Sandwich-Type Theorems for Locally Convex Cones

Bachelor Thesis

by

Phillip Baumann

August 2017

Contents

1 Introduction

In this thesis we shall prove sandwich-type theorems for superlinear functionals with values in $\mathbb{R} \cup \{\infty, -\infty\}$ on ordered cones. Therefor, we will use the concept of locally convex cones, introduced by K. Keimel and W. Roth in [1]. These yield a boundedness from below, necessary to handle superlinear functionals attaining the value $-\infty$.
In Section 2 we provide a sandwich theorem for functionals with values in $\mathbb{R} \cup \{\infty\}$, which will be used to prove a generalisation in Sections 3 and 4. There, we shall also give a short introduction to locally convex cones, which can be approached either by an abstract 0-neighborhood-system or by a convex quasiuniform structure.
In Section 5 we give a general Hahn-Banach-type theorem, which yields a variety of extension and seperation theorems as corollaries.
Those will find some application in our final section, where we investigate the range of linear functionals on an ordered cone. The Sup-Inf-Theorem proven in this section gives a different characterization of sub- and superharmonic elements of an ordered cone.

2 A Sandwich Theorem for Ordered Cones

Definition 2.1. A cone is a set P endowed with an addition $+ : P \times P \to P$, $(a, b) \mapsto a+b$ and a scalar multiplication $\cdot : P \times \mathbb{R}^+ \to P$, $(\alpha, a) \mapsto \alpha a$ for which the following conditions hold:

- $(a + b) + c = a + (b + c)$, for $a, b, c \in P$
- $a + b = b + a$, for $a, b \in P$
- there exists an element $0_P \in P$ such that $a + 0_P = a$, for $a \in P$
- $\alpha(\beta a) = (\alpha\beta)a$, for $a \in P$ and $\alpha, \beta \in \mathbb{R}^+$
- $(\alpha + \beta)a = \alpha a + \beta a$, for $a \in P$ and $\alpha, \beta \in \mathbb{R}^+$
- $\alpha(a + b) = \alpha a + \beta b$, for $a, b \in P$ and $\alpha \in \mathbb{R}^+$
- $1a = a$, for $a \in P$

Definition 2.2. An ordered cone $(P, \leq)$ is a cone carrying a reflexive and transitive relation $\leq$ satisfying the following conditions:

- $a \leq b \Longrightarrow a + c \leq b + c$, for $a, b, c \in P$
- $a \leq b \Longrightarrow \alpha a \leq \alpha b$, for $a, b \in P$ and $\alpha \in \mathbb{R}^+$

Remark 2.3. Let P be a cone. Since

$$0a = (0 \cdot 2)a = 0(2a) = 0((1 + 1)a) = 0(1a + 1a) = 0(a + a) = 0a + 0a$$

we conclude that $0a = 0_P$ holds for $a \in P$. In the following we shall write 0 instead of 0_P if it is clear, which neutral element is meant. Furthermore, for $a, b, x, y \in P$ satisfying $a \leq b$ and $x \leq y$ we infer

$$\begin{aligned} a + x &\leq b + x \\ x + b &\leq y + b \end{aligned}$$

hence, $a + x \leq b + y$ holds.

Example 2.4. In the following let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ denote the extended real line with the usual algebraic operations extended by:

- $\alpha + \infty = \infty$, for $\alpha \in \overline{\mathbb{R}}$
- $\alpha\infty = \infty$, for $\alpha > 0$
- $0\infty = 0$

It is easy to verify that $\overline{\mathbb{R}}$ together with the standard order relation $\leq$ forms an ordered cone.

Definition 2.5. Let P be an ordered cone. A sublinear functional on P is a map $p : P \to \overline{\mathbb{R}}$ satisfying

$$p(\alpha a) = \alpha p(a) \text{ and } p(a + b) \leq p(a) + p(b), \qquad \text{for } a, b \in P \text{ and } \alpha \in \mathbb{R}^+$$

Likewise a map $q : P \to \overline{\mathbb{R}}$ is called a superlinear functional on P, if

$$q(\alpha a) = \alpha q(a) \text{ and } q(a + b) \geq q(a) + q(b), \qquad \text{for } a, b \in P \text{ and } \alpha \in \mathbb{R}^+$$

A linear functional on P is a map $\mu : P \to \overline{\mathbb{R}}$ such that

$$\mu(\alpha a) = \alpha \mu(a) \text{ and } \mu(a + b) = \mu(a) + \mu(b), \qquad \text{for } a, b \in P \text{ and } \alpha \in \mathbb{R}^+$$

In the following we use the pointwise order relation for functionals f, g on a cone P, i.e. $f \leq g$ iff $f(a) \leq g(a)$ for $a \in P$.

Theorem 2.6. *Let P be an ordered cone. For a sublinear functional $p : P \to \overline{\mathbb{R}}$ and a superlinear functional $q : P \to \overline{\mathbb{R}}$ there exists a monotone linear functional $\mu : P \to \overline{\mathbb{R}}$ such that $q \leq \mu \leq p$ if and only if*

$$q(a) \leq p(b) \text{ whenever } a \leq b \qquad \text{for } a, b \in P \tag{1}$$

Proof. For the necessity of the condition assume there exists a monotone linear functional $\mu : P \to \overline{\mathbb{R}}$ satisfying $q \leq \mu \leq p$ and let $a \leq b$ for some $a, b \in P$. By the properties of μ we get $q(a) \leq \mu(a) \leq \mu(b) \leq p(b)$, which proves (1).
For the converse let $X := \{s : P \to \overline{\mathbb{R}} \mid s \text{ is monotone and sublinear}, q \leq s \leq p\}$ and define a functional $s : P \to \overline{\mathbb{R}}$ by

$$s(a) := \inf\{p(b) \mid a \le b, b \in P\}$$

Now we show that $s \in X$. Therefor let $a \le b$. By the transitivity of the order we have $b \le c \Rightarrow a \le c$, which implies $\{c \in P \mid b \le c\} \subseteq \{c \in P \mid a \le c\}$ and hence $s(a) \le s(b)$. Since $a \le b$ iff $\lambda a \le \lambda b$ for $\lambda > 0$ and p is sublinear, we conclude that $s(\lambda a) = \lambda s(a)$. Furthermore,

$$\begin{aligned}
s(a+b) &= \inf\{p(c) \mid a+b \le c, \quad c \in P\} \\
&= \inf\{p(c_1+c_2) \mid a+b \le c_1+c_2, \quad c_1, c_2 \in P\} \\
&\le \inf\{p(c_1)+p(c_2) \mid a+b \le c_1+c_2, \quad c_1, c_2 \in P\} \\
&\le \inf\{p(c_1)+p(c_2) \mid a \le c_1,\ b \le c_2, \quad c_1, c_2 \in P\} \\
&= s(a)+s(b)
\end{aligned}$$

shows that s is sublinear. Finally, our assumption implies that $q(a)$ is a lower bound for $\{p(b) \mid a \le b, \quad b \in P\}$ for $a \in P$. Together with the reflexivity of the order we infer

$$q(a) \le s(a) \le p(a), \qquad \text{for } a \in P$$

which proves $s \in X$. Now we want to apply Zorn's Lemma in order to show that X contains a minimal element. Therefor let $C \subseteq X$ be a totally ordered subset of X and set

$$s_0(a) := \inf\{s(a) \mid s \in C\}$$

Obviously $s_0(a) \le s(a)$ holds for $s \in C$ and $a \in P$, which yields $q \le s_0 \le p$. Now let $\epsilon > 0$ and $a, b \in P$. By the definition of s_0 there exist $s_a, s_b \in C$ such that

$$\begin{aligned}
s_a(a) \le s_0(a) + \tfrac{\epsilon}{2} \\
s_b(b) \le s_0(b) + \tfrac{\epsilon}{2}
\end{aligned}$$

Since C is totally ordered, we have either $s_a \le s_b$ or $s_b \le s_a$. Without loss of generality assume $s_a \le s_b$. Then

$$\begin{aligned}
s_0(a+b) &\le s_a(a+b) \\
&\le s_a(a) + s_a(b) \\
&\le s_a(a) + s_b(b) \\
&\le s_0(a) + s_0(b) + \epsilon
\end{aligned}$$

together with

$$s_0(\lambda a) = \inf\{s(\lambda a) \mid s \in C\} = \inf\{\lambda s(a) \mid s \in C\} = \lambda s_0(a), \qquad \text{for } a \in P \text{ and } \lambda \ge 0$$

ensures that s_0 is sublinear. As every $s \in C$ is monotone, we easily see that s_0 has to be monotone as well. Putting those results together, we conclude that $s_0 \in X$ is a lower bound of C. By Zorn's Lemma there exists a minimal element $\mu \in X$. To finish the proof we need to show that μ indeed is a linear functional. Let $a_0 \in P$ and set

$$\alpha_0 := sup\{q(c) - \mu(b) \mid b, c \in P, \mu(b) < \infty, c \le a_0 + b\}$$

As $c \leq a_0 + b$ implies $q(c) \leq \mu(c) \leq \mu(a_0 + b) \leq \mu(a_0) + \mu(b)$ and $a_0 \leq a_0 + 0$ we observe that

$$q(a_0) \leq \alpha_0 \leq \mu(a_0) \tag{2}$$

holds. Now define a functional $\tilde{\mu} : P \to \overline{\mathbb{R}}$ by

$$\tilde{\mu}(a) := \inf\{\mu(b) + \lambda\alpha_0 \mid b \in P, \lambda \geq 0, a \leq b + \lambda a_0\}$$

In order to show $q \leq \tilde{\mu}$ let $a \leq b + \lambda a_0$. If $\lambda = 0$ or $\mu(b) = \infty$, $q(a) \leq \mu(b) + \lambda\alpha_0$ easily follows. Otherwise $\frac{a}{\lambda} \leq a_0 + \frac{b}{\lambda}$ and the definition of α_0 proves $q(a) \leq \mu(b) + \lambda\alpha_0$, which implies $q \leq \tilde{\mu}$. Analogous to the beginning of the proof it can be shown that $\tilde{\mu}$ is monotone and sublinear. Together with $\tilde{\mu} \leq \mu \leq p$ this results in $\tilde{\mu} \in X$ and by the minimality of μ we get $\tilde{\mu} = \mu$.
Finally, $a_0 \leq 0 + 1a_0$ implies $\tilde{\mu}(a_0) \leq \alpha_0$, which combined with (2) yields $\tilde{\mu}(a_0) = \alpha_0$. Hence, the mapping $a_0 \mapsto \alpha_0$ coincides with $\tilde{\mu}$. To complete the proof we show that the map

$$\tilde{\mu} : P \to \overline{\mathbb{R}}, \qquad a \mapsto sup\{q(c) - \mu(b) \mid b, c \in P, \mu(b) < \infty, c \leq a + b\}$$

is indeed superlinear. Therefor let $a, b \in P$ and $\lambda \geq 0$. It is evident that

$$\tilde{\mu}(\lambda a) = \lambda\tilde{\mu}(a)$$

holds. Furthermore, let $b_i, c_i \in P$ for $i \in \{1, 2\}$, with

$$c_1 \leq a + b_1$$

$$c_2 \leq b + b_2$$

Then $c_1 + c_2 \leq a + b + b_1 + b_2$ holds and by the sublinearity of μ and the superlinearity of q we conclude

$$\begin{aligned} q(c_1 + c_2) - \mu(b_1 + b_2) &\geq q(c_1) + q(c_2) - (\mu(b_1) + \mu(b_2)) \\ &= q(c_1) - \mu(b_1) + q(c_2) - \mu(b_2) \end{aligned}$$

This implies superlinearity of $\tilde{\mu}$. Hence, linearity of μ is shown, which finishes the proof. ■

The following example shows a situation where the preceding Sandwich Theorem can not be applied and motivates our upcoming theory.

Example 2.7. Let P be the vector space of all sequences in $\mathbb{R}$ with only finitely many non-zero elements and endow P with the canonical order $\leq$, $(a_i)_{i\in\mathbb{N}} \leq (b_i)_{i\in\mathbb{N}}$ iff $a_i \leq b_i$ for $i \in \mathbb{N}$. For $a = (a_i)_{i\in\mathbb{N}} \in P$ set $n(a) := \max\{i \in \mathbb{N} \mid a_i \neq 0\}$, and define a functional $p : P \to \overline{\mathbb{R}}$ by

$$p(a) = \begin{cases} \infty & \text{if } a_i > 0 \text{ for some } i \in \mathbb{N} \\ n(a) \sum\limits_{i\in\mathbb{N}} a_i & \text{else} \end{cases}$$

A straightforward calculation shows that p is sublinear, and even monotone.Let us show that there exists no superlinear functional q with $q \leq p$. In particular, Theorem 2.6 can not be applied to any linear functional dominated by p. Assume on the contrary that there is a superlinear functional $q : P \to \overline{\mathbb{R}}$ with $q \leq p$. Let $e_n := (\delta_{ni})_{i\in\mathbb{N}}$, where δ_{ij} denotes the Kronecker Delta and for some $\lambda \geq 0$ let

$$a_n := -(\lambda e_1 + e_n)$$

Now, the superlinearity of q implies

$$\lambda q(-e_1) + q(-e_n) \leq q(a_n) \leq p(a_n) = -n(\lambda + 1)$$

and hence

$$q(-e_n) \leq -\lambda(q(-e_1) + n) - n$$

But if we let $n > -q(-e_1)$ we see that this can not hold true for every $\lambda \geq 0$.

3 Locally Convex Cones

In order to extend the field of application for sandwich-type theorems, we want to allow the superlinear functional q to attain the value $-\infty$. A theorem of this kind can be formulated using the concept of locally convex cones.

Definition 3.1. Let $(P, \leq)$ be an ordered cone and $V \subseteq P$ not containing 0_P. Then (P, V) is called a full locally convex cone, if V satisfies the following conditions:

- (V1) $0 \leq v$ for $v \in V$
- (V2) for $u, v \in V$ there exists $w \in V$ such that $w \leq u$ and $w \leq v$
- (V3) $\lambda v \in V$, $u + v \in V$ for $u, v \in V$ and $\lambda > 0$
- (V4) for $a \in P$ and $v \in V$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$

Definition 3.2. A subset $V \subseteq P$ of an ordered cone $(P, \leq)$ is called an abstract 0-neighborhood system if $0_P \notin V$ and V1-V3 hold. The elements of V are often referred to as neighborhoods.

Definition 3.3. Let $(Q, \leq)$ be an ordered cone and V a set. (Q, V) is called a locally convex cone if there exists an ordered cone $(P, \leq)$ such that $Q \subseteq P$ is a subcone and (P, V) is a full locally convex cone.

Remark 3.4. Let (P, V) be a locally convex cone. Then V gives rise to three different topologies in the following way:
For $v \in V$ and $a \in P$ we define

$$v(a) := \{b \in P \mid b \leq a + v\}$$

to be a neighborhood of a in the upper topology,

$$(a)v := \{b \in P \mid a \leq b + v\}$$

to be a neighborhood of a in the lower topology, and

$$v(a)v := v(a) \cap (a)v$$

to be a neighborhood of a in the symmetric topology. The name "locally convex cone" is derived from the fact that these sets are indeed convex and for every $a \in P$ the union of all $v \in V$ of those neighborhoods forms a neighborhood basis in the corresponding topology.

In the following example we shall motivate another, more intuitive approach to locally convex cones.

Example 3.5. Let (P, V) be a locally convex cone. For $v \in V$ set

$$\tilde{v} := \{(a, b) \in P \times P \mid a \leq b + v\}$$

We now prove that $\tilde{V} := \{\tilde{v} \mid v \in V\}$ satisfies the following conditions:

- every $\tilde{v} \in \tilde{V}$ is a convex set
- $\triangle := \{(a, a) \mid a \in P\} \subseteq \tilde{v}$, for $\tilde{v} \in \tilde{V}$
- for $\tilde{u}, \tilde{v} \in \tilde{V}$ there is $\tilde{w} \in \tilde{V}$ with $\tilde{w} \subseteq \tilde{u} \cap \tilde{v}$
- $(\lambda\tilde{u}) \circ (\mu\tilde{u}) \subseteq (\lambda + \mu)\tilde{u}$, for $\lambda, \mu > 0$ and $\tilde{u} \in \tilde{V}$
- $\lambda\tilde{v} \in \tilde{V}$, for $\lambda > 0$ and $\tilde{v} \in \tilde{V}$
- for $a \in P$ and $\tilde{v} \in \tilde{V}$ there is $\rho > 0$ such that $(0, a) \in \rho\tilde{v}$

It is straightforward to check that these sets are indeed convex. Since $\leq$ is reflexive and (V1) holds, we conclude that $a \leq a + v$ for $a \in P$ and $v \in V$. Hence, $\triangle \subseteq \tilde{v}$ holds for $\tilde{v} \in \tilde{V}$. Furthermore, for $\tilde{u}, \tilde{v} \in \tilde{V}$ by (V2), there exists $w \in V$ satisfying $w \leq u$ and $w \leq v$ and it is clear that $\tilde{w} \subseteq \tilde{u} \cap \tilde{v}$ holds.
Now let $\lambda, \mu > 0, \tilde{v} \in \tilde{V}$ and $(a, b) \in (\lambda\tilde{v}) \circ (\mu\tilde{v})$. By the definition of $\circ$ there exists $c \in P$ such that $(a, c) \in \lambda\tilde{v}$ and $(c, b) \in \mu\tilde{v}$. As $(a, c) \in \lambda\tilde{v}$ iff $\frac{a}{\lambda} \leq \frac{c}{\lambda} + v$ we get:

$$\begin{aligned} a &\leq c + \lambda v \\ c &\leq b + \mu v \end{aligned}$$

Dividing those inequalities by $(\lambda + \mu)$, adding $\frac{\lambda}{\lambda+\mu}v$ to the second one and using the transitivity of $\leq$ leads to

$$\frac{a}{\lambda + \mu} \leq \frac{b}{\lambda + \mu} + \frac{\mu}{\lambda + \mu}v + \frac{\lambda}{\lambda + \mu}v = \frac{b}{\lambda + \mu} + v$$

showing $(a, b) \in (\lambda + \mu)\tilde{v}$. Since $\lambda\tilde{v} = \widetilde{\lambda v}$ for $\lambda > 0$ and (V3) holds true, we conclude that $\lambda\tilde{v} \in \tilde{V}$ holds for $\tilde{v} \in \tilde{V}$ and $\lambda > 0$. Finally, let $a \in P$ and $v \in V$. Then the condition $(0, a) \in \rho\tilde{v}$ for some $\rho > 0$ transfers into $0 \leq a + \rho v$, which follows immediately from (V4).

Definition 3.6. Let P be a cone. A collection U of convex subsets of $P \times P$ is called a convex quasiuniform structure if the following conditions hold:

- (U1) $\quad \triangle \subseteq u, \qquad$ for $u \in U$
- (U2) $\quad$ for $u, v \in U$ there exists $w \in U$ such that $w \subseteq u \cap v$
- (U3) $\quad (\lambda u) \circ (\mu u) \subseteq (\lambda + \mu)u, \qquad$ for $u \in U$ and $\lambda, \mu > 0$
- (U4) $\quad \lambda u \in U, \qquad$ for $u \in U$ and $\lambda > 0$
- (U5) $\quad$ for $a \in P$ and $u \in U$ there exists $\rho > 0$ such that $(0_P, a) \in \rho u$

We have already seen above that every locally convex cone gives rise to a convex quasiuniform structure. Now we show how a full locally convex cone can be constructed, starting with a cone P and a convex quasiuniform structure.

Example 3.7. Let Q be a cone and U a convex quasiuniform structure on Q. Define

$$V := \{(r_u)_{u \in U} \mid r_u > 0 \text{ and } r_u = \infty \text{ for almost all u } \in U\}$$

By adding a zero element to V we obtain a cone V_0 endowed with the usual componentwise operations and order. Obviously, $P := Q \oplus V_0$ defines a cone as well. Now we shall introduce an order on P in the following way: For $a, b \in Q$ and $r, s \in V_0$ let

$$a \oplus r \preccurlyeq b \oplus s \quad \text{iff } r \leq s \text{ and } (a, b) \in \lambda u \text{ for every } \lambda > s_u - r_u, \text{ whenever } s_u < \infty$$

In the following we prove that $\preccurlyeq$ indeed is an order relation. Reflexivity is easily checked, since (U1) and (U4) hold. In order to show transitivity, let $x \oplus r \preccurlyeq y \oplus s \preccurlyeq z \oplus t$. Firstly, we notice that $r \leq s \leq t$ holds. Secondly, consider $u \in U$ such that $t_u < \infty$ and let $\lambda > t_u - r_u$. Then there exist $\lambda_1 > s_u - r_u, \lambda_2 > t_u - s_u$ satisfying $\lambda = \lambda_1 + \lambda_2$. As $(x, y) \in \lambda_1 u$ and $(y, z) \in \lambda_2 u$, using the property (U3), we conclude that $(x, z) \in \lambda u$ holds, which proves transitivity of $\preccurlyeq$.
Finally, we want to show that $(P, \preccurlyeq)$ is an ordered cone. Therefor, the only thing left to prove is the compatibility of $\preccurlyeq$ with the algebraic operations on P. Let $x \oplus r \preccurlyeq y \oplus s$. It is evident that $\lambda(x \oplus r) \preccurlyeq \lambda(y \oplus s)$ holds for $\lambda > 0$. Finally, let $z \oplus t \in P$. Obviously, $r+t \leq s+t$ holds, so let $u \in U$ such that $s_u+t_u < \infty$ and $\lambda > (s_u+t_u)-(r_u+t_u) = s_u-r_u$. Now choose $\epsilon > 0$ satisfying $\bar{\lambda} := \lambda - \epsilon > s_u - r_u$. We notice $(x, y) \in \bar{\lambda}u$ and $(z, z) \in \epsilon u$. This transfers into the equivalent formulation

$$a := (\frac{x}{\bar{\lambda}}, \frac{y}{\bar{\lambda}}) \in u$$

$$b := (\frac{z}{\epsilon}, \frac{z}{\epsilon}) \in u$$

By using convexity of u, we conclude that

$$\frac{\bar{\lambda}}{\bar{\lambda} + \epsilon} a + \frac{\epsilon}{\bar{\lambda} + \epsilon} b = (\frac{x+z}{\bar{\lambda} + \epsilon}, \frac{y+z}{\bar{\lambda} + \epsilon}) \in u$$

hence, $(x+z, y+z) \in (\bar{\lambda}+\epsilon)u = \lambda u$.
To finish proving our assertion, we now shall show that $\{0_Q\} \oplus V \subseteq P$ satisfies the conditions (V1) - (V4).
(V1) immediately results from (U1) and (U4) using the definition of $\preccurlyeq$. In order to show (V2) let $(r_u)_{u\in U}, (s_u)_{u\in U} \in V$ and set $t_u := \min\{r_u, s_u\}$ for $u \in U$. Then it is straightforward to check

$$0_Q \oplus (t_u)_{u\in U} \preccurlyeq 0_Q \oplus (r_u)_{u\in U}$$

$$0_Q \oplus (t_u)_{u\in U} \preccurlyeq 0_Q \oplus (s_u)_{u\in U}$$

Furthermore, since V is a cone without zero, we conclude that (V3) holds as well. Finally, let $a \oplus r \in P$ and $0_Q \oplus s \in \{0_Q\} \oplus V$. Assume $u_1, ..., u_n$ are those members of U, which satisfy $s_{u_i} < \infty$. (U5) now implies that for every $i \in \{1, ..., n\}$ there exists some $\lambda_i > 0$ satisfying $(0, a) \in \lambda_i u_i$. Set

$$\rho_i := \max\{\frac{\lambda_i - r_{u_i}}{s_{u_i}}, 0\}, \qquad \text{for } i \in \{1, ..., n\}$$

and

$$\rho := \max\{\rho_i \mid i \in \{1, ..., n\}\}$$

For some $i \in \{1, ..., n\}$ let $\lambda > r_i + \rho s_i$. Then $\lambda > \lambda_i$ holds and by the convexity of u_i together with $(0_Q, 0_Q) \in u_i$ we conclude that

$$\lambda_i u_i \subseteq \lambda u_i$$

Since $(0, a) \in \lambda_i u_i$ and $i \in \{1, ..., n\}$ was arbitrary, we see that

$$0_P \preccurlyeq (a \oplus r) + \rho(0 \oplus s)$$

holds, proving (V4). Thus, $(P, \{0\} \oplus V)$ endowed with $\preccurlyeq$ is a full locally convex cone.

We will use the preceding construction of such a full locally convex cone to prove some of our upcoming theorems.

4 Sandwich-Type Theorems for Locally Convex Cones

In the following we consider functionals attaining the value $-\infty$. We extend the algebraic operation defined on $\overline{\mathbb{R}}$ to $\overline{\overline{\mathbb{R}}} := \overline{\mathbb{R}} \cup \{-\infty\}$ by

$$a + (-\infty) = -\infty, \qquad \text{for } a \in \overline{\mathbb{R}}$$

$$\alpha \cdot (-\infty) = -\infty, \qquad \text{for } \alpha > 0$$

$$0 \cdot (-\infty) = 0$$

If we let $-\infty$ be the least element of $\overline{\overline{\mathbb{R}}}$, $(\overline{\overline{\mathbb{R}}}, \leq)$ again is an ordered cone.

Definition 4.1. Let P be a cone. A map $q : P \to \overline{\overline{\mathbb{R}}}$ is called an extended superlinear functional iff

$$q(a+b) \geq q(a) + q(b) \qquad \text{for } a, b \in P$$
$$q(\alpha a) = \alpha q(a) \qquad \text{for } a \in P \text{ and } \alpha > 0$$

Definition 4.2. Let (P, V) be a locally convex cone and $v \in V$. A linear, sublinear or superlinear map $\mu : P \to \overline{\mathbb{R}}$ is said to be uniformly continuous with respect to v iff

$$a \leq b + v \implies \mu(a) \leq \mu(b) + 1, \qquad \text{for } a, b \in P$$

The set of all uniformly continuous linear functionals in respect to a certain $v \in V$ is denoted by v°, the polar of v.
The union of all polars v° is called the dual cone P^*.

Remark 4.3. Let (P, V) be a locally convex cone, $v \in V$ a neighborhood and $\mu \in v^\circ$. Firstly, we notice that for $\lambda > 0$

$$a \leq b + \lambda v \implies \frac{a}{\lambda} \leq \frac{b}{\lambda} + v \implies \mu(\frac{a}{\lambda}) \leq \mu(\frac{b}{\lambda}) + 1 \implies \mu(a) \leq \mu(b) + \lambda$$

holds. If we let $a \leq b$ for $a, b \in P$, then for every $\epsilon > 0$

$$a \leq b + \epsilon v$$

holds and therefore we get

$$\mu(a) \leq \mu(b) + \epsilon$$

Hence, μ is monotone. Secondly, $(\overline{\mathbb{R}}, \mathbb{R}^+ \setminus \{0\})$ obviously forms a locally convex cone. Then μ is continuous with respect to the lower, upper and symmetric topology, which can be seen as follows: Let $a \in P$, $\lambda > 0$ and $\lambda(\mu(a))$ a neighborhood in the upper topology of $\mu(a)$

$$\lambda(\mu(a)) = \{c \in \overline{\mathbb{R}} \mid c \leq \mu(a) + \lambda\}$$

Choosing $\lambda v(a)$ as a neighborhood of a in the upper topology of P leads to

$$\mu(\lambda v(a)) \subseteq \lambda(\mu(a))$$

hence, μ is continuous with respect to the upper topology. An analogous proof shows continuity with respect to the lower and symmetric topology.

Theorem 4.4. *Let (P, V) be a locally convex cone and $v \in V$ some neighborhood. Furthermore, let $p : P \to \overline{\mathbb{R}}$ be a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional. Then there exists a linear functional $\mu \in v^\circ$ such that $q \leq \mu \leq p$ if and only if*

$$a \leq b + v \implies q(a) \leq p(b) + 1, \qquad \textit{for } a, b \in P \tag{3}$$

Proof. The necessity of the condition is evident, as for any $\mu \in v^\circ$ satisfying $q \leq \mu \leq p$ and $a, b \in P$, $a \leq b + v$

$$q(a) \leq \mu(a) \leq \mu(b) + 1 \leq p(b) + 1$$

holds. For the converse, assume that condition (3) holds. $\bar{V} := \{\lambda v \mid \lambda > 0\}$ obviously satisfies (V1)-(V4) and therefore $(P, \bar{V})$ forms a locally convex cone. Hence, following the notation of Example 3.5, $U := \{\tilde{u} \subseteq P \times P \mid u \in \bar{V}\}$ is a convex quasiuniform structure. Now we use the same method as shown in Example 3.7 to construct a full locally convex cone: Let

$$\bar{P} := P \oplus \mathbb{R}^+$$

and define an order relation $\preccurlyeq$ on $\bar{P}$ by

$$(a \oplus \alpha) \preccurlyeq (b \oplus \beta) \text{ iff } \alpha \leq \beta \text{ and } (a, b) \in \lambda\tilde{v}, \qquad \text{for } \lambda > \beta - \alpha$$

$(\bar{P}, \preccurlyeq)$ now forms a full locally convex cone, as elaborated in Example 3.7. We extend the sublinear functional p to $\bar{P}$ by

$$\bar{p}(a \oplus \alpha) := p(a) + \alpha$$

and define a map $\bar{q}$ on $\bar{P}$ by

$$\bar{q}(a \oplus \alpha) := sup\{q(d) - \lambda \mid d \in P, \lambda > 0 \text{ and } d \leq a + \lambda v\} + \alpha$$

Since (V4) holds, for every $a \in P$ there exists some $\lambda > 0$ such that $0 \leq a + \lambda v$. This implies

$$\bar{q}(a \oplus \alpha) \geq -\lambda + \alpha > -\infty$$

hence, $\bar{q} : \bar{P} \to \overline{\mathbb{R}}$. Now we will show that $\bar{q}$ is superlinear. It is straightforward to check that for $\lambda > 0$ and $(a \oplus \alpha) \in \bar{P}$

$$\bar{q}(\lambda(a \oplus \alpha)) = \lambda\bar{q}(a \oplus \alpha)$$

holds. In order to show superadditivity let $(a \oplus \alpha), (b \oplus \beta) \in \bar{P}$. Then

$$\begin{aligned} d_1 &\leq a + \lambda_1 v \\ d_2 &\leq b + \lambda_2 v \end{aligned}$$

implies

$$d_1 + d_2 \leq (a + b) + (\lambda_1 + \lambda_2)v$$

for some $d_1, d_2 \in P$ and $\lambda_1, \lambda_2 > 0$. Since q is superlinear, we get

$$q(d_1 + d_2) - (\lambda_1 + \lambda_2) \geq [q(d_1) - \lambda_1] + [q(d_2) - \lambda_2]$$

Hence, $\bar{q}$ is a superlinear functional. Now, to finish the proof, we want to apply Theorem 2.6 to the functionals $\bar{q}$ and $\bar{p}$ on the ordered cone $(\bar{P}, \preccurlyeq)$. As $\bar{p}$ obviously is sublinear, the only thing left to show is that condition (1) holds. Therefor let $(a \oplus \alpha), (b \oplus \beta) \in \bar{P}$ such that

$$(a \oplus \alpha) \preccurlyeq (b \oplus \beta)$$

From the definition of $\preccurlyeq$ we infer that

$$\alpha \leq \beta \text{ and } (a, b) \in \lambda\tilde{v}, \qquad \text{for } \lambda > \beta - \alpha$$

holds. Assume to the contrary $\bar{q}(a \oplus \alpha) > \bar{p}(b \oplus \beta)$. Then there exists $d \in P$ and $\bar{\lambda} > 0$ such that

$$d \leq a + \bar{\lambda}v \tag{4}$$

and

$$q(d) - \bar{\lambda} + \alpha > p(b) + \beta \tag{5}$$

holds. But, for any $\lambda > \beta - \alpha$, $(a, b) \in \lambda\tilde{v}$ transfers into $a \leq b + \lambda v$. Combined with (4), we infer

$$d \leq b + (\lambda + \bar{\lambda})v$$

and our condition implies

$$q(d) \leq p(b) + \lambda + \bar{\lambda}$$

Since $\lambda > \beta - \alpha$ was arbitrary, this is a contradiction to (5). Hence, condition (1) holds and we can apply Theorem 2.6. So there exists a monotone linear functional $\tilde{\mu}$ on $\bar{P}$ such that $\bar{q} \leq \bar{\mu} \leq \bar{p}$. Finally, we show that

$$\mu(a) := \bar{\mu}(a \oplus 0), \qquad \text{for } a \in P$$

has the desired properties. As $q(a) \leq \bar{q}(a \oplus 0)$ and $p(a) = \bar{p}(a \oplus 0)$ holds for $a \in P$, we infer $q \leq \mu \leq p$. Now let $a, b \in P$ satisfying $a \leq b + v$. By (V1) we observe that

$$a \leq b + \lambda v, \qquad \text{for } \lambda > 1$$

holds. Hence, $(a \oplus 0) \leq (b \oplus 1) = (b \oplus 0) + (0_P \oplus 1)$. Using the linearity of $\bar{\mu}$ we get

$$\mu(a) \leq \mu(b) + \bar{\mu}(0_P \oplus 1) \leq \mu(b) + \bar{p}(0_P \oplus 1) = \mu(b) + 1$$

Since μ obviously inherits linearity from $\bar{\mu}$, we have shown $\mu \in v^{\circ}$, which finishes the proof. ■

In the following we shall utilise the connection between the neighborhoods of a locally convex cone and left-absorbing sets of an ordered cone to formulate an algebraic version of Theorem 4.4.

Definition 4.5. Let $(P, \leq)$ be an ordered cone. A convex subset $L \subseteq P$ is called left-absorbing if $0_P \in L$ and for every $a \in P$ there is $l \in L$ and $\lambda \geq 0$ such that $\lambda l \leq a$.

Theorem 4.6. *Let $(P, \leq)$ be an ordered cone, $p : P \to \overline{\mathbb{R}}$ a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional. Then there exists a monotone linear functional $\mu : P \to \overline{\mathbb{R}}$ satisfying $q \leq \mu \leq p$ if and only if there is a left-absorbing subset $L \subseteq P$ such that*

$$a + l \leq b \implies q(a) \leq p(b) + 1, \qquad \textit{for } a, b \in P \textit{ and some } l \in L \tag{6}$$

Proof. First, assume there exists a monotone linear functional μ such that $q \leq \mu \leq p$. Set

$$L := \{b \in P \mid 0 \leq \mu(b) + 1\}$$

L obviously contains 0_P and it is straightforward to check convexity of L. For $a \in P$ let $\lambda > 0$ such that $0 \leq \mu(a) + \lambda$. Linearity of μ now guarantees $\frac{a}{\lambda} \in L$. As $\lambda \frac{a}{\lambda} \leq a$ we see that L indeed is a left-absorbing subset of P.
In order to prove (6) let $a, b \in P$ and $l \in L$ with $a + l \leq b$. Due to the properties of μ and the definition of L we get

$$\begin{array}{rcl} \mu(a) + \mu(l) & \leq & \mu(b) \\ 0 & \leq & \mu(l) + 1 \end{array}$$

Adding 1 to the first inequality, $\mu(a)$ to the second one and using transitivity of $\leq$, we obtain $\mu(a) \leq \mu(b) + 1$. Since $q \leq \mu \leq p$ holds, we conclude

$$q(a) \leq p(b) + 1$$

proving one direction of the equivalence.
For the converse assume there exists a left-absorbing subset $L \subseteq P$ such that (6) holds. Let

$$u := \{(a, b) \in P \times P \mid a + l \leq b \text{ for some } l \in L\}$$

and

$$U := \{\lambda u \mid \lambda > 0\}$$

Note that $\lambda u = \{(a, b) \in P \times P \mid a + \lambda l \leq b \text{ for some } l \in L\}$. We now shall prove that U is a convex quasiuniform structure:
Therefor let $\lambda, \mu > 0$ arbitrary. Convexity of λu follows from the convexity of L and (U1) is a direct consequence of $0_P \in L$. Now assume without loss of generality that $\lambda \leq \mu$. Then for $(a, b) \in \lambda u$ there exists $l \in L$ such that

$$\frac{a}{\lambda} + l \leq \frac{b}{\lambda}$$

Multiplying with $\frac{\lambda}{\mu} \leq 1$ yields

$$\frac{a}{\mu} + \frac{\lambda}{\mu} l \leq \frac{b}{\mu}$$

As $0_P \in L$ and L is convex, we conclude $\frac{\lambda}{\mu} l \in L$, hence, $(a, b) \in \mu u$. Therefore $\lambda u \subseteq \mu u$ holds, showing (U2). In order to prove (U3) let $(a, b) \in (\lambda u) \circ (\mu u)$. Then there exist $c \in P$, l_1, l_2 such that

$$\begin{array}{rcl} \dfrac{a}{\lambda} + l_1 & \leq & \dfrac{c}{\lambda} \\ \dfrac{c}{\mu} + l_2 & \leq & \dfrac{b}{\mu} \end{array}$$

Multiplication with $\frac{\lambda}{\lambda+\mu}$ (resp. $\frac{\mu}{\lambda+\mu}$) and adding $\frac{\mu}{\lambda+\mu} l_2$ to the second inequality yields

$$\frac{a}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} l_1 + \frac{\mu}{\lambda + \mu} l_2 \leq \frac{b}{\lambda + \mu}$$

Since L is convex, we conclude $\frac{\lambda}{\lambda+\mu}l_1 + \frac{\mu}{\lambda+\mu}l_2 \in L$ and therefore $(a,b) \in (\lambda+\mu)u$ holds. This proves (U3). (U4) is an immediate consequence of the definition of U. To show (U5) let $a \in P$. By the properties of a left-absorbing set, there exists $\rho > 0$ and $l \in L$ such that $\rho l \leq a$, which is equivalent to $(0_P, a) \in \rho u$. Hence, (U5) holds and U is a convex quasiuniform structure.
Now $Q := P \oplus \mathbb{R}^+$ endowed with an order relation $\preccurlyeq$ defined by

$$(a \oplus \alpha) \preccurlyeq (b \oplus \beta) \text{ iff } \alpha \leq \beta \text{ and } (a,b) \in \lambda u, \qquad \text{for } \lambda > \beta - \alpha$$

forms a full locally convex cone, as elaborated in Example 3.7. Therefore

$$\bar{P} := (P \oplus \{0\}, \{O_P\} \oplus (\mathbb{R}^+ \setminus \{0\}))$$

forms a locally convex cone. We now define a sublinear functional $\tilde{p}$ and a superlinear functional $\tilde{q}$ on $\bar{P}$ in a natural way: For $a \in P$ let

$$\tilde{p}(a \oplus 0) := p(a)$$

$$\tilde{q}(a \oplus 0) := q(a)$$

To finish the proof, we want to apply Theorem 4.4 to the neighborhood $0_P \oplus \frac{1}{2} \in Q$. Therefor we need to check if (3) holds. Let $a, b \in P$ such that

$$(a \oplus 0) \preccurlyeq (b \oplus 0) + (0_P \oplus \frac{1}{2}) = (b \oplus \frac{1}{2})$$

holds. By the definition of $\preccurlyeq$ we get

$$(a,b) \in \lambda u, \qquad \text{for } \lambda > \frac{1}{2}$$

This yields $(a,b) \in u$, hence, there exists $l \in L$ such that $a + l \leq b$. (6) now implies $q(a) \leq p(b) + 1$, which is equivalent to $\tilde{q}(a \oplus 0) \leq \tilde{p}(b \oplus 0) + 1$. Applying the theorem yields the existence of a linear functional $\tilde{\mu} \in (\frac{1}{2})^\circ$ on $\bar{P}$ satisfying $\tilde{q} \leq \tilde{\mu} \leq \tilde{p}$. Finally, define a linear functional μ on P by

$$\mu(a) := \tilde{\mu}(a \oplus 0) \qquad \text{for } a \in P$$

As $q \leq \mu \leq p$ is obvious, the only thing left to show is monotonicity of μ. Therefor let $a, b \in P$, $a \leq b$. Since $0_P \in L$, we observe $(a,b) \in \lambda u$ for every $\lambda > 0$, showing $(a \oplus 0) \preccurlyeq (b \oplus 0)$. Monotonicity of $\tilde{\mu}$ now yields

$$\mu(a) = \tilde{\mu}(a \oplus 0) \leq \tilde{\mu}(b \oplus 0) = \mu(b)$$

Hence, μ is monotone and the proof is finished. ■

Corollary 4.7. *Let (P, V) be a locally convex cone and $v \in V$ a neighborhood. For any sublinear functional $p : P \to \overline{\mathbb{R}}$ the following are equivalent:*

(i) *p is uniformly continuous with respect to v;*

(ii) *for every extended superlinear functional $q : P \to \overline{\overline{\mathbb{R}}}$ with $q \leq p$ there exists a monotone linear functional $\mu \in v^\circ$ such that $q \leq \mu \leq p$;*

(iii) $p(a) = \max\{\mu(a) \mid \mu \in v^\circ$ *and* $\mu \leq p\}$, *for* $a \in P$

Proof. Asssume p is continuous with respect to v. Let q be an extended superlinear functional satisfying $q \leq p$ and $a, b \in P$ such that $a \leq b + v$. Then

$$q(a) \leq p(a) \leq p(b) + 1$$

holds and Theorem4.4 yields the existence of a linear functional $\mu \in v^\circ$ with the desired properties.

Now assume that (ii) holds and let $a \in P$. Define an extended superlinear functional $q : P \to \overline{\overline{\mathbb{R}}}$ the following way:

$$q(b) := \begin{cases} \lambda p(a) & \text{if } b = \lambda a \text{ for some } \lambda \geq 0 \\ -\infty & \text{else} \end{cases}$$

Since p is positively homogeneous, we observe $q \leq p$. Our assumption now implies that there exists a monotone linear functional $\mu \in v^\circ$ such that $q \leq \mu \leq p$. This shows $q(a) \leq \mu(a) \leq p(a) = q(a)$, hence, $p(a) = \mu(a)$ holds. As

$$p(a) \leq \max\{\mu(a) \mid \mu \in v^\circ \text{ and } \mu \leq p\}$$

is evident, we infer (iii).

For the last part of the proof assume that (iii) holds. Let $a, b \in P$ such that $a \leq b + v$. By our assumption there exists a functional $\mu \in v^\circ$ satisfying $\mu \leq p$ and $\mu(a) = p(a)$. Hence,

$$p(a) = \mu(a) \leq \mu(b) + 1 \leq p(b) + 1$$

holds, which shows uniform continuity of p and therefore finishes the proof. ■

Theorem 4.8. *Let (P, V) be a locally convex cone, and $v \in V$ a neighborhood. Furthermore, let $p : P \to \overline{\mathbb{R}}$ be a sublinear functional that is uniformly continuous with respect to v. If p is unbounded on a subset $A \subseteq P$, then there exists a monotone linear functional $\mu \in v^\circ$, $\mu \leq p$ such that μ is also unbounded on A.*

Proof. Let p be a sublinear functional that is continuous with respect to v which is unbounded on a subset $A \subseteq P$. Since

$$q(b) := \begin{cases} 0 & \text{if } b = 0 \\ -\infty & \text{else} \end{cases}$$

obviously is an extended superlinear functional such that $q \le p$, Corollary 4.7 now guarantees that there exists at least one monotone linear functional $\mu \in v^\circ$ satisfying $\mu \le p$. If $\inf\{p(a) \mid a \in A\} = -\infty$, then the same holds true for any functional $\mu \le p$, hence, our claim is obvious.
Thus we assume that $sup\{p(a) \mid a \in A\} = \infty$ and every functional $\mu \in v^\circ$, $\mu \le p$ is bounded below on A. Now we will prove that in this case, there exists at least one such functional μ, which is unbounded above on A.
Therefor we will construct sequences of elements $a_n \in A$, of functionals $\mu_n \in v^\circ$ and of real numbers α_n in the following way:
Set $\alpha_1 := \frac{1}{2}$ and let $a_1 \in A$ such that $p(a_1) \ge 2$. Such an element exists, since we assumed that p is unbounded above on A. Furthermore, Corollary 4.7 yields the existence of a monotone linear functional $\mu_1 \in v^\circ$ such that $\mu_1 \le p$ and $\mu_1(a_1) = p(a_1)$.
For $n \ge 2$ we observe the following: By (V4), for every $i \in \{1, ..., n-1\}$ there exists $\lambda_i \ge 0$ such that $0 \le a_i + \lambda_i v$. Then $\lambda := \max\{\lambda_i \mid i \in \{1, ..., n-1\}\} \cup \{1\}$ satisfies

$$0 \le a_i + \lambda v \qquad \text{for } i \in \{1, ..., n-1\}$$

Multiplying with the strictly positive real number $\alpha_n := \frac{2^{-n}}{\lambda} \le 2^{-n}$ yields

$$0 \le \alpha_n a_i + 2^{-n} v \qquad \text{for } i \in \{1, ..., n-1\}$$

Note that for any $m \in v^\circ$

$$\alpha_n m(a_i) \ge -2^{-n} \tag{7}$$

holds. By our assumption, every μ_i, $i \in \{1, ..., n-1\}$ is bounded below i.e. there exist $c_i \in \mathbb{R}$ such that $\mu_i > c_i$. Hence, using part (iii) of Corollary 4.7 again, we can choose $a_n \in A$ and $\mu_n \in v^\circ$ satisfying $\mu_n \le p$ and

$$\sum_{i=0}^{n} \alpha_i \mu_i(a_n) \ge \sum_{i=0}^{n-1} \alpha_i c_i + \mu_n(a_n) \ge n \tag{8}$$

Now set

$$\alpha := \sum_{i=1}^{\infty} \alpha_i \le \sum_{i=1}^{\infty} 2^{-i} \le 1$$

Let $c \in P$. By (V4) there exists some $\lambda \ge 0$ such that $0 \le c + \lambda v$. As every μ_n is contained in v° we infer

$$-\lambda \le \mu_n(c) \le p(c)$$

Hence,

$$\mu(c) := \frac{1}{\alpha} \sum_{i=1}^{\infty} \alpha_i \mu_i(c) \qquad \text{for } c \in P$$

is convergent in $\overline{\mathbb{R}}$ and defines a linear functional on P. Now $\mu \le p$ is evident, and for $c, d \in P$ with $c \le d + v$

$$\mu(c) = \frac{1}{\alpha} \sum_{i=1}^{\infty} \alpha_i \mu_i(c) \le \frac{1}{\alpha} \sum_{i=1}^{\infty} \alpha_i(\mu_i(d) + 1) \le \mu(d) + 1$$

holds, which implies $\mu \in v^\circ$. Furthermore, by inequalities (7) and (8), we compute

$$\alpha\mu(a_n) = \sum_{i=1}^{\infty} \alpha_i\mu_i(a_n) = \sum_{i=1}^{n} \alpha_i\mu_i(a_{in}) = \sum_{i=n+1}^{\infty} \alpha_i\mu_i(a_{in}) \geq n - \sum_{i=n+2}^{\infty} 2^{-i} \geq n-1$$

Therefore, we infer

$$sup\{\mu(a) \mid a \in A\} \geq sup\{\mu(a_n) \mid n \in \mathbb{N}\} = \infty$$

which shows that μ indeed is unbounded above on A. ■

5 Hahn-Banach Type Theorems for Locally Convex Cones

We now come to our main Hahn-Banach-type theorems. First we show a generalised extension theorem, which has some interesting results as special cases.

Definition 5.1. Let P be a cone and $C \subseteq P$ a convex subset. A map $f : P \to \overline{\mathbb{R}}$ is called convex if

$$f(\lambda c_1 + (1-\lambda)c_2) \leq \lambda f(c_1) + (1-\lambda) f(c_2)$$

holds for all $c_1, c_2 \in C$ and $\lambda \in [0,1]$.
Likewise, a map $f : C \to \overline{\mathbb{R}}$ is said to be concave if

$$f(\lambda c_1 + (1-\lambda)c_2) \geq \lambda f(c_1) + (1-\lambda) f(c_2)$$

holds for all $c_1, c_2 \in C$ and $\lambda \in [0,1]$.
An affine function is a map that is both convex and concave.

Theorem 5.2. *Let (P,V) be a locally convex cone, C and D non-empty convex subsets of P, and $v \in V$ a neighborhood. Furthermore, let $p : P \to \overline{\mathbb{R}}$ be a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional.*
For a convex functional $f : C \to \overline{\mathbb{R}}$ and a concave functional $g : D \to \overline{\overline{\mathbb{R}}}$ there exists a monotone linear functional $\mu \in v^\circ$ satisfying

$$q \leq \mu \leq p, \quad g \leq \mu \text{ on } D \quad \text{and} \quad \mu \leq f \text{ on } C \tag{9}$$

if and only if

$$a + \rho d \leq b + \sigma c + v \implies q(a) + \rho g(d) \leq p(b) + \sigma f(c) + 1 \tag{10}$$

holds for $a, b \in P$, $c \in C$, $d \in D$ and $\rho, \sigma \geq 0$.

Proof. At first we will show the necessity of condition (10) for the existence of a linear functional $\mu \in v^\circ$ with the desired properties. Therefor assume there exists a monotone linear functional $\mu \in v^\circ$ such that (9) holds. Furthermore, let $a, b \in P$, $c \in C$, $d \in D$ and $\rho, \sigma \geq 0$ such that $a + \rho d \leq b + \sigma c + v$. We infer

$$q(a) + \rho q(d) \leq q(a + \rho d) \leq \mu(a + \rho d) \leq \mu(b + \sigma c) + 1 \leq p(b + \sigma c) + 1 \leq p(b) + \sigma p(c) + 1$$

Hence, condition (10) holds. For the converse, assume that our condition is valid and define two functionals $\tilde{p}$ and $\tilde{q}$ on P by

$$\tilde{p}(x) := \inf\{p(b) + \sigma f(c) + \lambda \mid b \in P, c \in C,\ \ \lambda, \sigma \geq 0,\ \text{and } x \leq b + \sigma c + \lambda v\}$$

$$\tilde{q}(x) := \sup\{q(a) + \rho g(d) \mid a \in P, d \in D,\ \ \rho \geq 0,\ \text{and } a + \rho d \leq x\}$$

As $x \leq x + 0c + 0v$ and $c \leq 0_P + 1c + 0v$ holds, we conclude

$$\tilde{p} \leq p \quad \text{and} \quad \tilde{p} \leq f \text{ on } C \tag{11}$$

Analogously it can be seen that

$$\tilde{q} \geq q \quad \text{and} \quad \tilde{q} \geq g \text{ on } D \tag{12}$$

holds. Let $x, b \in P$, $c \in C$ and $\lambda, \sigma \geq 0$ such that $x \leq b + \sigma c + \lambda v$. By (V4) there exists some $\rho \geq 0$ satisfying $0 \leq x + \rho v$. Combining those inequalities yields

$$0 \leq b + \sigma c + (\rho + \lambda)v$$

and condition (10) guarantees

$$0 \leq p(b) + \sigma f(c) + (\rho + \lambda)$$

hence, $\tilde{p}(x) \geq -\rho > -\infty$. To finish the proof, we want to apply Theorem 4.4 to the functionals $\tilde{p} : P \to \overline{\mathbb{R}}$ and $\tilde{q} : P \to \overline{\overline{\mathbb{R}}}$. Therefor we need to prove sublinearity for $\tilde{p}$ resp. extended superlinerity for $\tilde{q}$. We shall only show the required properties for $\tilde{p}$, as the proof of $\tilde{q}$ is analogous. Positive homogenity is easily seen, since

$$x \leq b + \sigma c + \lambda v \quad \text{iff} \quad \alpha x \leq \alpha b + \alpha\sigma c + \alpha\lambda v$$

holds for any $\alpha, \lambda \geq 0$, $x, b \in P$ and $c \in C$ and

$$p(\alpha b) + \alpha\sigma f(c) + \alpha\lambda = \alpha(p(b) + \sigma f(c) + \lambda)$$

In order to show subadditivity, let $x, y, b_1.b_2 \in P$, $c_1, c_2 \in C$ and $\sigma_1, \sigma_2, \lambda_1, \lambda_2 \geq 0$ such that

$$\begin{aligned} x &\leq b_1 + \sigma_1 c_1 + \lambda_1 v \\ y &\leq b_2 + \sigma_2 c_2 + \lambda_2 v \end{aligned}$$

Adding those inequalities together yields

$$x + y \leq (b_1 + b_2) + \sigma(\frac{\sigma_1}{\sigma}c_1 + \frac{\sigma_2}{\sigma}c_2) + (\lambda_1 + \lambda_2)$$

where $\sigma := \sigma_1 + \sigma_2$. By the convexity of C we infer $c := \frac{\sigma_1}{\sigma} c_1 + \frac{\sigma_2}{\sigma} c_2 \in C$. Thus, using the properties of p and f, we compute

$$p(b_1 + b_2) + \sigma f(c) + (\lambda_1 + \lambda_2) \leq p(b_1) + p(b_2) + \sigma_1 f(c_1) + \sigma_2 f(c_2) + \lambda_1 + \lambda_2$$

Hence, $\tilde{p}(x + y) \leq \tilde{p}(x) + \tilde{p}(y)$ holds, showing that $\tilde{p}$ indeed is a sublinear functional. In order to apply Theorem 4.4, the only thing left is to show that condition (3) holds. Therefor let $x, y \in P$ such that $x \leq y + v$. For any $a, b \in P$, $c \in C$, $d \in D$ and $\lambda, \rho, \sigma \geq 0$ with $a + \rho d \leq x$ and $y \leq b + \sigma c + \lambda v$ we infer

$$a + \rho d \leq b + \sigma c + (1 + \lambda) v$$

and, by condition (10),

$$q(a) + \rho g(d) \leq p(b) + \sigma f(c) + \lambda + 1$$

follows. Therefore, we can apply Theorem 4.4, which yields a monotone linear functional $\mu \in v^\circ$ such that $\tilde{q} \leq \mu \leq \tilde{p}$. From (11) and (12) we conclude that μ indeed has the desired properties. ∎

Remark 5.3. If $\alpha c \in C$, for any $c \in C$ and $\alpha \geq 0$, and if f is a linear functional, we observe that condition (10) needs to be verified only for $\sigma = 1$. Obviously, the same holds for D, g and ρ. Furthermore, if $f \equiv \infty$ or $g \equiv -\infty$, we have to consider the condition only for $\sigma = 0$ and $\rho = 0$ resp.

Similar to Theorem 4.6 we can formulate an algebraic version of Theorem 5.2 by considering the convex left-absorbing subset $L := \{b \in P \mid 0 \leq \mu(b) + 1\}$ for a monotone linear functional μ on an ordered cone $(P, \leq)$.

Theorem 5.4. *Let $(P, \leq)$ be an ordered cone and C, D non-empty convex subsets of P. Furthermore, let $p : P \to \overline{\mathbb{R}}$ be a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional. For a convex functional $f : C \to \overline{\mathbb{R}}$ and a concave functional $g : D \to \overline{\overline{\mathbb{R}}}$ there exists a monotone linear functional $\mu : P \to \overline{\mathbb{R}}$ satisfying*

$$q \leq \mu \leq p, \quad g \leq \mu \text{ on } D \quad \text{and} \quad \mu \leq f \text{ on } C$$

if and only if there is a left-absorbing convex subset $L \subseteq P$ such that

$$a + \rho d + l \leq b + \sigma c \implies q(a) + \rho g(d) \leq p(b) + \sigma f(c) + 1 \tag{13}$$

holds for $a, b \in P$, $l \in L$, $c \in C$, $d \in D$ and $\rho, \sigma \geq 0$.

Proof. Similar as Theorem 4.6. ∎

Corollary 5.5. *Let (P, V) be a locally convex cone, $v \in V$ a neighborhoods, and C, D non-empty convex subsets of P. For a convex functional $f : P \to \overline{\mathbb{R}}$ and a concave functional $g : P \to \overline{\overline{\mathbb{R}}}$ there exists a monotone linear functional $\mu \in v^\circ$ such that*

$$g \leq \mu \text{ on } D \quad \text{and} \quad \mu \leq f \text{ on } C$$

if and only if

$$\rho d \leq \sigma c + v \implies \rho g(d) \leq \sigma f(c) + 1 \tag{14}$$

holds for $c \in C$, $d \in D$ and $\rho, \sigma \geq 0$.

Proof. Apply Theorem 5.2 to the sublinear functional $p : P \to \overline{\mathbb{R}}$ and extended superlinear functional $q : P \to \overline{\overline{\mathbb{R}}}$ defined by

$$p(a) := \begin{cases} 0 & \text{if } a = 0 \\ \infty & \text{else} \end{cases}$$

$$q(a) := \begin{cases} 0 & \text{if } a = 0 \\ -\infty & \text{else} \end{cases}$$

■

Corollary 5.6. *Let (P, V) be a locally convex cone, $v \in V$ a neighborhood and $Q \subseteq P$ a subcone of P. Then for any linear, uniformly continuous functional $\mu : Q \to \overline{\mathbb{R}}$ there exists a linear functional $\tilde{\mu} \in v^\circ$ such that*

$$\mu = \tilde{\mu} \text{ on } Q$$

Proof. It is evident that every linear functional is concave and convex. Therefore, we can apply Corollary 5.5 to the functionals $f = g = \mu$ on the convex sets $C = D = Q$. As C, D, f and g fulfill the requirements of Remark 5.3, we observe that condition (14) reduces to

$$d \leq c + v \implies \mu(d) \leq \mu(c) + 1$$

for $c, d \in Q$, which coincides with the uniform continuity of μ. Hence, there exists a monotone linear functional $\tilde{\mu} \in v^\circ$ such that $\mu \leq \tilde{\mu} \leq \mu$ on Q. ■

Theorem 5.7. *Let (P, V) be a locally convex cone, $v \in V$ a neighborhood and C, D non-empty convex subsets of P. For $\alpha \in \mathbb{R}$ there exists a monotone linear functional $\mu \in v^\circ$ such that*

$$\mu(c) \leq \alpha \leq \mu(d) \qquad \text{for } c \in C \text{ and } d \in D$$

if and only if

$$\rho d \leq \sigma c + v \implies \alpha\rho \leq \alpha\sigma + 1 \tag{15}$$

Proof. Apply Corollary 5.5 with the maps $f \equiv \alpha$ and $g \equiv \alpha$. ■

Definition 5.8. Let $(P, \leq)$ be an ordered cone. A subset $C \subseteq P$ is called increasing, if $a \in C$ whenever $c \leq a$ for $a \in P$ and some $c \in C$.
Likewise, a subset D of P is called decreasing, if $a \in D$ whenever $a \leq d$ for $a \in P$ and some $d \in D$.

Corollary 5.9. *Let $(P, \leq)$ be an ordered cone, and let C, D be disjoint non-empty convex subsets of P. Furthermore, suppose that for every $a \in P$ there are $c \in C$, $d \in D$ and $\sigma, \rho \geq 0$ such that $\rho d \leq a + \sigma c$.*

(i) If C is decreasing and $0_P \in C$, then there exists a monotone linear functional $\mu : P \to \overline{\mathbb{R}}$ such that

$$\mu(c) \leq 1 \leq \mu(d) \qquad \textit{for } c \in C \textit{ and } d \in D$$

(ii) If D is increasing and $0_P \in D$, then there exists a monotone linear functional $\mu : P \to \overline{\mathbb{R}}$ such that

$$\mu(c) \leq -1 \leq \mu(d) \qquad \textit{for } c \in C \textit{ and } d \in D$$

Proof. We shall take a similar approach as in the proof of Theorem 4.6: Let

$$L := \{l \in P \mid \rho d \leq l + \sigma c \text{ for } c \in C,\, d \in D \text{ and } \sigma, \rho \geq 0,\, \sigma + \rho \leq 1\}$$

At first, we will show that L is a left-absorbing set. Notice that by our condition there exist $\rho, \sigma \geq 0$ and $c \in C$, $d \in D$ such that $\rho d \leq 0_P + \sigma c$. By letting $\lambda := \rho + \sigma$, we observe $\frac{\rho}{\lambda} d \leq 0_P + \frac{\sigma}{\lambda} c$ and $\frac{\rho}{\lambda} + \frac{\sigma}{\lambda} \leq 1$. Hence, $0_P \in L$.
In order to prove convexity of L, let $l_1, l_2 \in L$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. By the definition of L there exist $\rho_1, \rho_2, \sigma_1, \sigma_2 \geq 0$, $c_1, c_2 \in C$ and $d_1, d_2 \in D$ satisfying

$$\begin{aligned} \rho_1 d_1 &\leq l_1 + \sigma_1 c_1 \\ \rho_1 d_1 &\leq l_1 + \sigma_1 c_1 \end{aligned}$$

and

$$\begin{aligned} \rho_1 + \sigma_1 &\leq 1 \\ \rho_2 + \sigma_2 &\leq 1 \end{aligned}$$

Hence,

$$\alpha\rho_1 d_1 + \beta\rho_2 + d_2 \leq \alpha l_1 + \beta l_2 + \alpha\sigma_1 c_1 + \beta\sigma_2 c_2 \tag{16}$$

holds. Now set $\lambda_d := \alpha\rho_1 + \beta\rho_2$ and $\lambda_c := \alpha\sigma_1 + \beta\sigma_2$. By the convexity of C, resp. D we infer $d := \frac{\alpha\rho_1}{\lambda_d} d_1 + \frac{\beta\rho_2}{\lambda_d} d_2 \in D$ and $c := \frac{\alpha\sigma_1}{\lambda_c} c_1 + \frac{\beta\sigma_2}{\lambda_c} c_2 \in C$. Therefore, (16) yields $\lambda_d d \leq \alpha l_1 + \beta l_2 + \lambda_c c$ and since

$$\lambda_d + \lambda_c = \alpha(\rho_1 + \sigma_1) + \beta(\rho_2 + \sigma_2) \leq \alpha + \beta = 1$$

we see that $\alpha l_1 + \beta l_2 \in L$.
In order to prove the left-absorbing property of L, let $a \in P$. Then there exist $\rho, \sigma \geq 0$, $c \in C$ and $d \in D$ such that $\rho d \leq a + \sigma c$. If $\rho = \sigma = 0$, we infer $0_P \leq a$, hence, the proof

is evident. Otherwise, dividing by $\lambda := \rho + \sigma$ shows $\frac{1}{\lambda} a \in L$. Since $\lambda(\frac{1}{\lambda} a) \leq a$ holds, we conclude that L indeed is a left-absorbing set.
Now $U := \{\lambda u \mid \lambda > 0\}$ where $u := \{(a,b) \in P \times P \mid a + l \leq b \text{ for some } l \in L\}$ defines a convex quasiuniform structure as elaborated before. Let $(P \oplus \mathbb{R}^+, \{0_P\} \oplus \mathbb{R}^+ \setminus \{0\})$ be the full locally convex cone generated by U, endowed with an order $\preccurlyeq$ defined by

$$(a \oplus \alpha) \preccurlyeq (b \oplus \beta) \text{ iff } \alpha \leq \beta \text{ and } (a,b) \in \lambda u, \qquad \text{for } \lambda > \beta - \alpha$$

for $a, b \in P$ and $\alpha, \beta \geq 0$. For a detailed proof of the required properties see Example 3.7. In the following we will consider the subcone $Q := P \oplus \{0\}$ of $\bar{P} := P \oplus \mathbb{R}^+$.
In order to prove part (i) we want to apply Theorem 5.7 to the locally convex cone $(Q, \{0_P\} \oplus \mathbb{R}^+ \setminus \{0\})$, with the neighborhood $v := 0_p \oplus 1$ and $\alpha := 1$. Furthermore, we shall identify the required convex subsets $\bar{C}$ and $\bar{D}$ with $C \oplus \{0\}$ and $D \oplus \{0\}$.
Assume that, contrary to condition (15), there exist $c \in C$, $d \in D$ and $\sigma, \rho \geq 0$ such that

$$\rho(d \oplus 0) \preccurlyeq \sigma(c \oplus 0) + v \tag{17}$$

and

$$\rho > \sigma + 1 \tag{18}$$

holds. This implies $(\rho d, \sigma c) \in \lambda u$ for every $\lambda > 1$. Let $\lambda > 1$ arbitrary. Then there exist $l \in L$, $c' \in C$, $d' \in D$ and $\sigma', \rho' \geq 0$, $\rho' + \sigma' \leq 1$ such that

$$\rho d + \lambda l \leq \sigma c$$
$$\rho' d' \leq l + \sigma' c'$$

Combining those inequalities yields

$$\lambda \rho' d' + \rho d \leq \lambda \sigma' c' + \sigma c \tag{19}$$

Since C and D are convex, we observe

$$d'' := \frac{\lambda \rho' d' + \rho d}{\lambda \rho' + \rho} \in D \quad \text{and} \quad c'' := \frac{\lambda \sigma' c' + \sigma c}{\lambda \sigma' + \sigma} \in C$$

Furthermore, plugging c'' and d'' into (19) yields

$$d'' \leq \sigma'' c'' \tag{20}$$

where $\sigma'' := \frac{\sigma + \lambda \sigma'}{\rho + \lambda \rho'}$. By our assumption, we have

$$\sigma + \sigma' \leq \sigma + 1 < \rho \leq \rho + \rho'$$

Hence, if we choose $\lambda > 1$ small enough, $\sigma'' < 1$ holds. Since C is convex and contains 0_P, we infer $\sigma'' c'' \in C$. But this contradicts our assumption that C is decreasing and disjoint from D.
Now Theorem 5.7 yields the existence of a linear functional $\bar{\mu} \in v^\circ$ on Q such that

$$\bar{\mu}(c) \leq 1 \leq \bar{\mu}(d) \qquad \text{for } c \in \bar{C} \text{ and } d \in \bar{D}$$

Finally, $\mu : P \to \overline{\mathbb{R}}$ defined by

$$\mu(a) := \bar{\mu}(a \oplus 0) \qquad \text{for } a \in P$$

has the desired properties.
The proof of part (ii) is similar, if we let $\alpha = -1$ in Theorem 5.7. ■

Definition 5.10. Let (P, V) be a locally convex cone. An element $a \in P$ is called upper bounded, if for every $v \in V$ there exists $\alpha > 0$ such that $a \leq \alpha v$.

Theorem 5.11. *Let (P, V) be a locally convex cone and $B \subseteq P$ a non-empty convex subset of P such that $0_P \in B$.*

(i) If B is closed with respect to the lower topology on P, then for every $a \in B^{\mathrm{c}}$ there exists a linear functional $\mu \in P^*$ such that

$$\mu(b) \leq 1 \leq \mu(a) \qquad \textit{for } b \in P \in B \tag{21}$$

and indeed $1 < \mu(a)$ if a is upper bounded.

(ii) If B is closed with respect to the upper topology on P, then for every $a \in B^{\mathrm{C}}$ there exists a linear functional $\mu \in P^*$ such that

$$\mu(a) < -1 \leq \mu(a) \qquad \textit{for } b \in P \in B \tag{22}$$

Proof. In order to prove part (i), let $a \in B^{\mathrm{C}}$. Hence, there exists $u \in V$ such that $(a)u \cap C = \emptyset$. We shall apply Theorem 5.7 with the neighborhood $v := \frac{1}{2}u \in V$, the convex sets B and $(a)v$, and $\alpha = 1$.
Assume that, contrary to condition (15), there are $b \in B$, $c \in (a)v$ and $\sigma, \rho \geq 0$ such that

$$\begin{aligned} \rho c &\leq \sigma b + v & (23) \\ \rho &> \sigma + 1 & (24) \end{aligned}$$

Since $\frac{\sigma}{\rho} < 1$ and $0_P \in B$, we infer $b' := \frac{\sigma}{\rho} b \in B$. Furthermore, as $\frac{1}{\rho} < 1$, inequality (23) yields

$$c \leq b' + \frac{1}{\rho} v \leq b' + v$$

Now $c \in (a)v$ transfers into $a \leq c + v$. Therefore, we observe $a \leq b' + 2v = b' + u$, showing $b' \in (a)u$. But this contradicts our assumption $(a)u \cap B = \emptyset$. Hence, condition (15) holds and Theorem 5.7 guarantees the existence of a linear functional $\mu \in v^\circ$ satisfying

$$\mu(b) \leq 1 \leq \mu(c) \qquad \text{for } b \in B \text{ and } c \in (a)v \tag{25}$$

which proves the first statement of (i). Now assume that a is even upper bounded. Then there is $\alpha > 0$ such that $\alpha a \leq v$, hence, $(1 + \alpha)a \leq a + v$. Now $a' := \frac{1}{1+\alpha} a$ satisfies $a \leq a' + \frac{1}{1+\alpha} v \leq a' + v$, that is $a' \in (a)v$. Therefore, (25) yields

$$1 \leq \mu(a') = \frac{1}{1 + \alpha} \mu(a)$$

hence, $1 < 1 + \alpha \leq \mu(a)$ holds as claimed.
For part (ii), an analogous argument shows that there is a neighborhood $v \in V$ and a linear functional $\mu \in v^\circ$ such that

$$\mu(c) \leq -1 \leq \mu(b) \qquad \text{for } b \in B \text{ and } c \in v(a) \tag{26}$$

By (V4), there is some $\lambda \geq 0$ such that $0 \leq a + \lambda v$. Since $0 \leq v$, we can choose $\lambda > 1$. Therefore, we infer

$$0 \leq \frac{1}{\lambda} a + v$$

and hence

$$a' := \frac{\lambda - 1}{\lambda} a \leq a + v$$

Hence, $a' \in v(a)$. Now (26) yields

$$\mu(a) \leq -\frac{\lambda}{\lambda - 1} < -1$$

finishing the proof. ■

6 The Sup-Inf Theorem

Theorem 6.1. *Let (P, V) be a locally convex cone, $p : P \to \overline{\mathbb{R}}$ a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional. Furthermore, suppose that there exists at least one linear functional $\mu \in P^*$ satisfying $q \leq \mu \leq p$. Then for all $a \in P$*

$$\sup_{\substack{\mu \in P^* \\ q \leq \mu \leq p}} \mu(a) = \sup_{v \in V} \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b + v\} \tag{27}$$

and for all $a \in P$ such that $\mu(a)$ is finite for at least one $\mu \in P^*$ satisfying $q \leq \mu \leq p$

$$\inf_{\substack{\mu \in P^* \\ q \leq \mu \leq p}} \mu(a) = \inf_{v \in V} \sup\{q(c) - p(b) \mid b, c \in P,\ p(b) \in \mathbb{R},\ c \leq a + b + v\} \tag{28}$$

Proof. Let $a \in P$. In order to abbreviate our notation, we shall use α and $\bar{\alpha}$ to denote the left-hand side and the right-hand side of equation (27). In the same way we will use β and $\bar{\beta}$ for equation (28).
From our assumptions we infer that $\alpha > -\infty$ and $\beta < \infty$. Now let $\mu \in p^*$ such that $q \leq \mu \leq p$. By the definition of P^* there exists $v \in V$ such that $\mu \in v^\circ$. For $\epsilon > 0$ let $w := \epsilon v \in V$. Then $a + c \leq b + w$ and $q(c) \in \mathbb{R}$ imply

$$\mu(a) + q(c) \leq \mu(a) + \mu(c) \leq \mu(b) + \epsilon \leq p(b) + \epsilon$$

for $b, c \in P$. Hence, $\mu(a) \leq p(b) - q(c) + \epsilon$. This shows that

$$\mu(a) \leq \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b + w\} + \epsilon$$

As $\epsilon > 0$ can be chosen arbitrarily small, we get $\mu(a) \leq \bar{\alpha}$. Likewise, for some $b, c \in P$ such that $c \leq a + b + w$ and $p(b) \in \mathbb{R}$ we conclude that

$$q(c) \leq \mu(c) \leq \mu(a) + \mu(b) + \epsilon \leq \mu(a) + p(b) + \epsilon$$

holds. Therefore, we infer $\mu(a) \geq q(c) - p(b) - \epsilon$, and

$$\mu(a) \geq \sup\{q(c) - p(b) \mid b, c \in P,\ q(c) \in \mathbb{R},\ c \leq a + b + w\} - \epsilon$$

Again, this shows $\mu(a) \geq \bar{\beta}$. Combining the previous results yields

$$\bar{\beta} \leq \mu(a) \leq \bar{\alpha} \tag{29}$$

$$\bar{\beta} \leq \beta \leq \alpha \leq \bar{\alpha} \tag{30}$$

We shall proceed to show that $\bar{\alpha} \leq \alpha$ holds. For $\alpha = \infty$ this is obvious. Thus we may assume that $\alpha \in \mathbb{R}$ and let $v \in V$. For $0 < \epsilon \leq \frac{1}{4}$ there is $\mu \in P^*$ such that $q \leq \mu \leq p$ and $\mu(a) \geq \alpha - \epsilon$. Hence, we find a neighborhood $u \in V$ such that $\mu \in u^\circ$. By (V2) there exists $o \in V$ satisfying $o \leq v$ and $o \leq u$. Now we shall apply Theorem 5.2 to the neighborhood $w := \frac{1}{2}o$, the convex sets $C = D = \{a\}$ and the functionals $f \equiv \infty$ and $g \equiv \alpha + \epsilon$.
Since there is no linear functional $\eta \in w^\circ$ such that $q \leq \eta \leq p$ and $\eta(a) \geq \alpha + \epsilon$, we conclude that condition (10) must fail. By Remark 5.3 there are $b, c \in P$ and $\rho \geq 0$ such that

$$c + \rho a \leq b + w \tag{31}$$

and

$$q(c) + \rho(\alpha + \epsilon) > p(b) + 1 \tag{32}$$

This yields $p(b) < \infty$. Since $\mu \in u^\circ$ and $c + \rho a \leq b + w = b + \frac{1}{2}o \leq b + \frac{1}{2}u$ we compute

$$q(c) + \rho(\alpha - \epsilon) \leq \mu(c) + \rho\mu(a) \leq \mu(b) + \frac{1}{2} \leq p(b) + \frac{1}{2} \tag{33}$$

Combining inequalities (32) and (33) implies $q(c) \in \mathbb{R}$ and $p(b) + 1 < p(b) + \frac{1}{2} + 2\rho\epsilon$. Hence,

$$1 \leq \frac{1}{4\epsilon} < \rho$$

As $\epsilon \leq \frac{1}{4}$, we infer $\rho > 1$. Multiplying inequality (31) by $\frac{1}{\rho} < 1$ yields

$$c' + a \leq b' + \frac{1}{\rho}w \leq b' + w \tag{34}$$

with $c' = \frac{1}{\rho}c$ and $b' = \frac{1}{\rho}b$. Furthermore, using (32), we observe

$$p(b') \leq p(b') + \frac{1}{\rho} = \frac{1}{\rho}(p(b) + 1) < q(c') + (\alpha + \epsilon) \tag{35}$$

hence, $p(b') - q(c') \leq \alpha + \epsilon$. Thus, we obtain

$$\begin{aligned} \alpha + \epsilon &\geq \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b + w\} \\ &\geq \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b + v\} \end{aligned}$$

showing $\alpha \geq \bar{\alpha}$. Together with (30), this proves the first part of our theorem. For the second part, the only thing left to show is $\beta \leq \bar{\beta}$. If $\beta = -\infty$, the proof is obvious. Therefore, we assume $\beta \in \mathbb{R}$ and let $v \in V$. For $0 < \epsilon \leq \frac{1}{4}$ there exists $\mu \in P^*$ such that $\mu(a) \leq \beta + \epsilon$. Thus, there are $u, o \in V$ satisfying $\mu \in u^\circ$, $o \leq u$ and $o \leq v$.
Now we shall apply Theorem 5.2 to the neighborhood $w := \frac{1}{2}o \in V$, the convex sets

$C = D = \{a\}$ and the functionals $g \equiv -\infty$, $f \equiv \beta - \epsilon$. Again, by Remark 5.3 there exist $b, c \in P$ and $\sigma \geq 0$ such that

$$c \leq b + \sigma a + w \tag{36}$$

and

$$q(c) > p(b) + \sigma(\beta - \epsilon) + 1 \tag{37}$$

Furthermore, we compute

$$q(c) \leq \mu(c) \leq \mu(b) + \sigma\mu(a) + \frac{1}{2} \leq p(b) + \sigma(\beta + \epsilon) + \frac{1}{2} \tag{38}$$

This shows $q(c), p(b) \in \mathbb{R}$ and combining the last two inequalities yields

$$\sigma > \frac{1}{4\epsilon} \geq 1 \tag{39}$$

Now multiplying inequality (36) with $\frac{1}{\sigma} \leq 1$ results in

$$c' \leq b' + a + w \tag{40}$$

with $c' = \frac{1}{\sigma}c$ and $b' = \frac{1}{\sigma}b$. Finally, (37) yields

$$q(c') > p(b') + (\beta - \epsilon) + \frac{1}{\sigma} \geq p(b') + (\beta - \epsilon)$$

hence, $q(c') - p(b') \geq \beta - \epsilon$. Thus, we infer

$$\begin{aligned} \beta - \epsilon &\leq \sup\{q(c) - p(b) \mid b, c \in P,\ p(b) \in \mathbb{R},\ c \leq a + b + w\} \\ &\leq \sup\{q(c) - p(b) \mid b, c \in P,\ p(b) \in \mathbb{R},\ c \leq a + b + v\} \end{aligned}$$

As $v \in V$ was arbitrary, we conclude that $\beta \leq \bar{\beta}$ holds. ■

Definition 6.2. Let (P, V) be a locally convex cone, $p : P \to \overline{\mathbb{R}}$ a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional. Then $w \in V$ is said to satisfy condition (wp) iff

$$\text{for all } a, b \in P,\ a \leq b + w \text{ there is } w' \in P \text{ such that } p(w') \leq 1 \text{ and } a \leq b + w' \tag{41}$$

Likewise, we say $w \in V$ satisfies condition (wq) iff

$$\text{for all } a, b \in P,\ a \leq b + w \text{ there is } w' \in P \text{ such that } q(w') \geq -1 \text{ and } a + w' \leq b \tag{42}$$

Remark 6.3. Let (P, V) be a locally convex cone, and P^* its dual cone. We shall endow P^* with the topology $\omega(P^*, P)$ of pointwise convergence. Furthermore, let $p : P \to \overline{\mathbb{R}}$ be a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional. Now assume that a neighborhood $w \in V$ satisfies (wp). Then, for a linear functional $\mu \in P^*$, $\mu \leq p$ we infer

$$a \leq b + w \implies a \leq b + w' \implies \mu(a) \leq \mu(b) + \mu(w') \leq \mu(b) + p(w') \leq \mu(b) + 1$$

for $a, b \in P$ and some $w' \in V$, hence, $\mu \in w^\circ$. Thus, $M := \{\mu \in P^* \mid q \leq \mu \leq p\} \subseteq w^\circ$ holds. Since M is a closed subset of the compact set w° ([1, Proposition II.2.4]), we conclude that M is $\omega(P^*, P)$-compact as well. Thus, the infimum and the supremum on the left hand sides of equation (27) and (28) turn into a minimum and maximum. Furthermore, we observe that

$$\sup_{v \in V} \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b + v\}$$
$$\leq \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b\}$$

holds. On the other hand, for $a, b, c \in P$, $\epsilon > 0$ and $v \in V$, $v \leq \epsilon w$ satisfying $a + c \leq b + v$ there is $w' \in P$ such that

$$a + c \leq b + v \leq b + \epsilon w \leq b + \epsilon w'$$

and

$$p(b + \epsilon w') - q(c) \leq (p(b) - q(c)) + \epsilon$$

But this yields the reverse inequality

$$\sup_{v \in V} \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b + v\}$$
$$\geq \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b\}$$

An analogous computation shows that equality (28) can be simplified in a similar way. Moreover, the above results remain unchanged if we replace condition (wp) with (wq). This leads to the following corollary:

Corollary 6.4. *Let (P, V) be a locally convex cone, $p : P \to \overline{\mathbb{R}}$ a sublinear functional and $q : P \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional such that*

$$a \leq b \implies q(a) \leq p(b) \tag{43}$$

holds for $a, b \in P$. If either (wp) or (wq) holds for a certain neighborhood $w \in V$, then

$$\max_{\substack{\mu \in P^* \\ q \leq \mu \leq p}} \mu(a) = \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b\} \tag{44}$$

$$\min_{\substack{\mu \in P^* \\ q \leq \mu \leq p}} \mu(a) = \sup\{q(c) - p(b) \mid b, c \in P,\ p(b) \in \mathbb{R},\ c \leq a + b\} \tag{45}$$

holds for all $a \in P$.

Proof. We shall apply Theorem 6. 1 to the functionals p and q on P. Then, Remark 6.3 yields the desired equations. In order to apply the theorem, we have to verify some additional assumptions. We will only carry out the proof for (wp), as the proof for (wq) is similar. Therefore, assume there is a neighborhood $w \in V$ satisfying (wp). Now $a \leq b + w$ guarantees $q(a) \leq p(b) + 1$. Hence, Theorem 4.4 yields a linear functional $\mu \in w^\circ$ such that $q \leq \mu \leq p$. Thus, there is at least one linear functional $\mu \in P^*$ satisfying $q \leq \mu \leq p$. For the second equation, let $a \in P$. Assume $\mu(a) = \infty$ for every $\mu \in P^*$, $q \leq \mu \leq p$.

Then, for every $n \in \mathbb{N}$ there is no $\mu \in w^{\circ}$ satisfying $q \leq \mu \leq p$ and $\mu(a) \leq n$. Hence, applying Theorem 5.2 to $C = D = \{a\}$ and the functionals $g \equiv -\infty$ and $f \equiv n$, yields $a_n, b_n \in P$ and $\sigma_n \geq 0$ such that

$$a_n \leq b_n + \sigma_n a + w \tag{46}$$

and

$$q(a_n) > p(b_n) + \sigma_n n + 1 \tag{47}$$

Furthermore, if (wp) holds, there are $w'_n \in P$ satisfying $p(w'_n) \leq 1$ and $a_n \leq b_n + \sigma_n a + w'_n$. We observe $\sigma_n > 0$, since otherwise our assumption would yield $q(a_n) \leq p(b_n) + 1$. Thus, we can divide by σ_n and infer

$$q(\frac{a_n}{\sigma_n}) - p((\frac{b_n + w'_n}{\sigma_n}) \geq \frac{1}{\sigma_n}(q(a_n) - p(b_n) - 1) > n \tag{48}$$

This shows

$$\sup\{q(c) - p(b) \mid b, c \in P,\ p(b) \in \mathbb{R},\ c \leq a + b\} = \infty = \min_{\substack{\mu \in P^* \\ q \leq \mu \leq p}} \mu(a)$$

Hence, equation (45) holds also in this case. ■

Definition 6.5. Let (P, V) be a locally convex cone, $C \subseteq P$ a subcone of P and $\mu \in P^*$ a linear functional. An element $a \in P$ is said to be C-subharmonic in μ if for every $\eta \in P^*$

$$\eta(a) \geq \mu(a) \quad \text{holds whenever} \quad \eta(c) \geq \mu(c) \text{ for all } c \in C \tag{49}$$

Likewise, $a \in P$ is called C-superharmonic in μ if for every $\eta \in P^*$

$$\eta(a) \leq \mu(a) \quad \text{holds whenever} \quad \eta(c) \leq \mu(c) \text{ for all } c \in C \tag{50}$$

Corollary 6.6. *Let (P, V) be a locally convex cone, $\mu \in P^*$ and $C \subseteq P$ a subcone of P. An element $a \in P$ is C-superharmonic in μ if and only if*

$$\mu(a) = \sup_{v \in V} \inf\{\mu(c) \mid c \in C,\ a \leq c + v\} \tag{51}$$

Similarly, an element $a \in P$ such that $\mu(a) < \infty$ is C-subharmonic in μ if and only if

$$\mu(a) = \inf_{v \in V} \sup\{\mu(c) \mid c \in C,\ c \leq a + v\} \tag{52}$$

Proof. Let $\mu \in P^*$. Hence, there is $w \in V$ such that $\mu \in w^{\circ}$. Now

$$p(a) = \begin{cases} \mu(a) & \text{if } a \in C \\ \infty & \text{else} \end{cases}$$

$$q(a) = \begin{cases} 0 & \text{if } a = 0_P \\ -\infty & \text{else} \end{cases}$$

defines a sublinear resp. superlinear functional on P. Since $q \leq \mu \leq p$ obviously holds true, we can apply Theorem 6.1 to p and q. For a linear functional $\eta \in P^*$, $q \leq \eta \leq p$ holds if and only if $\eta(c) \leq \mu(c)$ for all $c \in C$. Therefore, by equation (27) we observe that an element $a \in P$ is C-superharmonic in μ if and only if

$$\mu(a) \geq \sup_{v \in V} \inf\{p(b) - q(c) \mid b, c \in P,\ q(c) \in \mathbb{R},\ a + c \leq b + v\} = \sup_{v \in V} \inf\{\mu(c) \mid c \in C,\ \ a \leq c + v\}$$

As the reverse inequality is evident, we conclude that (51) holds. The proof of part two is similar, if we apply Theorem 6.1 to the functionals p and q on P defined by

$$p(a) = \begin{cases} 0 & \text{if } a = 0_P \\ \infty & \text{else} \end{cases}$$

$$q(a) = \begin{cases} \mu(a) & \text{if } a \in C \\ -\infty & \text{else} \end{cases}$$

■

References

[1] K. Keimel and W. Roth: *Ordered cones and approximation*, Springer, Berlin, 1992

[2] L. Nachbin: *Topology and order*, Van Nostrand, Princeton, 1965

[3] W. Roth: *Hahn-Banach type theorems for locally convex cones*, Australian Mathematical Society, 2000