcal{A}/\mathcal{L}_\rho$. If $A \in \mathcal{L}_\rho$ and $B \in \mathcal{A}$, we know from before

$$\rho((BA)^*BA) = \rho((B^*BA)^*A) = 0$$

which means $BA \in \mathcal{L}_\rho$. Therefore, $\mathcal{L}_\rho$ is a left ideal and closed since ρ is continuous. ■

We call $\mathcal{L}_\rho$ the *left kernel* of ρ for a state ρ of a C^*-algebra.

5.7 Theorem. *Let $\mathcal{A}$ be a C^*-algebra and ρ a state of $\mathcal{A}$. Then there exists a cyclic representation π_ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\rho$ and a unit cyclic vector x_ρ for π_ρ that satisfies $\rho = \omega_{x_\rho} \circ \pi_\rho$ which means*

$$\forall A \in \mathcal{A} : \rho(A) = \langle \pi_\rho(A)x_\rho, x_\rho \rangle.$$

Proof. By Proposition 5.6, we know that for the left kernel $\mathcal{L}_\rho$ of ρ there is an inner product on $\mathcal{A}/\mathcal{L}_\rho$ defined by

$$\langle A + \mathcal{L}_\rho, B + \mathcal{L}_\rho \rangle = \rho(B^* A)$$

for $A, B \in \mathcal{A}$. Thus, $\mathcal{A}/\mathcal{L}_\rho$ is a pre-Hilbert space which means that its completion $\mathcal{H}_\rho$ is a Hilbert space.
Let $A, B_1, B_2 \in \mathcal{A}$ with $B_1 + \mathcal{L}_\rho = B_2 + \mathcal{L}_\rho$. We know $B_1 - B_2 \in \mathcal{L}_\rho$ and therefore $A(B_1 - B_2) \in \mathcal{L}_\rho$ since $\mathcal{L}_\rho$ is a left ideal of $\mathcal{A}$. Therefore, the operator

$$\pi(A) : \mathcal{A}/\mathcal{L}_\rho \to \mathcal{A}/\mathcal{L}_\rho, B + \mathcal{L}_\rho \mapsto AB + \mathcal{L}_\rho$$

is well-defined and we see that it is linear as well. Using Proposition 3.5 (ii), we get

$$\|A\|^2 I - A^* A = \|A^* A\| I - A^* A \in \mathcal{A}^+.$$

Corollary 3.9 tells us $B^*(\|A\|^2 I - A^* A)B$ is also positive. Then

$$\begin{aligned}
\|A\|^2 \|B + \mathcal{L}_\rho\|^2 - \|\pi(A)(B + \mathcal{L}_\rho)\|^2 &= \|A\|^2 \|B + \mathcal{L}_\rho\|^2 + \|AB + \mathcal{L}_\rho\|^2 \\
&= \|A\|^2 \langle B + \mathcal{L}_\rho, B + \mathcal{L}_\rho \rangle - \langle AB + \mathcal{L}_\rho, AB + \mathcal{L}_\rho \rangle \\
&= \|A\|^2 \rho(B^* B) - \rho(B^* A^* AB) \\
&= \rho(B^*(\|A\|^2 I - A^* A)B) \geq 0.
\end{aligned}$$

This shows that $\pi(A)$ is a bounded linear operator on $\mathcal{A}/\mathcal{L}_\rho$ with $\|\pi(A)\| \leq \|A\|$. Therefore, $\pi(A)$ extends uniquely to a bounded linear operator $\pi_\rho(A)$ on $\mathcal{H}_\rho$ with $\|\pi_\rho(A)\| \leq \|A\|$ as well.
$\pi(I)$ is the identity operator on $\mathcal{A}/\mathcal{L}_\rho$ which means $\pi_\rho(I)$ must be the identity operator on $\mathcal{H}_\rho$. If we have $A, B, C \in \mathcal{A}$ and $s, t \in \mathbb{C}$, then

$$\begin{aligned}
\pi_\rho(sA + tB)(C + \mathcal{L}_\rho) &= (sA + tB)C + \mathcal{L}_\rho \\
&= s(AC + \mathcal{L}_\rho) + t(BC + \mathcal{L}_\rho) \\
&= (s\pi_\rho(A) + t\pi_\rho(B))(C + \mathcal{L}_\rho), \\
\pi_\rho(AB)(C + \mathcal{L}_\rho) &= ABC + \mathcal{L}_\rho = \pi_\rho(A)(BC + \mathcal{L}_\rho) \\
&= \pi_\rho(A)\pi_\rho(B)(C + \mathcal{L}_\rho), \\
\langle \pi_\rho(A)(B + \mathcal{L}_\rho), C + \mathcal{L}_\rho \rangle &= \langle AB + \mathcal{L}_\rho, C + \mathcal{L}_\rho \rangle = \rho(C^* AB) \\
&= \rho((A^* C)^* B) = \langle B + \mathcal{L}_\rho, A^* C + \mathcal{L}_\rho \rangle \\
&= \langle B + \mathcal{L}_\rho, \pi_\rho(A^*)(C + \mathcal{L}_\rho) \rangle.
\end{aligned}$$

Since $\mathcal{A}/\mathcal{L}_\rho$ is everywhere dense in $\mathcal{H}_\rho$, we get

$$\begin{aligned}
\pi_\rho(sA + tB) &= s\pi_\rho(A) + t\pi_\rho(B), \\
\pi_\rho(AB) &= \pi_\rho(A)\pi_\rho(B), \\
\pi_\rho(A)^* &= \pi_\rho(A^*).
\end{aligned}$$

This shows that π_ρ is representation of $\mathcal{A}$ on $\mathcal{H}$ indeed.
We define $x_\rho := I + \mathcal{L}_\rho \in \mathcal{H}_\rho$. Then

$$\pi(\mathcal{A})x_\rho = \{\pi_\rho(A)x_\rho : A \in \mathcal{A}\} = \{\pi_\rho(A)(I + \mathcal{L}_\rho) : A \in \mathcal{A}\} = \{A + \mathcal{L}_\rho : A \in \mathcal{A}\} = \mathcal{A}/\mathcal{L}_\rho.$$

Therefore, $\pi(\mathcal{A})x_\rho = \mathcal{A}/\mathcal{L}_\rho$ is everywhere dense in $\mathcal{H}_\rho$ which means that x_ρ is a cyclic vector of the cyclic representation π_ρ. For $A \in \mathcal{A}$, we have

$$\langle \pi_\rho(A)x_\rho, x_\rho \rangle = \langle A + \mathcal{A}/\mathcal{L}_\rho, I + \mathcal{A}/\mathcal{L}_\rho \rangle = \rho(A)$$

which shows $\rho = \omega_{x_\rho} \circ \pi_\rho$ and $\|x_\rho\|^2 = \rho(I) = 1$. ■

This method used in Theorem 5.7 to get a cyclic representation from a state is called the *Gelfand-Naimark-Segal construction* or in short *GNS construction*. It is an important tool in C^*-algebra theory. Based on the GNS construction, we can explore a few further results.

5.3 Further results

5.8 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and ρ a state of $\mathcal{A}$. Let $\mathcal{H}_\rho, \pi_\rho$ and x_ρ be the Hilbert space, cyclic representation and unit cyclic vector from the GNS construction. If π is a cyclic representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with $\rho = \omega_x \circ \pi$ for x a unit cyclic vector of π, then we find an isomorphism U from $\mathcal{H}_\rho$ to $\mathcal{H}$ with*

$$x = Ux_\rho \wedge \forall A \in \mathcal{A} : \pi(A) = U\pi_\rho(A)U^*.$$

Proof. If $A \in \mathcal{A}$, then

$$\begin{aligned}\|\pi(A)x\|^2 &= \langle \pi(A)x, \pi(A)x \rangle = \langle \pi(A^*A)x, x \rangle \\ &= \rho(A^*A) = \langle \pi_\rho(A^*A)x_\rho, x_\rho \rangle = \|\pi_\rho(A)x_\rho\|^2.\end{aligned}$$

Now for $A, B \in \mathcal{A}$ with $\pi_\rho(A)x_\rho = \pi_\rho(B)x_\rho$, we use the above statement for $A - B$ and get $\pi(A)x = \pi(B)x$. Therefore,

$$U_0 : \pi_\rho(\mathcal{A})x_\rho \to \pi(\mathcal{A})x, \pi_\rho(A)x_\rho \mapsto \pi(A)x$$

is well-defined. We immediately see that U_0 is linear and norm-preserving. Since $\pi_\rho(\mathcal{A})x_\rho$ is everywhere dense in $\mathcal{H}_\rho$ and $\pi(\mathcal{A})x$ is everywhere dense in $\mathcal{H}$, we can extend U_0 to an isometric isomorphism from $\mathcal{H}_\rho$ to $\mathcal{H}$. We get

$$Ux_\rho = U_0\pi_\rho(I)x_\rho = \pi(I)x = x$$

and for $A, B \in \mathcal{A}$

$$U\pi_\rho(A)\pi_\rho(B)x_\rho = U\pi_\rho(AB)x_\rho = \pi(AB)x = \pi(A)\pi(B)x = \pi(A)U\pi_\rho(B)x_\rho.$$

Since the set $\pi_\rho(\mathcal{A})x_\rho = \{\pi_\rho(B)x_\rho : B \in \mathcal{A}\}$ is everywhere dense in $\mathcal{H}_\rho$, we know $U\pi_\rho(A) = \pi(A)U$ from which we follow $\pi(A) = U\pi_\rho(A)U^{-1}$. Since U is norm-preserving, U^{-1} is norm-preserving as well and because of the parallelogram rule, we know that U^{-1} is preserving the inner product as well. Then we see for $x \in \mathcal{H}_\rho, y \in \mathcal{H}$

$$\langle Ux, y\rangle = \langle U^{-1}Ux, U^{-1}y\rangle = \langle x, U^{-1}y\rangle.$$

This shows $U^{-1} = U^*$ and $\pi(A) = U\pi_\rho(A)U^*$ for every $A \in \mathcal{A}$. ■

5.9 Definition. Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{H}, \mathcal{K}$ Hilbert spaces and φ and ψ a representation of $\mathcal{A}$ on $\mathcal{H}$ and $\mathcal{K}$, respectively. We call φ and ψ *(unitarily) equivalent* if there exists an isomorphism U from $\mathcal{H}$ to $\mathcal{K}$ such that $\psi(A) = U\varphi(A)U^*$ for every $A \in \mathcal{A}$.

Proposition 5.8 tells us that if π is a representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, ρ a state of $\mathcal{A}$ and $x \in \mathcal{H}$ a unit cyclic vector of π with $\rho = \omega_x \circ \pi$ then π and π_ρ given by the GNS construction are equivalent. For the element x_ρ obtained by the GNS construction, we can choose the isomorphism U for the equivalence such that $Ux_\rho = x$. The following corollary is a special case of this property.

5.10 Corollary. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, $x \in \mathcal{H}$ a unit vector and $\rho := \omega_x|_\mathcal{A}$ the vector state to x. Then the representation π_ρ from the GNS construction is (unitarily) equivalent to the representation $A \mapsto A|_{\overline{\mathcal{A}x}}$ of $\mathcal{A}$ on $\overline{\mathcal{A}x}$. The isomorphism U that achieves the equivalence can be chosen so that $Ux_\rho = x$ with x_ρ the unit cyclic vector of π_ρ from the GNS construction.*

Proof. x is a unit cyclic vector for the representation $\pi : \mathcal{A} \to \mathcal{B}(\overline{\mathcal{A}x}), A \mapsto A|_{\overline{\mathcal{A}x}}$ of $\mathcal{A}$ on $\overline{\mathcal{A}x}$ and we have $\rho = \omega_x \circ \pi$. Proposition 5.8 tells us that π_ρ and π are (unitarily) equivalent with the isomorphism U that satisfies $Ux_\rho = x$. ■

5.4 The Gelfand-Naimark Theorem

In the next proposition we prove that the representations obtained via the GNS construction from pure states of a C^*-algebra are enough to separate the elements of $\mathcal{A}$.

5.11 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and $A \in \mathcal{A}$ with $A \neq 0$. Then we find a pure state of $\mathcal{A}$ such that $\pi_\rho(A) \neq 0$ with π_ρ the representation in the GNS construction.*

Proof. Let π_ρ, x_ρ be the representation and unit cyclic vector given by the GNS construction. Theorem 4.17 (i) tells us that there is a pure state ρ of $\mathcal{A}$ which satisfies $\langle \pi_\rho(A)x_\rho, x_\rho\rangle = \rho(A) \neq 0$. Therefore, $\pi_\rho(A) \neq 0$. ■

Now we will recount the definition of a sum of Hilbert spaces.

5.12 Definition. Let $\mathbb{B}$ be a set and $(\mathcal{H}_b)_{b\in\mathbb{B}}$ a family of Hilbert spaces. Then we define

$$\sum_{b\in\mathbb{B}} \mathcal{H}_b := \left\{ (x_b)_{b\in\mathbb{B}} \in \prod_{b\in\mathbb{B}} \mathcal{H}_b : \sum_{b\in\mathbb{B}} \|x_b\|^2 < \infty \right\}.$$

If $(T_b)_{b\in\mathbb{B}}$ is a family of uniformly bounded linear operators with $T_b \in \mathcal{B}(\mathcal{H}_b)$, then we can define

$$\sum_{b\in\mathbb{B}} T_b : \sum_{b\in\mathbb{B}} \mathcal{H}_b \to \prod_{b\in\mathbb{B}} \mathcal{H}_b, (x_b)_{b\in\mathbb{B}} \mapsto (T_b x_b)_{b\in\mathbb{B}}.$$

5.13 Proposition. *Let $\mathbb{B}$ be a set and $(\mathcal{H}_b)_{b\in\mathbb{B}}$ a family of Hilbert spaces. Then $\sum_{b\in\mathbb{B}} \mathcal{H}_b$ is a Hilbert space with the inner product*

$$\langle (x_b)_{b\in\mathbb{B}}, (y_b)_{b\in\mathbb{B}} \rangle := \sum_{b\in\mathbb{B}} \langle x_b, y_b \rangle$$

for $(x_b)_{b\in\mathbb{B}}, (y_b)_{b\in\mathbb{B}} \in \sum_{b\in\mathbb{B}} \mathcal{H}_b$. For $(T_b)_{b\in\mathbb{B}}$ a family of uniformly bounded linear operators with $T_b \in \mathcal{B}(\mathcal{H}_b)$ the function $\sum_{b\in\mathbb{B}} T_b$ is a bounded linear operator on $\sum_{b\in\mathbb{B}} \mathcal{H}_b$. Also there is $\left(\sum_{b\in\mathbb{B}} T_b\right)^* = \sum_{b\in\mathbb{B}} T_b^*$.

Proof. In the same way that $\ell^2(\mathbb{B})$ is a Hilbert space we see that $\sum_{b\in\mathbb{B}} \mathcal{H}_b$ is a Hilbert space with the given inner product. If $(T_b)_{b\in\mathbb{B}}$ is a family of uniformly bounded linear operators with $T_b \in \mathcal{B}(\mathcal{H}_b)$, we have $C > 0$ such that $\|T_b x_b\| \leq C\|x_b\|$ for all $b \in \mathbb{B}$ and $x_b \in \mathcal{H}_b$. Then we know for $(x_b)_{b\in\mathbb{B}} \in \sum_{b\in\mathbb{B}} \mathcal{H}_b$ that

$$\|(\sum_{b\in\mathbb{B}} T_b)((x_b)_{b\in\mathbb{B}})\|^2 = \|(T_b x_b)_{b\in\mathbb{B}}\|^2 = \sum_{b\in\mathbb{B}} \|T_b x_b\|^2 \leq \sum_{b\in\mathbb{B}} C^2\|x_b\|^2 = C^2\|(x_b)_{b\in\mathbb{B}}\|.$$

This shows that $\sum_{b\in\mathbb{B}} T_b$ is a function from $\sum_{b\in\mathbb{B}} \mathcal{H}_b$ to $\sum_{b\in\mathbb{B}} \mathcal{H}_b$ and bounded (by the uniform bound of the T_b). It is linear since T_b is linear for every $b \in \mathbb{B}$. Therefore, $\sum_{b\in\mathbb{B}} T_b$ is a bounded linear operator on $\sum_{b\in\mathbb{B}} \mathcal{H}_b$. Now let $(x_b)_{b\in\mathbb{B}}, (y_b)_{b\in\mathbb{B}} \in \sum_{b\in\mathbb{B}} \mathcal{H}_b$. Then

$$\begin{aligned}
\left\langle (\sum_{b\in\mathbb{B}} T_b)((x_b)_{b\in\mathbb{B}}), (y_b)_{b\in\mathbb{B}} \right\rangle &= \langle (T_b x_b)_{b\in\mathbb{B}}, (y_b)_{b\in\mathbb{B}} \rangle = \sum_{b\in\mathbb{B}} \langle T_b x_b, y_b \rangle \\
&= \sum_{b\in\mathbb{B}} \langle x_b, T_b^* y_b \rangle = \langle (x_b)_{b\in\mathbb{B}}, (T_b^* y_b)_{b\in\mathbb{B}} \rangle \\
&= \left\langle (x_b)_{b\in\mathbb{B}}, (\sum_{b\in\mathbb{B}} T_b^*)((y_b)_{b\in\mathbb{B}}) \right\rangle.
\end{aligned}$$

This shows $\left(\sum_{b\in\mathbb{B}} T_b\right)^* = \sum_{b\in\mathbb{B}} T_b^*$. ■

5.14 Definition. Let $\mathcal{A}$ be a C^*-algebra, $\mathbb{B}$ a set, $(\mathcal{H}_b)_{b\in\mathbb{B}}$ a family of Hilbert spaces and $(\varphi_b)_{b\in\mathbb{B}}$ a family where φ_b is a representation of $\mathcal{A}$ on $\mathcal{H}_b$ for $b \in \mathbb{B}$. Then we define

$$\sum_{b\in\mathbb{B}} \varphi_b : \mathcal{A} \to \sum_{b\in\mathbb{B}}, A \mapsto \sum_{b\in\mathbb{B}} \varphi_b(A))$$

and we call $\sum_{b\in\mathbb{B}} \varphi_b$ the direct sum of the family $(\varphi_b)_{b\in\mathbb{B}}$ of representations of $\mathcal{A}$.

5.15 Proposition. *Let $\mathcal{A}$ be a C^*-algebra, $\mathbb{B}$ a set, $(\mathcal{H}_b)_{b\in\mathbb{B}}$ a family of Hilbert spaces and $(\varphi_b)_{b\in\mathbb{B}}$ a family where φ_b is a representation of $\mathcal{A}$ on $\mathcal{H}_b$ for $b \in \mathbb{B}$. Then $\sum_{b\in\mathbb{B}} \varphi_b$ is a representation of $\mathcal{A}$ on $\sum_{b\in\mathbb{B}} \mathcal{H}_b$.*

Proof. Let $\varphi := \sum_{b\in\mathbb{B}} \varphi_b$. Since $\|\varphi_b(A)\| \leq \|A\|$ for every $A \in \mathcal{A}, b \in \mathbb{B}$, we know that $(\varphi_b(A))_{b\in\mathbb{B}}$ is a family of uniformly bounded (by $\|A\|$) linear operators with $\varphi_b(A) \in \mathcal{B}(\mathcal{H}_b)$ for every $b \in \mathbb{B}$ and therefore $\varphi(A) = \sum_{b\in\mathbb{B}} \varphi_b(A)$ is a bounded (by $\|A\|$) linear operator on $\sum_{b\in\mathbb{B}} \mathcal{H}_b$. For $A, B \in \mathcal{A}$, $s, t \in \mathbb{C}$ and $(x_b)_{b\in\mathbb{B}} \in \sum_{b\in\mathbb{B}} \mathcal{H}_b$, we have

$$\begin{aligned}\varphi(sA + tB)((x_b)_{b\in\mathbb{B}}) &= (\sum_{b\in\mathbb{B}} \varphi_b(sA + tB))(x_b)_{b\in\mathbb{B}} = (\varphi_b(sA + tB)(x_b))_{b\in\mathbb{B}} \\ &= s(\varphi_b(A)(x_b))_{b\in\mathbb{B}} + t(\varphi_b(B)(x_b))_{b\in\mathbb{B}} \\ &= s(\sum_{b\in\mathbb{B}} \varphi_b(A))(x_b)_{b\in\mathbb{B}} + t(\sum_{b\in\mathbb{B}} \varphi_b(B))(x_b)_{b\in\mathbb{B}} \\ &= (s\varphi(A) + t\varphi(B))((x_b)_{b\in\mathbb{B}}),\end{aligned}$$

$$\begin{aligned}\varphi(AB)((x_b)_{b\in\mathbb{B}}) &= (\sum_{b\in\mathbb{B}} \varphi_b(AB))(x_b)_{b\in\mathbb{B}} = (\varphi_b(AB)(x_b))_{b\in\mathbb{B}} \\ &= (\varphi_b(A)\varphi_b(B)(x_b))_{b\in\mathbb{B}} = (\sum_{b\in\mathbb{B}} \varphi_b(A))(\varphi_b(B)(x_b))_{b\in\mathbb{B}} \\ &= (\sum_{b\in\mathbb{B}} \varphi_b(A))(\sum_{b\in\mathbb{B}} \varphi_b(B))(x_b)_{b\in\mathbb{B}} = \varphi(A)\varphi(B)(x_b)_{b\in\mathbb{B}}, \\ \varphi(A^*) = \sum_{b\in\mathbb{B}} \varphi_b(A^*) &= \sum_{b\in\mathbb{B}} \varphi_b(A)^* = \left(\sum_{b\in\mathbb{B}} \varphi_b(A)\right)^* = \varphi(A)^*, \\ \varphi(I)((x_b)_{b\in\mathbb{B}}) &= (\sum_{b\in\mathbb{B}} \varphi_b(I))((x_b)_{b\in\mathbb{B}}) = (\sum_{b\in\mathbb{B}} I)((x_b)_{b\in\mathbb{B}}) = (x_b)_{b\in\mathbb{B}}.\end{aligned}$$

Therefore, we see that φ is a C^*-homomorphism from $\mathcal{A}$ to $\mathcal{B}(\sum_{b\in\mathbb{B}} \mathcal{H}_b)$ which means φ is a representation of $\mathcal{A}$ on $\sum_{b\in\mathbb{B}} \mathcal{H}_b$. ■

5.16 Theorem (The Gelfand-Naimark Theorem)**.** *Every C^*-algebra has a faithful representation.*

Proof. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{S}_0$ a set of states containing all pure states. Let $\mathcal{H}_\rho$ and π_ρ be the Hilbert space and representation obtained by the GNS construction for every state $\rho \in \mathcal{S}_0$. Let $\varphi = \sum_{\rho\in\mathcal{S}_0} \pi_\rho$ be the direct sum of the family $(\pi_\rho)_{\rho\in\mathcal{S}_0}$ of representations, hence φ is a representation of $\mathcal{A}$ on $\sum_{\rho\in\mathcal{S}_0} \mathcal{H}_\rho$. Let $A \in \mathcal{A}$ with $\varphi(A) = 0$. Then $\sum_{\rho\in\mathcal{S}_0} \pi_\rho(A) = 0$ from which we see that $\pi_\rho(A) = 0$ for every $\rho \in \mathcal{S}_0$. By Proposition 5.11 we get $A = 0$. Therefore, φ is injective and a faithful representation of $\mathcal{A}$ on $\sum_{b\in\mathbb{B}} \mathcal{H}_b$. ■

If φ is a faithful representation of a C^*-algebra $\mathcal{A}$ on some Hilbert space $\mathcal{H}$, we know from Proposition 2.13 that $\varphi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. With Theorem 5.16 we have shown that such a faithful representation exists. Therefore, we proved that every C^*-algebra $\mathcal{A}$ is C^*-isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

5.17 Definition. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{S}(\mathcal{A})$ its state space. Let $\mathcal{H}_\rho$ and π_ρ be the Hilbert space and representation obtained by the GNS construction for every state

$\rho \in \mathcal{S}(\mathcal{A})$. Then the faithful representation

$$\Phi := \sum_{\rho \in \mathcal{S}(\mathcal{A})} \pi_\rho$$

of $\mathcal{A}$ on $\mathcal{H}_\Phi := \sum_{\rho \in \mathcal{S}(\mathcal{A})} \mathcal{H}_\rho$ is called the *universal representation* of $\mathcal{A}$.

5.18 Proposition. *Let $\mathcal{A}$ be a C^*-algebra and Φ its universal representation on $\mathcal{H}_\Phi$. Then every state of $\mathcal{A}$ has the form $\omega_y \circ \Phi$ for a unit vector $y \in \mathcal{H}_\Phi$ and every state of $\Phi(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H}_\Phi)$ is a vector state.*

Proof. Let $\mathcal{S}$ be the state space of $\mathcal{A}$ and $\sigma \in \mathcal{S}$ be a state of $\mathcal{A}$. Let $\mathcal{H}_\rho, \pi_\rho$ and x_ρ be the Hilbert space, representation and unit vector given by the GNS construction for $\rho \in \mathcal{S}$. Then we know $\sigma = \omega_{x_\sigma} \circ \pi_\sigma$. We define

$$y := (y_\rho)_{\rho \in \mathcal{S}} \in \mathcal{H}_\Phi, y_\rho := \begin{cases} x_\sigma, & \text{if } \rho = \sigma \\ 0, & \text{if } \rho \neq \sigma. \end{cases}$$

Then $\|y\|^2 = \sum_{\rho \in \mathcal{S}} \|y_\rho\|^2 = \|x_\sigma\|^2 = 1$ which shows y is a unit vector. For $A \in \mathcal{A}$, we have

$$\begin{aligned} \sigma(A) &= (\omega_{x_\sigma} \circ \pi_\rho)(A) = \langle \pi_\sigma(A) x_\sigma, x_\sigma \rangle \\ &= \sum_{\rho \in \mathcal{S}} \langle \pi_\rho(A) y_\rho, y_\rho \rangle = \langle (\pi_\rho(A) y_\rho)_{\rho \in \mathcal{S}}, (y_\rho)_{\rho \in \mathcal{S}} \rangle \\ &= \langle \Phi(A)(y_\rho)_{\rho \in \mathcal{S}}, (y_\rho)_{\rho \in \mathcal{S}} \rangle = (\omega_y \circ \Phi)(A). \end{aligned}$$

This shows $\sigma = \omega_y \circ \Phi$.
If τ is a state of $\Phi(\mathcal{A})$, then $\tau \circ \Phi$ is a state of $\mathcal{A}$. From before we know there is a unit vector $y \in \mathcal{H}_\Phi$ such that $\tau \circ \Phi = \omega_y \circ \Phi$. By using Φ^{-1} from the right on both sides, we know $\tau = \omega_y$ and therefore τ is a vector state. ∎

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