Sets universal in measure

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1 Introduction

In this text, we study universal sets and present a few results related to a conjecture of Paul Erdös (1913-1996). What is a universal set?

Definition 1. *A set $E \subseteq \mathbb{R}$ is called* universal (in measure)*, if every measurable $S \subseteq \mathbb{R}$ of positive Lebesgue measure contains an affine copy of E: there is a pair $(t, r) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ satisfying $t + rE \subseteq S$. A function $\varphi : \mathbb{R} \to \mathbb{R}$ of the form $\varphi(x) = t + rx$ is called a* similarity mapping.

It is known that all finite sets are universal (see Theorem 2) and certain types of infinite sets have been shown to be non-universal. Paul Erdös conjectured [7, chap. 4, p. 29], that no infinite universal set exists.

In Section 2 we discuss properties of universal sets and provide a proof that all finite sets are universal. In Section 3 we construct a Cantor-like set that does not contain any affine copy of a given slowly converging sequence. The construction of a set avoiding translation copies of a given infinite set at almost every scale can be found in Section 4. To conclude, in Section 5 we present two equivalent formulations of the conjecture.

The most recent attempt towards Erdös' conjecture was published shortly after this seminar paper was written: In [1], the authors endeavor to construct a Cantor-like set not containing any affine copy of a sequence converging arbitrarily fast, which would prove the conjecture. However, the paper was shortly revoked as it was found to contain a gap. Thus, the so-called *similarity conjecture* remains unproven.

2 Properties of universal sets

We start by stating a few simple observations regarding universal sets.

Lemma 1. *If $E \subseteq \mathbb{R}$ is universal, the following statements are true:*

1. *Every subset of E is universal.*
2. *If φ is a similarity mapping, then $\varphi(E)$ is universal.*
3. *The closure of E is universal. [9, Lemma 2.1]*

4. *E is a bounded set of empty interior.*

Proof.

1. For an arbitrary $S \subseteq \mathbb{R}$ with $\lambda(S) > 0$, there is a similarity mapping $\varphi : \mathbb{R} \mapsto \mathbb{R}$ satisfying $\varphi(E) \subseteq S$. Needless to say, $\varphi(E') \subseteq S$ for any subset $E' \subseteq E$.

2. This is owing to the fact that the inverse of φ is a similarity mapping, as is the composition of similarity mappings.

3. Let $S \subseteq \mathbb{R}$ be some set of positive measure. Choose S' to be a closed subset of S which satisfies $\lambda(S') > 0$. Since E is universal, we find $(t, r) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ with $t + rE \subseteq S'$. Because $t + r\overline{E} = \overline{t + rE} \subseteq \overline{S'} = S' \subseteq S$, the set $\overline{E}$ is universal as well.

4. If E is unbounded, then so is every set of the form $t+rE$, if only $r \neq 0$. If E contains an open interval, then so does any set $t + rE$. In both cases, $S := [0, 1] \setminus \mathbb{Q}$ is a possible witness of non-universality.

□

Before we turn our attention to infinite sets, let us verify that all finite sets are indeed universal. The following theorem slightly extends this result: If $E \subseteq \mathbb{R}$ is finite and $S \subseteq \mathbb{R}$ has positive measure, we show that if only the scaling parameter r is chosen small enough, there is a $t \in \mathbb{R}$ so that $t + rE \subseteq S$.

Theorem 2. *If $E \subseteq \mathbb{R}$ is finite, then E is universal* [1]*. Furthermore, for every $S \subseteq \mathbb{R}$ of positive measure, the set $\{r \mid \exists t \in \mathbb{R} : t + rE \subseteq S\}$ contains an interval $(0, R)$ with $R > 0$.*

Proof. If E consists only of a single point, there is not a lot to show. Assume $\#E \geq 2$ and write $E = \{e_1, \ldots, e_m\}$, where $e_i < e_j$ for $i < j$. We may also assume $e_1 = 0$ and $e_m = 1$, since if the statement is true for an appropriately scaled and shifted version of E, then it holds for E as well, owing to the fact that the composition of similarity mappings is once again a similarity mapping.

Let $S \subseteq \mathbb{R}$ be an arbitrary set of positive measure. Assume, towards a contradiction, the following:

$$\forall R > 0 \exists r \in (0, R) : \forall t \in \mathbb{R} : t + rE \nsubseteq S. \tag{1}$$

Chose a sequence $(r_n)_{n \in \mathbb{N}}$ of positive numbers converging to zero, so that for every $n \in \mathbb{N}$ we have $\forall t \in \mathbb{R} : t + r_n E \nsubseteq S$. Lebesgue's Density Theorem (see Theorem 13 in the appendix) guarantees the existence of some $s \in S$ satisfying

$$\lim_{r \searrow 0} \frac{\lambda\left(S \cap [s - r, s + r]\right)}{2s} = 1.$$

We will now show that each interval $[s - r_n, s + r_n]$ contains a set that does not belong to S and has measure at least δr_n, with $\delta > 0$ fixed and not depending on r_n. This yields

$$\lim_{n \to \infty} \frac{\lambda\left(S \cap [s - r_n, s + r_n]\right)}{2s} \leq \lim_{n \to \infty} \frac{2r_n - \delta r_n}{2r_n} = 1 - \frac{\delta}{2} < 1,$$

[1] This observation is sometimes attributed to Hugo Steinhaus (1887-1972), for example in [9], where *Sur les distances des points dans les ensembles de measure positive*, Fund. Math., 1(1920), 93-104, is cited. For an alternative proof, see [5].

contradicting s being a point of density in S.

Fix $n \in \mathbb{N}$ and define

$$\delta := \min_{i \in \{1,\ldots,m-1\}} (e_{i+1} - e_i).$$

Consider the similarity mapping $\varphi_0(x) := (s - r_n) + r_n x$, which maps e_1 to $s - r_n$ and e_m to s. For every $t \in \mathbb{R}$, $\varphi_t(x) := \varphi_0(x) + t$ is once again a similarity mapping with scaling parameter r_n. The sets

$$E_k := \{\varphi_t(e_k) | t \in [0, \delta r_n)\} \cap S^c = [\varphi_0(e_k), \varphi_0(e_k) + \delta r_n) \cap S^c, k = 1, \ldots, n$$

are disjoint, owing to our choice of δ. By our assumption that S contains no translate of E scaled by r_n, we know that

$$\forall t \in [0, \delta r_n) \exists k \in \{1, \ldots, n\} : \varphi_t(e_k) \in E_k.$$

Reformulating this using the translations

$$\begin{aligned} \tau_k \colon [\varphi_0(e_k), \varphi_0(e_k) + \delta r_n) &\to [0, \delta r_n) \\ x &\mapsto x - \varphi_0(e_k), \end{aligned}$$

we obtain

$$[0, \delta r_n) = \bigcup_{k=1}^{n} \tau_k(E_k).$$

By the subadditivity and translation invariance of λ and because the E_k are disjoint, we now deduce the following inequality:

$$\delta r_n = \lambda([0, \delta r_n)) = \lambda\left(\bigcup_{k=1}^{n} \tau_k(E_k)\right) \leq \sum_{k=1}^{n} \lambda(\tau_k(E_k)) = \sum \lambda(E_k) = \lambda\left(\bigcup_{k=1}^{n} E_k\right).$$

Thus, for arbitrary $n \in \mathbb{N}$, we have found a set of measure greater than δr_n fully contained in $S^c \cap [s - r_n, s + r_n]$, namely the union of the sets E_k. As a result, we get the already mentioned contradiction and conclude that the negation of (1) - which is the statement of the theorem - holds. □

Remark 1. It is interesting to note that if $S \subseteq \mathbb{R}$ contains a similar copy of every finite set, $\lambda(S)$ is not necessarily greater than zero - see [8] for a construction of a Lebesgue null set with this property.

We have already seen that the set of universal sets is closed under similarity mappings. The same can be said for the set of sets that do not contain a similar copy of a given set. In fact, if we know of the existence of one such 'witness' of non-universality, then we can already assume that there is a closed subset of $[0, 1]$ of measure arbitrarily close to 1 that contains no copy of the given set, as the following Lemma shows:

Lemma 3. *If $E \subseteq \mathbb{R}$ is not universal, then for every $\epsilon > 0$ there is a closed set $S_\epsilon \subseteq [0, 1]$ of measure $\lambda(S_\epsilon) > 1 - \epsilon$ that does not contain any affine copy of E.*

Proof. E is not universal, so there is a set S of positive measure not containing any affine copy of E. Let $s \in S$ be a point of density in S. Choose $r > 0$ so that $\frac{1}{2r}\lambda(S \cap [s-r, s+r]) > 1 - \epsilon$ and define

$$S_\epsilon := \left((\frac{r-s}{2r}) + \frac{1}{2r} S \right) \cap [0,1]$$

S_ϵ is a subset of a scaled and shifted version of S and therefore does not contain any affine copy of E. By shifting S_ϵ by $\frac{s-r}{2r}$, it is readily verified that $\lambda(S_\epsilon) > 1 - \epsilon$. Since λ is a regular measure, we may assume S_ϵ to be closed (otherwise simply take a closed subset of S_ϵ of sufficiently large measure). □

The following lemma mentioned in [5] shows that while the sets S_ϵ from the previous Lemma may be of measure arbitrarily close to 1, they are never of full measure in $[0,1]$. It also gives a rough idea about what kind of sets we will have to construct if we want to prove that a given set is non-universal.

Lemma 4 ([5], Theorem 5)**.** *Let $E \subseteq \mathbb{R}$ be a countable and bounded set. If $S \subseteq \mathbb{R}$ satisfies $\lambda(O \setminus S) = 0$ for some open set $O \subseteq \mathbb{R}$, then S contains an affine copy of E.*

Proof. Without loss of generality, we may assume $E = \{e_n \mid n \in \mathbb{N}\} \subseteq [0,1]$ (see Lemma 1). Suppose S is of full measure in some interval $[s-\epsilon, s+\epsilon]$ and define $S' := S \cap [s-\epsilon, s+\epsilon]$. We now show that for any $r \in (0, \epsilon)$

$$\bigcap_{n \in \mathbb{N}} (S' - re_n) \neq \emptyset. \tag{2}$$

This will prove the statement, since $t \in \bigcap_{n \in \mathbb{N}} (S' - re_n)$ implies $t + rE \subseteq S' \subseteq S$.

Note that for every $n \in \mathbb{N}$, the set $(S' - re_n)$ is of full measure in $[s-\epsilon, s]$. Therefore:

$$\lambda\left(\bigcap_{n \in \mathbb{N}} (S' - re_n) \right) \geq \lambda([s-\epsilon, s]) - \lambda\left(\bigcup_{n \in \mathbb{N}} [s-\epsilon, s] \setminus (S' - re_n) \right) > \epsilon, \tag{3}$$

which shows that not only is the set in question not empty, but it even has positive measure. □

3 A set of positive measure not containing a copy of a slowly converging sequence

In order to show that all infinite sets are not universal, it would suffice to show that any strictly decreasing zero-sequence does not have this property: $E \subseteq \mathbb{R}$ can only be universal, if E is bounded (Lemma 1). In that case, let $x \in E$ be an accumulation point of E and $(e_n)_{n \in \mathbb{N}}$ a strictly decreasing (or increasing) sequence in E converging to x. Since the composition of similarity mappings is once again a similarity mapping and subsets of universal sets are universal, if an appropriately shifted sequence can be shown to be non-universal, then so is E.

This motivates the following theorem of Kenneth J. Falconer, to be found in [3]. The comparatively simple proof stems from [9].

Theorem 5 (Falconer[2]). *Let $E = (e_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers converging to* 0 *such that*

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = 1.$$

Then E is not universal.

Proof. (Svetic) Let $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary strictly decreasing zero-sequence[3] satisfying $0 < \epsilon_n < 1$ and $\prod_{n=1}^{\infty}(1-\epsilon_n) > 0$. For each $n \in \mathbb{N}$ define $N(n) \in \mathbb{N}$, so that $m \geq N(n)$ implicates $e_{m+1}/e_m > 1 - \epsilon_n$.

We construct a set S of positive measure, which contains no similar copy of E. Define $S_0 := [0,1]$ and recursively construct a sequence of sets $S_0 \supseteq S_1 \supseteq S_2 \ldots$, and finally $S := \bigcap_{n\in\mathbb{N}} S_n$. Each set S_n is obtained by removing a fraction from S_{n-1} in the following way:

Assume S_{n-1} is given as the distinct union of closed intervals of the same length. From each of these, we remove a certain number k_n of evenly spaced open intervals, so that the fractional amount removed is ϵ_n and k_{n+1} closed intervals of equal length remain. The number of intervals to be removed is chosen so that $n\lambda(I_n) < e_{N(n)}$ holds, where I_n denotes an arbitrary one of the remaining closed intervals.

Each set S_n is a finite union of closed intervals, therefore S is closed. Because λ is continuous from above, we have

$$\lambda(S) = \lim_{n\in\mathbb{N}} \lambda(S_n) = \prod_{n=1}^{\infty}(1-\epsilon_n) > 0,$$

so S has positive measure.

We now show that S contains no similar copy of E: Let a pair $(t,r) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ be given. Assume $r < 0$ (a similar argument can be made if $r > 0$). Since $\lim_{n\to\infty}(t + re_n) = t$ and S is closed, we may assume $t \in S$, as otherwise $t + rE \nsubseteq S$ is straightforward. This is also apparent if t is the far left point of some closed interval of any of the S_n, so assume this is not the case.

Choose $n \in \mathbb{N}$ satisfying $\frac{1}{n} < -r$. Let I_n denote the closed interval constructed at the n-th step with $t \in I_n$ and O_n the open interval removed from it to the left. Let $m \in \mathbb{N}$ be minimal so that $t + re_m \in I_n$. By showing $-r(e_{m-1} - e_m) < \lambda(O_n)$, we see that $t + re_{m-1} \in O_n$ and therefore $t + re_{m-1} \notin S$:

Because t and $t + re_m$ are both in I_n and because of our choice of n, we have $e_m/n < -re_m \leq \lambda(I_n)$ and therefore $e_m < n\lambda(I_n) < e_{N(n)}$. Since our sequence $(e_n)_{n\in\mathbb{N}}$ is monotone, this implies $m > N(n)$ and thus $-re_m/-re_{m-1} > 1 - \epsilon_n$, or written more conveniently for future purposes:

$$\frac{-re_m}{1-\epsilon_n} > -re_{m-1} \tag{4}$$

We also have $\lambda(O_n)/(\lambda(O_n) - re_m) \geq \lambda(O_n)/(\lambda(O_n) + \lambda(I_n)) > \epsilon_n$ (whereby the second inequality is true because ϵ_n was the fractional amount removed at the nth step and can

[2]According to [9], this theorem was also proven independently by S. J. Eigen in *Putting Convergent Sequences into Measurable Sets*, Studia Sci. Math. Hung., 20(1985), 411-412, MR 88f:28003.

[3]e.g. $\epsilon_n := \frac{1}{(n+1)^2}$

be written as $\epsilon_n = k_n\lambda(O_n)/(k_n\lambda(O_n) + k_{n+1}\lambda(I_n)))$. This can be rewritten as $\lambda(O_n) \geq \lambda(O_n)\epsilon_n - re_m\epsilon_n$, from which we can deduce

$$\lambda(O_n) > \frac{-re_m\epsilon_n}{1-\epsilon_n}. \tag{5}$$

Putting these results together, we get

$$-r(e_{m-1} - e_m) \overset{4}{<} \frac{-re_m}{1-\epsilon_n} - (-re_m) = \frac{-re_m\epsilon_n}{1-\epsilon_n} \overset{5}{<} \lambda(O_n), \tag{6}$$

which, as mentioned, shows $t + re_{m-1} \notin S$. □

As pointed out in [9], Theorem 5 can be used to show any universal set to be a λ^*-zero set:

Corollary 6. *If $E \subseteq \mathbb{R}$ is universal, then $\lambda^*(E) = 0$. If E is measurable (as well as universal), then for every $S \subseteq \mathbb{R}$ of positive measure, the set*

$$\{(t,r) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \mid t + rE \subseteq S\} \tag{7}$$

is uncountable.

Proof. Suppose $\lambda^*(E) > 0$ and assume that 0 is a point of density in E (otherwise shift E accordingly so this is true). Because E is universal only if its closure is universal, we may also assume that E is closed. We now extract a sequence from E, which meets the premise of Theorem 5. Define

$$e_n := \sup\left(E \cap \left[0, \frac{1}{n}\right]\right).$$

Because E is closed and 0 is a point of density in E, $e_n \in E$. The sequence converges sufficiently slowly:

$$\lim_{n\in\mathbb{N}} \frac{e_{n+1}}{e_n} \geq \lim_{n\in\mathbb{N}} ne_{n+1} = \lim_{n\in\mathbb{N}} (n+1)e_{n+1} = 1.$$

The last equation was true, because for all $n \in \mathbb{N}$

$$1 \geq ne_n = \frac{\lambda([0,e_n])}{\frac{1}{n}} \geq \frac{\lambda\left(E \cap [0,\frac{1}{n}]\right)}{\frac{1}{n}},$$

where the last term tends to 1 for $n \to \infty$. According to Theorem 5, E is not universal. This proves the first statement.

Let E be measurable and let $S \subseteq \mathbb{R}$ with $\lambda(S) > 0$ be given. Suppose $(t_n, r_n)_{n\in\mathbb{N}}$ is an enumeration of all possible pairs in 7. Because each set $(t_n + r_nE)$ is a zero set, the set $S' := S \setminus \bigcup_{n=0}^{\infty}(t_n + r_nE)$ must be of positive measure. Because E is universal, there exist $(t,r) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ with $t + rE \subseteq S'$, but since $S' \subseteq S$, we have $(t,r) = (t_n, r_n)$ for some $n \in \mathbb{N}$. This yields a contradiction, since $t_n + r_nE \nsubseteq S'$. □

Refinements to Falconers result have been made, for instance by Mihail Kolountzakis, who showed the following:

Theorem 7. *Let $E \subseteq \mathbb{R}$ be an infinite set which contains, for arbitrarily large $n \in \mathbb{N}$, a subset $\{e_1, \dots, e_n\}$ with $e_1 > \dots > e_n > 0$ and*

$$-\log \delta_n = o(n), \tag{8}$$

where

$$\delta_n := \min_{i=1,\dots,n-1} \frac{e_i - e_{i+1}}{e_1}. \tag{9}$$

Then E is not universal.

Falconers theorem can be proven to be a corollary of the above, while another consequence is the non-universality of sets of the form $E + E$, with $E = \{\frac{1}{2^{n^\alpha}} \mid n \in \mathbb{N}\}$ for $\alpha \in (0, 2)$. [6, Section 4.4]

Another result is due to Jean Bourgain, who proved[4] that sets of the form $E_1 + E_2 + E_3$, where each E_i is an infinite set, are non-universal. The question however remains open for faster converging sequences, most prominently $E = \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$ (see [9]).

4 A set containing no translation copy of a given infinite set at almost every scale

Another characterization of universality is the following, pointed out in [6]:

Lemma 8. *Let $E \subseteq \mathbb{R}$ be an infinite set. E is universal if and only if every $S \subseteq \mathbb{R}$ with $\lambda(S) > 0$ satisfies*

$$\lambda^*(\{t \in \mathbb{R} \,| \exists r \in \mathbb{R} \setminus \{0\} : t + rE \subseteq S\}) > 0. \tag{10}$$

Proof. If (10) is true for any S of positive measure, then E is obviously universal.

Suppose E is an infinite set and $S \subseteq \mathbb{R}$ is such that $\lambda(S) > 0$ and $\lambda^*(\{t \in \mathbb{R} \mid \exists r \in \mathbb{R} \setminus \{0\} : t + rE \subseteq S\}) = 0$. Since λ is a regular measure, we may choose S to be closed, and because of Lemma 1, we may assume 0 to be an accumulation point in E. Under these assumptions, $S' := \{t \in \mathbb{R} \mid \exists r \in \mathbb{R} \setminus \{0\} : t + rE \subseteq S\}$ is a subset of S. Removing an open set of small measure containing S' from S, we get a set of positive measure which contains no similar copies of E. $\square$

Motivated by this result, Kolountzakis showed the following (again, see [6]):

Theorem 9 (Kolountzakis)**.** *Let $E \subseteq \mathbb{R}$ be an infinite set. For any $\epsilon > 0$ there is a set $S \subseteq [0, 1]$ satisfying $\lambda(S) > 1 - \epsilon$, such that*

$$\lambda\left(\{r \in \mathbb{R} \setminus \{0\} \mid \exists t \in \mathbb{R} : t + rE \subseteq S\}\right) = 0. \tag{11}$$

Proof. It suffices to prove the theorem for $E = \{e_1, e_2, \dots\}$ with $e_m \searrow 0$. We restrict the scaling parameter r to intervals $[\alpha, \beta]$ not containing zero and construct sets $S_{\alpha,\beta} \subseteq [0, 1]$ of measure as close to 1 as needed, so that $\lambda\left(\{r \in [\alpha, \beta] \mid \exists t \in \mathbb{R} : t + rE \subseteq S_{\alpha,\beta}\}\right) = 0$. If we intersect countably many such sets where the intervals of the scaling parameter cover

[4] In [9], Svetic cites J. Bourgain, *Construction Of Sets Of Positive Measure Not Containing An Affine Image Of A Given Infinite Structure*, Israel J. of Math., 60(1987).

$\mathbb{R} \setminus \{0\}$ and the sets are taken to be of sufficiently large measure[5], we will get a set S satisfying (11).

It is worth noting that it is enough to construct the sets $S_{\alpha,\beta}$ only for $\alpha, \beta > 0$, since for such a set, $(1 - S_{\alpha,\beta})$ will avoid almost all translation copies of E scaled by $r \in [-\beta, -\alpha]$.

Fix α, β so that $0 < \alpha < \beta < \infty$. We construct sets $S_n \subseteq [0,1]$ satisfying $\lambda(S_n) \to 1$ as well as $\lambda\left(\{r \in [\alpha,\beta] \mid \exists t \in \mathbb{R} : t + rE \subseteq S_n\}\right) \to 0$ for $n \to \infty$. If we define $S_{\alpha,\beta}$ as an intersection of countably many of these S_n of measure sufficiently close to 1, it will have the necessary properties.

We define each set S_n to be T_n-periodic with

$$S_n \cap [0, T_n] = (\epsilon_n T_n, T_n), \tag{12}$$

where $(T_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ are yet to be defined sequences of positive reals converging to zero. Evidently, $\lambda(S_n) \to 1$. Fix $n \in \mathbb{N}$. We now figure out how T_n and ϵ_n have to be defined, so that

$$\lambda\left(\{r \in [\alpha,\beta] \mid \exists t \in \mathbb{R} : t + rE \subseteq S_n\}\right) \to 0 \tag{13}$$

holds:

Choose $e_{n(1)}, \ldots, e_{n(n)} \in E$ so that

$$\frac{e_{n(j+1)}}{e_{n(j)}} < T_n,\ j = 1, \ldots, n-1. \tag{14}$$

Let us call a scaling parameter $r \in [\alpha, \beta]$ 'bad', if the maximum gap between the numbers

$$re_{n(1)} \bmod T_n, re_{n(2)} \bmod T_n, \ldots, re_{n(n)} \bmod T_n, \tag{15}$$

which we consider as points on a circle of length T_n, exceeds $\epsilon_n T_n$. Note that the gaps between the points do not change if we instead consider

$$(t + re_{n(1)}) \bmod T_n, (t + re_{n(2)}) \bmod T_n, \ldots, (t + re_{n(n)}) \bmod T_n,$$

where $t \in \mathbb{R}$ can be arbitrary. Because of this, if $r \in [\alpha, \beta]$ is not 'bad', then for any $t \in \mathbb{R}$, there is at least one $j \in \{1, \ldots n\}$ so that $(t + re_{n(j)}) \bmod T_n$ is in $[0, \epsilon_n T_n]$ (and therefore $t + re_{n(j)} \notin S_n$), as otherwise the maximum gap between the points in (15) would be greater than $\epsilon_n T_n$. Our goal now is to show that the measure of the 'bad' scaling parameters tends to zero for $n \to \infty$ (and appropriately defined T_n and ϵ_n), which will prove (13).

Let $k := \lceil \frac{1}{\epsilon_n} \rceil$ and define $I_i := \left[\frac{i}{k} T_n, \frac{i+1}{k} T_n\right], i = 0, \ldots, k-1$. Because these intervals have length $\leq \epsilon_n T_n$, any $r \in [\alpha, \beta]$ which is 'bad' belongs to $B^n := \bigcup_{i=0}^{k-1} B_i^n$, where

$$B_i^n := \{r \in [\alpha, \beta] \mid \forall j \in \{1, \ldots, n\} : (re_{n(j)} \bmod T_n) \notin I_i\}. \tag{16}$$

We will now show that

$$\lambda(B_i^n) \to (\beta - \alpha)(1 - \frac{1}{k})^n \text{ for } T_n \to 0. \tag{17}$$

[5] $\sum_{n\in\mathbb{N}} \lambda(A_n^c \cap [0,1]) < \epsilon$ for some $A_n \subseteq [0,1]$ implies $\lambda(\bigcap_{n\in\mathbb{N}} A_n) > 1 - \epsilon$, which can be seen by applying De Morgan's law.

Casually speaking, this means that when looked at modulo some sufficiently small number, the mappings $t \mapsto te_{n(j)}$ behave similar to independent and uniformly distributed random variables.

Fix $i \in \{0, \dots, k-1\}$. For $j = 1, \dots, n$ consider the sets

$$S^i_j := \left\{ r \in [\alpha, \beta] \mid \forall l \in \{j, j+1, \dots, n\} : re_{n(l)} \bmod T_n \notin I_i \right\}. \tag{18}$$

We prove, first for $j = n$ and then inductively for $j = n-1, \dots, 1$, that S^i_j consists of disjoint intervals, let us call them $J_{j,l}$, each of length

$$\lambda(J_{j,l}) = \frac{(1 - \frac{1}{k})T_n}{e_{n(j)}}, \tag{19}$$

plus a maximum of two smaller intervals, so that the total length is

$$\lambda(S^i_j) = (\beta - \alpha)(1 - \frac{1}{k})^{n-j-1} + \mathcal{O}(T_n). \tag{20}$$

As $S^i_1 = B^n_i$, this will show (17).

We start with the case $j = n$: When r moves from α to β, $re_{n(n)}$ traverses $\lfloor \frac{(\beta-\alpha)e_{n(n)}}{T_n} \rfloor$ full periods of T_n and for each period, the size of the corresponding interval $J_{n,l}$ of the scaling parameter, during which $re_{n(n)} \bmod T_n \notin I_i$, is $\lambda(J_{n,l}) = \frac{(1-(1/k))T_n}{e_{n(n)}}$. Multiplying the two also yields (20) for S^i_n.

Suppose (19) and (20) are true for some $j > 0$. For each interval $J_{j,l}$, the subset for which $re_{n(j-1)} \bmod T_n \notin I_i$ consists of disjoint intervals $J_{j-1,l}$ each of length $\frac{(1-(1/k))T_n}{e_{n(j-1)}}$, plus a maximum of two smaller intervals. The total size of S^i_{j-1} is

$$\lambda(S^i_{j-1}) = (\#\text{intervals } J_{j,l})(\#\text{subintervals } J_{j-1,l})(\text{length of each subinterval}), \tag{21}$$

where

$$\text{number of intervals } J_{j,l} = \left((\beta - \alpha)(1 - \frac{1}{k})^{n-j-1} + \mathcal{O}(T_n) \right) \frac{e_{n(j)}}{(1 - \frac{1}{k})T_n} \tag{22}$$

and

$$\text{number of subintervals } J_{j-1,l} = \frac{(1 - \frac{1}{k})T_n}{e_{n(j)}} \frac{e_{n(j-1)}}{T_n} + \mathcal{O}(T_n). \tag{23}$$

After multiplying (and some algebra), we get

$$\lambda(S^i_{j-1}) = (\beta - \alpha) \left(1 - \frac{1}{k}\right)^{n-(j-1)-1} + C \frac{e_{n(j)}}{e_{n(j-1)}} \frac{1}{T_n} \mathcal{O}(T_n), \tag{24}$$

where C is some constant that does not fit the page. Here we need (14), with the help of which we see that the last term is $\mathcal{O}(T_n)$. This proves (17).

We can now choose T_n sufficiently small so as to have

$$\lambda(B^n) = \lambda(\bigcup_{i=0}^{k-1} B^n_i) \leq \sum_{i=0}^{k-1} \lambda(B^n_i) \approx k(\beta - \alpha)(1 - \frac{1}{k})^n. \tag{25}$$

If we define $\epsilon_n := \frac{1}{\sqrt{n}}$, we further get

$$\dots \leq (\beta-\alpha)(\sqrt{n}+1)(1-\frac{1}{\sqrt{n}+1})^n \leq (\beta-\alpha)(\sqrt{n}+1)\exp(-\sqrt{n}), \tag{26}$$

which tends to 0 for $n \to \infty$. This completes the proof.

□

An immediate corollary of the above theorem is the following:

Corollary 10. *Let $E \subseteq [0,1]$ be an infinite set. For any $\epsilon > 0$, there is a set $S \subseteq [0,1]$ satisfying $\lambda(S) > 1-\epsilon$, which contains no translation copy of E.*[6]

Proof. Let S' be the set constructed in Theorem 9 with $\lambda(S') > 1-\frac{\epsilon}{2}$. Choose $r \in (1-\frac{\epsilon}{2}, 1)$ so that $\nexists t \in \mathbb{R} : t + rE \subseteq S'$ and define $S := \frac{1}{r}(S' \cap [0,r])$. The set S contains no translate of E and $S \subseteq [0,1]$ with $\lambda(S) = \frac{1}{r}(\lambda(S') - \lambda((r,1])) > 1-\epsilon$. □

5 Equivalent formulations to Erdös' similarity conjecture

Erdös' conjecture has been reformulated in various ways. Two equivalent formulations are presented in this section. The first one is due to Jakub Jasinski, who views the problem as a sort of "tiling puzzle" (see [5]):

Theorem 11 (Jasinski)**.** *For $E \subseteq (0,1)$, the following statements are equivalent:*

1. *E is not universal:*

$$\exists S_1 \subseteq \mathbb{R}, \lambda(S_1) > 0 : \forall (t,r) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} : t + rE \cap S_1^c \neq \emptyset. \tag{27}$$

2. *There is a closed set $S_2 \subseteq (0,1)$ of positive measure such that*

$$\forall t \in (0,1) : S_2^c + tE = \mathbb{R}. \tag{28}$$

3. *There is an open set $G \subseteq \mathbb{R}$ such that*

$$\lambda((0,1) \setminus G) > 0 \text{ and } \forall t \in (0,1) : G + tE = \mathbb{R}. \tag{29}$$

Proof. (2) ⇔ (3) is evident.
(1) ⇒ (2): For a fixed $r \in \mathbb{R}$, the first statement is equivalent to

$$\forall t \neq 0 \exists y \in S_1^c, e \in E : y - re = t,$$

which in turn is equivalent to

$$S_1^c - rE = \mathbb{R}.$$

[6] According to Kolountzakis, this was previously shown by Péter Komjáth in *Large Sets not Containing Images of a Given Sequence*, 1983, Canadian Mathematical Bulletin, 26(1), 41-43. doi:10.4153/CMB-1983-007-7

Shift S_1 by $\tau \in \mathbb{R}$, so that $(S_1 + \tau) \cap (0,1)$ is of positive measure. Define S_2 to be some closed subset of $S_1 + \tau \cap (0,1)$ with $\lambda(S_2) > 0$. For any $t \in (0,1)$, we have $S_2^c + tE \supseteq \tau + S_1^c + tE = \tau + \mathbb{R} = \mathbb{R}$.
(2) $\Rightarrow$ (1): Let $s \in S_2$ be a point of density in S_2. As s is also a point of density in $(2s - S_2)$, we have $\lambda(S_2 \cap (2s - S_2)) > 0$. Define $S_1 \subseteq (S_2 \cap (2s - S_2))$ so that $\lambda(S_1) > 0$ and $diam(S_1)$[7] $< diam(E)$.

The first statement is true for $|r| \geq 1$, since $diam(t + rE) \geq diam(E) > diam(S_1)$ for any $t \in \mathbb{R}$. If $r \in (-1,0)$, then $S_1^c - rE \supseteq S_2^c - rE = \mathbb{R}$ implies (1), as seen in the proof of "(1) $\Rightarrow$ (2)". In the case of $r \in (0,1)$, multiplying the equation $S_2 + rE = \mathbb{R}$ by -1 and adding $2s$ to both sides yields $\mathbb{R} = (2s - S_2^c) - rE$. As $(2s - S_2^c)$ is contained in S_1^c, this proves (1). □

The "puzzle"-idea is contained in (29): We are looking for an open set G with the above property. As every open subset of the real line can be written as a countable union of disjoint open intervals G_n, we might as well look for disjoint $G_n \subseteq [0,1]$, where the union of $G_n + rE$ covers $\mathbb{R}$ for any $r \in (0,1)$, and define

$$G := \bigcup_{n \in \mathbb{N}} G_n \cup (-\infty, 0) \cup (1, \infty). \tag{30}$$

To conclude this survey of results regarding universal sets, I would only like to mention one rather different but equivalent formulation to the problem, a proof of which can be found in [4]:

Theorem 12 (Humke and Laczkovich)**.** *Let $E \subseteq [0,1]$ be such that $\inf E = 0$ and $\sup E = 1$. Define Λ_n as the cardinality of the smallest set $B \subseteq \mathbb{N}_n := \{1, 2, \dots, n\}$ that intersects every set of the form $E_{x,y} := \{x + \lfloor ey \rfloor | e \in E\}$, where $x, y, x + y \in \mathbb{N}_n$ and $y \geq \lfloor \frac{n}{2} \rfloor$. Then the following statements are equivalent:*

1. *E is not universal.*
2. *$\lim\limits_{n \in \mathbb{N}} \frac{\Lambda_n}{n} = 0$.*
3. *$\liminf\limits_{n \in \mathbb{N}} \frac{\Lambda_n}{n} = 0$.*

Appendix

We state Lebegue's Density Theorem. A proof can for example be found in [2, chap. 7, p. 324].

Definition 2. *We call $x \in \mathbb{R}$ a* point of density *in $A \subseteq \mathbb{R}$, if*

$$\lim_{r \searrow 0} \frac{\lambda^* (A \cap [x - r, x + r])}{2r} = 1,$$

where λ^* denotes the outer Lebesgue measure.

Theorem 13 (Lebesgue Density Theorem)**.** *For an arbitrary set $A \subseteq \mathbb{R}$, almost every point $x \in A$ is a point of density in A.*

[7] For $A \subseteq \mathbb{R}$, we define the diameter of A to be $diam(A) := \sup\{|x - y| : x, y \in A\}$.

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