VIENNA UNIVERSITY OF TECHNOLOGY

On Convex and Positively Convex Modules

by

Stefan Koller

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Contents

Chapter 1

Convex Modules

The notion of convex modules has first been introduced by Neumann and Morgenstern in [9]. Convex modules give a generalization of convex subsets of linear spaces. In the first section basic definitions and properties of convex modules as given in [1], [3], [4], [5] and [6] are presented. A criterion is given for a convex module to be isomorphic to a convex subset of a linear space, namely being preseparated. Then, an affine function is constructed, that maps a convex module into a preseparated convex module and that is initial under such functions.

In the second part a semimetric on convex modules, introduced by Pumplün in [11], is discussed. Criteria on the semimetric being a metric are given.

In the final section superconvex modules are introduced and the construction of a completion functor, as described in [11], are displayed.

1.1 Convex Modules

Definition 1.1.1. A set X and a function $c : [0,1] \times X \times X \mapsto X$ are called a convex module and c is called convex combination if they satisfy the following conditions:

1. $c(\lambda, x, y) = c(1-\lambda, y, x)$ for all $x, y \in X$ and $\lambda \in [0,1]$.
2. $c(\lambda, x, x) = x$ for all $x \in X$ and $\lambda \in [0,1]$.
3. $c(0, x, y) = x$ for all $x, y \in X$.
4. $c(\lambda, x, c(\mu, y, z)) = (\lambda\mu, c(\frac{\lambda(1-\mu)}{1-\lambda\mu}, x, y), z)$ for all $x, y, z \in X$ and $\lambda, \mu \in (0,1)$.

Remark 1.1.2. For a convex module (X, c), c it is also said to be a convex structure on X. Some authors call convex modules ”semiconvex sets”.

The above notation is rather unintuitive. The following notation will make the implications of these conditions much clearer.

Definition 1.1.3. Let (X, c) be a convex module, let $\alpha_j \in (0, 1]$ for all $j \in \{1, .., n\}$ such that $\sum_{j=1}^{n} \alpha_j = 1$ and let $x_j \in X$ for all $j \in \{1, .., n\}$. Define the formal sum $\sum_{j=1}^{n} \alpha_j x_j$ inductively as $\sum_{j=1}^{m+1} \frac{\alpha_j}{\sum_{k=1}^{m+1} \alpha_k} x_j := c(\frac{\alpha_{m+1}}{\sum_{k=1}^{m+1} \alpha_k}, \sum_{j=1}^{m} \frac{\alpha_j}{\sum_{k=1}^{m} \alpha_k} x_j, x_{m+1})$ and $\sum_{j=1}^{1} x_j = x_1$.

Let (X, c) be a convex module, let $\alpha_j \in [0, 1]$ for all $j \in \mathbb{N}$ such that all but finitely many α_j are equal to zero and such that $\sum_{j=1}^{\infty} \alpha_j = 1$. Let $x_j \in X$ for all $j \in \{1, .., n\}$. Define $\sum_{j=1}^{\infty} \alpha_j x_j$ by omitting those indices j with $\alpha_j = 0$ in the above induction.

In this new notation the conditions in definition 1.1.1 read as follows:

1. $(1 - \lambda)x + \lambda y = \lambda y + (1 - \lambda)x$.
2. $(1 - \lambda)x + \lambda x = x$.
3. $1x + 0y = x$.
4. $(1 - \lambda)x + \lambda((1 - \mu)y + \mu z) = ((1 - \lambda)x + \lambda(1 - \mu)y) + \lambda\mu z$.

Thus, the notation as a formal sum is justified. For the remainder of this paper, this notation will be used.

Example 1.1.4. Any convex subset of a real vector space is a convex module by identifying the formal addition with the vector space addition.

For the remainder of this paper, whenever a subset of a real vector space is considered, it is assumed to be equipped with this convex structure. As the following example shows, there are convex modules which cannot be equivalent to a convex subsets of a real vector spaces.

Example 1.1.5. Let $X := \{x, y\}$, let $(1-\lambda)x+\lambda x = x$ for $\lambda \in [0, 1]$, let $(1-\lambda)y+\lambda y = y$ for $\lambda \in [0, 1]$, let $(1 - \lambda)x + \lambda y = x$ for $\lambda \in [0, 1)$, let $(1 - \lambda)x + \lambda y = x$ for $\lambda \in (0, 1]$, let $0x + 1y = y$ and let $1y + 0x = y$. It is easily seen, that X is a convex module. Since $x = \frac{1}{2}x + \frac{1}{2}y$ and $x \neq y$, it cannot be a subset of a real vector space.

The following definition characterizes those convex modules that can be considered as convex subsets of real vector spaces.

Definition 1.1.6. A convex module X is called preseparated if for all $x, y, z \in X$ and $\lambda \in [0, 1)$:

$$(1-\lambda)x + \lambda z = (1-\lambda)y + \lambda z \text{ implies } x = y.$$

Remark 1.1.7. preseparated convex modules are called cancellative by some authors.

Lemma 1.1.8. *Let X be a convex module, let $x, y, z \in X$ and let $\lambda \in [0, 1)$, such that $(1-\lambda)x + \lambda z = (1-\lambda)y + \lambda z$. Then $(1-\alpha)x + \alpha z = (1-\alpha)y + \alpha z$ for all $\alpha \in (0, 1]$.*

Proof. For $\alpha = 1$ and $\alpha = \lambda$ the equation is trivial. Let x, y, z, λ be as above. Show that the equation holds for all $\alpha \in (\lambda, 1)$:

$$(1-\alpha)x + \alpha z = (1-\alpha)x + \frac{\lambda(1-\alpha)}{1-\lambda}z + (1 - \frac{1-\alpha}{1-\lambda})z =$$

$$= \frac{1-\alpha}{1-\lambda}((1-\lambda)x + \lambda z) + (1 - \frac{1-\alpha}{1-\lambda})z = \frac{1-\alpha}{1-\lambda}((1-\lambda)y + \lambda z) + (1 - \frac{1-\alpha}{1-\lambda})z =$$

$$= (1-\alpha)y + \frac{\lambda(1-\alpha)}{1-\lambda}z + (1 - \frac{1-\alpha}{1-\lambda})z = (1-\alpha)y + \alpha z$$

Next let $\lambda^{(1)} := \frac{\lambda}{2-\lambda}$, then:

$$(1-\lambda^{(1)})x + \lambda^{(1)}z = \frac{1-\lambda}{2-\lambda}x + \frac{1}{2-\lambda}((1-\lambda)x + \lambda z) = \frac{1-\lambda}{2-\lambda}x + \frac{1}{2-\lambda}((1-\lambda)y + \lambda z) =$$

$$= \frac{1-\lambda}{2-\lambda}x + \frac{1-\lambda}{2-\lambda}y + \frac{\lambda}{2-\lambda}z = \frac{1-\lambda}{2-\lambda}y + \frac{1}{2-\lambda}((1-\lambda)x + \lambda z) =$$

$$= \frac{1-\lambda}{2-\lambda}y + \frac{1}{2-\lambda}((1-\lambda)y + \lambda z) = (1-\lambda^{(1)})y + \lambda^{(1)}z$$

Since the equation holds for $\lambda^{(1)}$, the equation holds for all $\alpha \in [\lambda^{(1)}, 1]$. Now we can iterate this process with $\lambda^{(n+1)} := \frac{\lambda^{(n)}}{2-\lambda^{(n)}}$. Since $\lambda \leq 1$ and $\lambda^{(n)} = \frac{\lambda^{(n-1)}}{2-\lambda^{(n-1)}} \leq \frac{\lambda^{(n-1)}}{2-\lambda} \leq \frac{\lambda}{(2-\lambda)^n}$ the equation holds for all $\alpha \in (0, 1]$. □

Definition 1.1.9. Let X and Y be convex modules. A mapping $f : X \to Y$ is called affine, if for all $x_j \in X$ and $\sum_{j=1}^{\infty} \alpha_j = 1$, such that $\alpha_j \geq 0$ for all $j \in \mathbb{N}$ and $\alpha_j = 0$ for all but finitely many $j \in \mathbb{N}$:

$$f(\sum_{j=1}^{\infty} \alpha_j x_j) = \sum_{j=1}^{\infty} \alpha_j f(x_j)$$

Clearly, the convex modules together with affine mappings as morphisms form a category. Let **Conv** denote this category and let $\text{Conv}(X, Y)$ denote the set of all affine mappings from X to Y.

Lemma 1.1.10. *Any bijective, affine mapping is an isomorphism.*

Proof. $f(\sum_{j=1}^{\infty}\alpha_j f^{-1}(x_j)) = \sum_{j=1}^{\infty}\alpha_j x_j$ and thus applying f^{-1} to both sides results in $f^{-1}(\sum_{j=1}^{\infty}\alpha_j x_j) = \sum_{j=1}^{\infty}\alpha_j f^{-1}(x_j)$. □

Lemma 1.1.11. *Let X be a convex module and let Y be a real vector space. Let $f : X \to Y$ be affine, then $f(X)$ is convex.*

Proof. Let $x, y \in X$ and $\lambda \in [0,1]$, then $(1-\lambda)f(x) + \lambda f(y) = f((1-\lambda)x + \lambda y) \in f(X)$. □

In the following let $\mathbb{R}_+$ denote the positive reals including 0 and let $\mathbb{R}_{++}$ denote the positive reals excluding 0.

Theorem 1.1.12. *A convex module X is isomorphic to a convex subset of a real vector space, if and only if X is preseparated.*

Proof. Every convex subset of a real vector space is preseparated, so clearly any convex module isomorphic to a convex subset of a real vector space is preseparated.

For the other direction, let X be a preseparated convex module. Let $C := (\mathbb{R}_{++}) \times X$. For $(\mu, x) \in C$ and $\lambda \in \mathbb{R}_{++}$ let $\lambda(\mu, x) := (\lambda\mu, x)$. For $(\mu, x) \in C$ and $(\nu, y) \in C$ let $(\mu, x) + (\nu, y) := (\mu + \nu, \frac{\mu}{\mu+\nu}x + \frac{\nu}{\mu+\nu}y)$. Define on $C \times C$ an equivalence relation: $(w, x) \sim (y, z)$ iff $w + z = x + y$. Let V be the set of all equivalence classes and let $[(x, y)]$ denote the equivalence class of (x, y) for $x, y \in C$. To define a vector space structure on V let $[(w, x)] + [(y, z)] := [(w + y, x + z)]$. If $(s, t) \sim (u, v)$ and $(w, x) \sim (y, z)$, then $(s, t) + (w, x) \sim (u, v) + (y, z)$, thus addition on V is well-defined and with $-[(x, y)] := [(y, x)]$ it is an abelian group. For $\lambda \in \mathbb{R}_{++}$ define $\lambda[(x, y)] := [(\lambda x, \lambda y)]$, for $\lambda \in -\mathbb{R}_{++}$ define $\lambda[(x, y)] := [(-\lambda y, -\lambda x)] = -[(-\lambda x, -\lambda y)]$ and define $0[(x, y)] = [(x, x)]$. Clearly addition and scalar multiplication are distributive and hence V is a real vector space.

Let $(\lambda, z) \in C$ be fixed. Let $f : X \to V$ be defined as $f(x) := [((1, x) + (\lambda, z), (\lambda, z))]$.

$$f(\sum_{j=1}^{\infty}\alpha_j x_j) = [((1, \sum_{j=1}^{\infty}\alpha_j x_j) + (\lambda, z), (\lambda, z))] = [(\sum_{j=1}^{\infty}((\alpha_j, x_j) + (\alpha_j\lambda, z)), \sum_{j=1}^{\infty}(\alpha_j\lambda, z))] =$$

$$= \sum_{j=1}^{\infty}[((\alpha_j, x_j) + (\alpha_j\lambda, z), (\alpha_j\lambda, z))] = \sum_{j=1}^{\infty}\alpha_j[((1, x_j) + (\lambda, z), (\lambda, z))] = \sum_{j=1}^{\infty}\alpha_j f(x_j)$$

Thus, f is affine and its image is convex. For injectivity, let $x \neq y$ with $x, y \in X$ and assume $f(x) = f(y)$. Then, $((1, x) + (\lambda, z), (\lambda, z)) \sim ((1, y) + (\lambda, z), (\lambda, z))$, which implies

$$(1 + 2\lambda, \frac{1}{1+2\lambda}x + \frac{2\lambda}{1+2\lambda}z) = (1, x) + (\lambda, z) + (\lambda, z) =$$

$$= (1, y) + (\lambda, z) + (\lambda, z) = (1 + 2\lambda, \frac{1}{1+2\lambda} y + \frac{2\lambda}{1+2\lambda} z)$$

But $\frac{1}{1+2\lambda} x + \frac{2\lambda}{1+2\lambda} z = \frac{1}{1+2\lambda} y + \frac{2\lambda}{1+2\lambda} z$ contradicts X being preseparated, hence f is injective. Thus, f is an affine bijection. The inverse of any affine bijection is also affine, hence f is an isomorphism. □

Definition 1.1.13. A subset C of a real vector space X is called a cone if the following conditions are satisfied:

1. $\alpha x \in C$, for all $x \in C$, $\alpha \in \mathbb{R}_+$.
2. $x + y \in C$, for all $x, y \in C$.

Remark 1.1.14. Clearly, any cone is convex.

Definition 1.1.15. Let X be a real vector space. A cone C is called proper if $C \cap (-C) = \{0\}$. A cone C is called generating if $C - C = X$.

Definition 1.1.16. A real vector space X is called an ordered vector space with order $\leq$, if $\leq$ is a partial order that satisfies:

1. $x \leq y$ implies $\alpha x \leq \alpha y$, for all $x, y \in X$, $\alpha \in \mathbb{R}_+$.
2. $x \leq y$ implies $x + z \leq y + z$, for all $x, y, z \in X$.

Remark 1.1.17. For any ordered vector space X, $C := \{x \in X : x \geq 0\}$ defines a proper cone. On the other hand, for any given proper cone C, $x \leq y :\leftrightarrow (y - x) \in C$ defines a partial order satisfying the conditions of an ordered vector space. Hence an ordered vector space can equally be defined by defining a proper cone.

Definition 1.1.18. A real vector space X is called base ordered vector space with base B, if $B \subseteq X$ is convex, if for all $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $b_1, b_2 \in B$: $\alpha_1 b_1 = \alpha_2 b_2$ implies $\alpha_1 = \alpha_2$, and if $\mathbb{R}_+ B - \mathbb{R}_+ B = X$.

Remark 1.1.19. In the definition above $C := \mathbb{R}_+ B$ is a cone and the conditions imply that C is proper and generating, i.e. $C \cap (-C) = \{0\}$ and $C - C = X$. Thus, any base ordered vector space is an ordered vector space.

Lemma 1.1.20. *For any base ordered vector space X with base B, the affine hull of B is proper, i.e. it does not include* 0.

Proof. Assume $x, y \in B$, $\lambda \in \mathbb{R}$ such that $x + \lambda(y - x) = 0$. If $\lambda \geq 1$, then $(\lambda - 1)x = \lambda y$ implies $\lambda = \lambda - 1$, thus a contradiction. If $\lambda \leq 0$, then $(1 - \lambda)x = -\lambda y$ implies $\lambda = \lambda - 1$ aswell. If $0 \leq \lambda \leq 1$, then $x + \lambda(y - x) = 0$, implies $0 \in B$. But then, $0 \cdot 0 = 1 \cdot 0$ implies $0 = 1$. Thus, the affine hull of B does not contain 0. □

Definition 1.1.21. Let **BOVec** denote the category of base ordered vector spaces with morphisms linear mappings which satisfy that the domain's base is mapped into the codomain's base. Let **Bs** : **BOVec** → **Conv** denote the functor which assigns to each base ordered vector space its base and to each linear function the restriction to its base.

Remark 1.1.22. Obviously this defines a functor, since the restriction of a linear mapping is still linear and hence affine. The composition of morphism is the usual composition of functions in both categories.

Remark 1.1.23. **Bs** can also be treated as a functor to the full and faithful subcategory of preseparated convex modules **PresepConv**.

Definition 1.1.24. Let X be a convex module. $\mathrm{Conv}(X,\mathbb{R})$ is a real vector space. Let $\mathrm{Conv}(X,\mathbb{R})^*$ denote its algebraic dual. Define $\tilde{\rho} : X \to \mathrm{Conv}(X,\mathbb{R})^*$ as $\tilde{\rho}(x)(f) = f(x)$. Let $R(X)$ denote the subspace of $\mathrm{Conv}(X,\mathbb{R})^*$ generated by $\tilde{\rho}(X)$. Let ρ be defined as the corestriction of $\tilde{\rho}$ to $R(X)$ and let $\hat{\rho}$ be defined as the corestriction of $\tilde{\rho}$ to its image.

Lemma 1.1.25. *ρ is affine and $R(X)$ is a base ordered vector space with base $\rho(X)$.*

Proof. $\rho(x)$ is defined by acting on affine functions f therefore,

$$\rho(\sum_{j=1}^{\infty}\alpha_j x_j)(f) = f(\sum_{j=1}^{\infty}\alpha_j x_j) = \sum_{j=1}^{\infty}\alpha_j f(x_j) = (\sum_{j=1}^{\infty}\alpha_j \rho(x_j))(f)$$

Thus ρ is affine and $\rho(X)$ is convex. The constant function $c_1 : x \mapsto 1$ is an element of $\mathrm{Conv}(X,\mathbb{R})$. Let $x,y \in X$ and $\alpha_1,\alpha_2 \in \mathbb{R}_+$ with $\alpha_1\rho(x) = \alpha_2\rho(y)$, then $\alpha_1 = \alpha_1\rho(x)(c_1) = \alpha_2\rho(y)(c_1) = \alpha_2$. $R(X)$ is generated by $\rho(X)$ and $\rho(X)$ is convex, thus $\mathbb{R}_+\rho(X) - \mathbb{R}_+\rho(X) = X$. □

Lemma 1.1.26. *X is preseparated if and only if ρ is injective.*

Proof. Assume that $x \neq y$ and $\alpha x + (1-\alpha)z = \alpha y + (1-\alpha)z$ with $\alpha \in (0,1]$, then

$$\alpha\rho(x) + (1-\alpha)\rho(z) = \rho(\alpha x + (1-\alpha)z) = \rho(\alpha y + (1-\alpha)) = \alpha\rho(x) + (1-\alpha)\rho(z)$$

Therefore $\rho(x) = \rho(y)$ and consequently ρ is not injective. On the other hand if X is preseparated, then X is isomorphic to a convex subset of some real vector space and thus for any $x \neq y$ with $x,y \in X$ there exists a function $f \in \mathrm{Conv}(X,\mathbb{R})$ such that $f(x) \neq f(y)$. Hence $\rho(x)(f) \neq \rho(y)(f)$. □

Corollary 1.1.27. *X is preseparated if and only if $\hat{\rho}$ is an isomorphism.*

Proof. This follows directly from lemmata 1.1.26 and 1.1.10. □

Lemma 1.1.28. *Let X_1, X_2 be base ordered vector spaces with bases B_1, B_2. Let $f : B_1 \to B_2$ be affine. Then there exists a unique linear extension $F : X_1 \to X_2$.*

Proof. Let $x, y \in B$ and let $\lambda, \mu \in \mathbb{R}_+$. Define $F(\lambda x - \mu y) := \lambda f(x) - \mu f(y)$. Then:

$$F(\lambda_1 x_1 - \mu_1 y_1 + \lambda_2 x_2 - \mu_2 y_2) =$$

$$= F((\lambda_1 + \lambda_2)(\frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2) - (\mu_1 + \mu_2)(\frac{\mu_1}{\mu_1 + \mu_2} y_1 + \frac{\mu_2}{\mu_1 + \mu_2} y_2)) =$$

$$= (\lambda_1 + \lambda_2) f(\frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2) - (\mu_1 + \mu_2) f(\frac{\mu_1}{\mu_1 + \mu_2} y_1 + \frac{\mu_2}{\mu_1 + \mu_2} y_2) =$$

$$= \lambda_1 f(x_1) + \lambda_2 f(x_2) - \mu_1 f(y_1) - \mu_2 f(y_2) = F(\lambda_1 x_1 - \mu_1 y_1) + F(\lambda_2 x_2 - \mu_2 y_2)$$

$$F(\nu(\lambda x - \mu y)) = \nu\lambda f(x) - \nu\mu f(y) = \nu F(\lambda x - \mu y)$$

Hence, F is additive and homogeneous. Thus, it suffices to show that F is independent of representation at 0, for F to be well-defined. Let $\lambda x - \mu y = 0$, then since B_1 is a base, $\lambda = \mu$ and hence $x = y$. Therefore, $F(\lambda x - \mu y) = \lambda f(x) - \lambda f(x) = 0$. Since $F(\nu(\lambda x - \mu y)) = \nu\lambda f(x) - \nu\mu f(y) = \nu F(\lambda x - \mu y)$, F is linear. Since any other linear extension has to satisfy the defining equation of F, the extension is unique. □

Theorem 1.1.29. *Let X be a convex module, let Y be a base ordered vector space with base B and let $f : X \to B$ be affine. Then there is a unique linear mapping $F : R(X) \to Y$, such that $F \circ \rho = f$.*

Proof. First, show that there is a unique affine mapping $\tilde{F} : \rho(X) \to Y$ satisfying $\tilde{F} \circ \rho = f$. For such a function to be well-defined $f(x) \neq f(y)$ has to imply $\rho(x) \neq \rho(y)$, for all $x, y \in X$. Let x and y be, such that $f(x) \neq f(y)$. Since Y is a real linear space there is a linear functional $g : Y \to \mathbb{R}$, such that $g(f(x)) \neq g(f(y))$. Since $g \circ f$ is an affine function to the reals, $\rho(x) \neq \rho(y)$. In order to show that $\tilde{F}$ is affine, let $x_j \in \rho(X)$ and let $y \in X$ such that $\rho(y_j) = x_j$ for all $j \in \mathbb{N}$. Thereby, $\tilde{F}(\sum_{j=1}^{\infty} \alpha_j x_j) = \tilde{F}(\rho(\sum_{j=1}^{\infty} \alpha_j y_j)) = f(\sum_{j=1}^{\infty} \alpha_j y_j) = \sum_{j=1}^{\infty} \alpha_j f(y_j) = \sum_{j=1}^{\infty} \alpha_j \tilde{F}(x_j)$. Since $R(X)$ is a base ordered vector space with base $\rho(X)$, there is a unique linear extension F of $\tilde{F}$, according to the preceding lemma. □

Definition 1.1.30. Let $\mathbf{R} : \mathbf{Conv} \to \mathbf{BOVec}$ denote the functor along ρ, which assigns to each convex module X the base ordered vector space $R(X)$ and to each affine mapping $f : X \to Y$, the unique linear mapping $F : R(X) \to R(Y)$ from the preceding theorem.

Remark 1.1.31. Note that $\hat{\rho} : X \to \mathbf{Bs} \circ \mathbf{R}(X)$ constitutes an affine function.

Corollary 1.1.32. ***R** : **Conv** $\to$ **BOVec** is left adjoint to **Bs** : **BOVec** $\to$ **Conv**, i.e. for each convex module X, for each base ordered vector space Y and for each affine*

$f : X \to \boldsymbol{Bs}(Y)$, there is a unique linear function $g : \boldsymbol{R}(X) \to Y$ that maps $\rho(X)$ into the base of Y, such that $\boldsymbol{Bs}(g) \circ \hat{\rho} = f$.

Proof. Let ι denote the inclusion of $\mathbf{Bs}(Y)$ into Y. According to the preceding theorem, there is a unique linear function $g : \mathbf{R}(X) \to Y$, such that $\rho(X)$ is mapped into the base of Y and such that $g \circ \rho = \iota \circ f$. Let $x \in X$, then $\mathbf{Bs}(g)(\hat{\rho}(x)) = g(\hat{\rho}(x)) = g(\rho(x)) = \iota(f(x)) = f(x)$. According to lemma 1.1.28, g is the unique function with this property. □

Corollary 1.1.33. *The categories* ***PresepConv*** *and* ***BOVec*** *are equivalent.*

Proof. For each $X \in \mathbf{PresepConv}$ the affine function $\hat{\rho}_X : X \to \mathbf{Bs} \circ \mathbf{R}(X)$ is an isomorphism. For each $Y \in \mathbf{BOCVec}$, let $F_Y : \mathbf{R} \circ \mathbf{Bs}(Y) \to Y$ be the unique linear and base preserving function that extends the inclusion of $\mathbf{Bs}(Y)$ into Y. F_Y is an isomorphism. □

1.2 A Semimetric on Convex Modules

Definition 1.2.1. Let X be convex module. Let the convex semimetric $d : X \times X \to \mathbb{R}$ of X be defined as:

$$d(x, y) := \inf\{\frac{\alpha}{1-\alpha} : \alpha \in [0, 1]; \tilde{x}, \tilde{y} \in X; (1-\alpha)x + \alpha\tilde{x} = (1-\alpha)y + \alpha\tilde{y}\}$$

Proposition 1.2.2. *d is a semimetric.*

Proof. Clearly, d is symmetric. For the triangle inequality let X be a convex module, let $x, y, z \in X$ and let $\epsilon > 0$ be fixed. Let $\frac{\alpha_x}{1-\alpha_x} \leq d(x, z) + \epsilon$, let $\frac{\alpha_y}{1-\alpha_y} \leq d(y, z) + \epsilon$ and let $\tilde{x}, \tilde{y}, \tilde{z_x}, \tilde{z_y} \in X$, such that $(1-\alpha_x)x + \alpha_x\tilde{x} = (1-\alpha_x)z + \alpha\tilde{z_x}$ and that $(1-\alpha_y)y + \alpha_y\tilde{y} = (1-\alpha_y)z + \alpha\tilde{z_y}$. Let $\alpha := \frac{\alpha_x+\alpha_y-2\alpha_x\alpha_y}{1-\alpha_x\alpha_y}$, let $\hat{x} := \frac{\alpha_x(1-\alpha_y)}{\alpha_x+\alpha_y-2\alpha_x\alpha_y}\tilde{x} + \frac{(1-\alpha_x)\alpha_y}{\alpha_x+\alpha_y-2\alpha_x\alpha_y}\tilde{z_y}$ and let $\hat{y} = \frac{(1-\alpha_x)\alpha_y}{\alpha_x+\alpha_y-2\alpha_x\alpha_y}\tilde{y} + \frac{\alpha_x(1-\alpha_y)}{\alpha_x+\alpha_y-2\alpha_x\alpha_y}\tilde{z_x}$. Then:

$$(1-\alpha)x + \alpha\hat{x} = \frac{(1-\alpha_x)(1-\alpha_y)}{1-\alpha_x\alpha_y}x + \frac{\alpha_x(1-\alpha_y)}{1-\alpha_x\alpha_y}\tilde{x} + \frac{(1-\alpha_x)\alpha_y}{1-\alpha_x\alpha_y}\tilde{z_y} =$$

$$= \frac{(1-\alpha_x)(1-\alpha_y)}{1-\alpha_x\alpha_y}z + \frac{\alpha_x(1-\alpha_y)}{1-\alpha_x\alpha_y}\tilde{z_x} + \frac{(1-\alpha_x)\alpha_y}{1-\alpha_x\alpha_y}\tilde{z_y}$$

$$(1-\alpha)y + \alpha\hat{y} = \frac{(1-\alpha_x)(1-\alpha_y)}{1-\alpha_x\alpha_y}y + \frac{(1-\alpha_x)\alpha_y}{1-\alpha_x\alpha_y}\tilde{y} + \frac{\alpha_x(1-\alpha_y)}{1-\alpha_x\alpha_y}\tilde{z_x} =$$

$$= \frac{(1-\alpha_x)(1-\alpha_y)}{1-\alpha_x\alpha_y}z + \frac{(1-\alpha_x)\alpha_y}{1-\alpha_x\alpha_y}\tilde{z_y} + \frac{\alpha_x(1-\alpha_y)}{1-\alpha_x\alpha_y}\tilde{z_x}$$

Thus, we get the equations:

$$(1-\alpha)x + \alpha\hat{x} = (1-\alpha)y + \alpha\hat{y}$$

$$\frac{\alpha}{1-\alpha} = \frac{\alpha_x}{1-\alpha_x} + \frac{\alpha_y}{1-\alpha_y}$$

Since ϵ was arbitrary, $d(x,y) \leq d(x,z) + d(z,y)$. □

Remark 1.2.3. Let $\tilde{x} = y$, let $\tilde{y} = x$ and let $\alpha = \frac{1}{2}$. Then the equation $(1-\alpha)x + \alpha\tilde{x} = (1-\alpha)y + \alpha\tilde{y}$ becomes $\frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}y + \frac{1}{2}x$. Hence $d(x,y) \leq 1$ for all $x, y \in X$.

Lemma 1.2.4. *Every affine mapping $f : X_1 \to X_2$ satisfies for all $x, y \in X_1$:*

$$d_2(f(x), f(y)) \leq d_1(x,y)$$

Proof. Let $\epsilon > 0$ be fixed and let $x, y \in X_1$. Let $\alpha \in [0,1]$, $\tilde{x}, \tilde{y} \in X_1$, such that $\frac{\alpha}{1-\alpha} \leq d_1(x,y) + \epsilon$ and that $(1-\alpha)x + \alpha\tilde{x} = (1-\alpha)y + \alpha\tilde{y}$. Then $(1-\alpha)f(x) + \alpha f(\tilde{x}) = f((1-\alpha)x + \alpha\tilde{x}) = f((1-\alpha)y + \alpha\tilde{y}) = (1-\alpha)f(y) + \alpha f(\tilde{y})$. Thus, $d_2(x,y) \leq \frac{\alpha}{1-\alpha}$ and since ϵ was arbitrary $d_2(f(x), f(y)) \leq d_1(x,y)$. □

Corollary 1.2.5. *Any affine mapping is continuous. Any isomorphism in **Conv** is an isometry.*

Proposition 1.2.6. *Let X be a convex module. If the convex semimetric is a metric, X is preseparated.*

Proof. Let $x, y, z \in$ X and let $\lambda \in [0,1)$, such that $x \neq y$ and $(1-\lambda)x + \lambda z = (1-\lambda)y + \lambda z$. Because of lemma 1.1.8, $(1-\alpha)x + \alpha z = (1-\alpha)y + \alpha z$ for all $\alpha \in (0,1]$ and therefore $d(x,y) = 0$. Hence, (X,d) cannot be metric. □

Definition 1.2.7. A convex module X is called linearly bounded if all affine mappings $f : \mathbb{R}_{++} \to X$ are constant.

Lemma 1.2.8. *Let X be a linearly bounded convex module. Then:*

- *Any convex submodule Y of X, i.e. a subset that carries the same convex structure, is linearly bounded.*
- *If X is a subset of a real vector space that carries the inherited convex structure and $\lambda \in \mathbb{R}$, then λX is linearly bounded, too.*

Proof. Let $f : \mathbb{R}_{++} \to Y$ be affine. Let $\iota : Y \to X$ denote the inclusion. ι obviously is affine and injective. Since X is linearly bounded, $\iota \circ f$ has to be constant. Since ι is injective, f is constant, too.

First assume $\lambda \neq 0$. Let $f : \mathbb{R}_{++} \to Y$ be affine. Let $g_\lambda : \lambda \cdot X \to X$ be defined as: $g_\lambda(x) = \frac{1}{\lambda}x$. Since g_λ is an affine bijection and linearly boundedness is invariant under isomorphisms, λX is linearly bounded if and only if X is linearly bounded. In case $\lambda = 0$, $0X$ consist of a single point and hence any function with codomain $0X$ is constant. □

Lemma 1.2.9. *Let X be a convex module, let I be an open, possibly infinite, real interval and let $f : I \to X$ be an affine mapping and let $\lambda, \mu \in I$, such that $\lambda \neq \mu$. If $f(\lambda) = f(\mu)$, then f is constant.*

Proof. Without loss of generality, let $\lambda < \mu$ and let $\xi \in I$, with $\xi > \mu$. Then the following equation holds:

$$\frac{\mu - \xi}{\lambda - \xi} f(\lambda) + \frac{\lambda - \mu}{\lambda - \xi} f(\xi) = f(\frac{\mu - \xi}{\lambda - \xi}\lambda + \frac{\lambda - \mu}{\lambda - \xi}\xi) =$$

$$= f(\mu) = f(\lambda) = \frac{\mu - \xi}{\lambda - \xi} f(\lambda) + \frac{\lambda - \mu}{\lambda - \xi} f(\lambda)$$

According to lemma 1.1.8, $\alpha f(\lambda) + (1 - \alpha) f(\xi) = \alpha f(\lambda) + (1 - \alpha) f(\lambda) = f(\lambda)$ for all $\alpha \in (0, 1]$. Since $\alpha f(\lambda) + (1 - \alpha) f(\xi) = f(\alpha\lambda + (1 - \alpha)\xi)$, this implies $f(\zeta) = f(\lambda)$ for all $\zeta \in [\lambda, \xi)$. Since $\xi > \mu$ was arbitrary this holds for all $\zeta \geq \lambda$. The proof for $\zeta \leq \mu$ is analogous. □

Lemma 1.2.10. *Let X be a convex module and let $f : \mathbb{R}_{++} \to X$ be an affine mapping. For all $x, y \in f(\mathbb{R}_{++})$: $d(x, y) = 0$.*

Proof. Let $\mu_1, \mu_2 \in \mathbb{R}_{++}$, such that $\mu_1 < \mu_2$ and let $\alpha \in (0, 1]$, then:

$$(1 - \alpha) f(\mu_1) + \alpha f(\frac{\mu_2 - (1 - \alpha)\mu_1}{\alpha}) = f(\mu_2) = (1 - \alpha) f(\mu_2) + \alpha f(\mu_2)$$

Thus, $d(f(\mu_1), f(\mu_2)) = 0$. □

Lemma 1.2.11. *A convex and balanced subset C of a real vector space X is linearly bounded if and only if for all $x \in X \backslash \{0\}$ there exists a $\lambda \in \mathbb{R}$, such that λx is not in C.*

Proof. Assume that there is an $x \in C$ such that $\mathbb{R}x \subseteq C$. Define $f : \mathbb{R}_{++} \to C$ by $f(\lambda) := \lambda x$. Thus, C cannot be linearly bound. For the other direction, assume that there is an affine and nonconstant $f : \mathbb{R}_{++} \to C$. Let $\lambda > 0$ be arbitrary and let $x := \frac{1}{2} f(1) - \frac{1}{2} f(0) \in C$. Then, $\frac{1}{\lambda} f(\lambda) + (1 - \frac{1}{\lambda}) f(0) = f(1)$ and $\lambda x = \frac{\lambda}{2} f(1) + \frac{1-\lambda}{2} f(0) - \frac{1}{2} f(0) = \frac{1}{2}(\lambda f(1) + (1 - \lambda) f(0)) - \frac{1}{2} f(0) = \frac{1}{2} f(\lambda) - \frac{1}{2} f(0) \in C$. Since $\lambda > 0$ was arbitrary and C is convex and balanced, $\mathbb{R}x \subseteq C$. □

Lemma 1.2.12. *Let X be a real vector space and let $\|\cdot\|$ denote the seminorm induced by the Minkowski functional of a convex, balanced and absorbing subset C. Let $x \in X$. Then, $\|x\| = 0$ if and only if $\mathbb{R}x \subseteq C$.*

Proof. If $\|x\| = 0$, then for all $\lambda \in \mathbb{R}_{++}$, $x \in \lambda C$ and hence $\frac{1}{\lambda}x \in C$. If $\mathbb{R}x \subseteq C$, then $x \in \lambda C$ for all $\lambda \in \mathbb{R}_{++}$ and hence $\|x\| = 0$. □

Definition 1.2.13. Let X be a real vector space and let $B \subseteq X$. Let $\mathrm{conv}(B)$ denote the convex closure of B, i.e. the intersection of all convex sets containing B.

Remark 1.2.14. The convex closure of B is the smallest convex set containing B.

Definition 1.2.15. For any base ordered vector space X with base B let $\|\cdot\|$ denote the seminorm induced by the Minkowski functional of $\mathrm{conv}(B \cup (-B))$. Call $\|\cdot\|$ the base seminorm of X. If $\|\cdot\|$ is a norm call it the base norm of X and call X a base normed vector space.

Remark 1.2.16. Clearly, $\mathrm{conv}(B \cup (-B))$ is convex and balanced and, since B is generating, $\mathrm{conv}(B \cup (-B))$ is absorbing too. Hence $\|\cdot\|$ is indeed a seminorm.

Corollary 1.2.17. *Any base ordered vector space X with base B is a base normed vector space if and only if $\mathrm{conv}(B \cup (-B))$ is linearly bounded.*

Proof. Follows directly from lemmata 1.2.12 and 1.2.11. □

Lemma 1.2.18. *Let X be a base ordered vector space with base B and let $B_\lambda := \{\frac{1}{2}(1+\lambda)x - \frac{1}{2}(1-\lambda)y : x, y \in B\}$. Then:*

- *$\mathrm{conv}(B \cup (-B)) = \{\alpha x - (1-\alpha)y : x, y \in B, \alpha \in [0,1]\} = \bigcup_{\lambda \in [-1,1]} B_\lambda$*
- *There is a linear function $\pi : X \to \mathbb{R}$, such that $\pi(B_\lambda) = \{\lambda\}$.*
- *The affine spaces generated by B_λ, $\lambda \in \mathbb{R}$ are pairwise disjoint.*

Proof. To show the first equation, let $w, x, y, z \in B$ and let $\alpha, \beta, \gamma \in [0,1]$. Then:

$$(1-\alpha)(\beta w-(1-\beta)x)+\alpha(\gamma y-(1-\gamma)z) = ((1-\alpha)\beta w+\alpha\gamma y)-((1-\alpha)(1-\beta)x+\alpha(1-\gamma)z) =$$

$$= (\beta - \alpha + \alpha\gamma)\left(\frac{\beta - \alpha\beta}{\beta - \alpha + \alpha\gamma}w + \frac{\alpha\gamma}{\beta - \alpha + \alpha\gamma}y\right)+$$

$$+(1 - \beta + \alpha\beta - \alpha\gamma)\left(\frac{1 - \alpha - \beta + \alpha\beta}{1 - \beta + \alpha\beta - \alpha\gamma}x + \frac{\alpha - \alpha\gamma}{1 - \beta + \alpha\beta - \alpha\gamma}z\right)$$

Since $(\beta-\alpha+\alpha\gamma) \in [-1,1]$ the convex combination is contained in $D := \{\alpha x-(1-\alpha)y : x, y \in B, \alpha \in [0,1]\}$ and thus D is convex. Let $v = \alpha x - (1-\alpha)y \in D$ be arbitrary.

Since v is a convex combination of x and $-y$, v must be in $\mathrm{conv}(B \cup (-B))$ and since $\mathrm{conv}(B \cup (-B))$ is the smallest convex set containing $B \cup (-B)$, the first equation holds true. The second equation is trivial.

For the second part let $b \in B$ be fixed and let $x \in X$ be arbitrary. X is generated by $B - B$ and thereby there are $y, z \in B$ and $\mu, \nu \in \mathbb{R}_+$ such that $x = \mu y - \nu z$. Define $\pi(x) := \mu - \nu$. To show that π is well-defined assume $x = \mu y - \nu z = \zeta u - \eta v$, with $\mu, \nu, \zeta, \eta \in \mathbb{R}_+$ and $y, z, u, v \in B$. Thus, $(\mu+\eta)(\frac{\mu}{\mu+\eta}y + \frac{\eta}{\mu+\eta}v) = (\nu+\zeta)(\frac{\nu}{\nu+\zeta}z + \frac{\zeta}{\nu+\zeta}u)$ and since $(\frac{\mu}{\mu+\eta}y + \frac{\eta}{\mu+\eta}v) \in B$ and $(\frac{\nu}{\nu+\zeta}z + \frac{\zeta}{\nu+\zeta}u) \in B$, definition 1.1.18 implies $\mu + \eta = \nu + \zeta$. To show that π is linear let $\mu, \nu, \zeta, \eta, \lambda \in \mathbb{R}_+$, let $y, z, u, v \in B$ and let $x = \mu y - \nu z, w = \zeta u - \eta v$. Then:

$$\pi(x + \lambda y) = \pi(\mu y - \nu z + \lambda(\zeta u - \eta v)) =$$

$$= \pi((\mu + \lambda\zeta)(\frac{\mu}{\mu + \lambda\zeta}y + \frac{\lambda\zeta}{\mu + \lambda\zeta}u) - (\nu + \lambda\eta)(\frac{\nu}{\nu + \lambda\eta}z + \frac{\lambda\eta}{\nu + \lambda\eta}v)) = (\mu - \nu) + \lambda(\zeta - \eta)$$

Thus π is linear. Obviously, this implies $\pi(B_\lambda) = \{\lambda\}$ and hence the third statement is true. □

Proposition 1.2.19. *Let X be a base ordered vector space with base B. For any $x, y \in B$ the base seminorm and the convex semimetric d on B satisfy:*

$$\|x - y\| = 2d(x, y)$$

Proof. Let $x, y, \tilde{x}, \tilde{y} \in B$ and $\alpha \in [0, 1)$, such that $(1 - \alpha)x + \alpha\tilde{x} = (1 - \alpha)y + \alpha\tilde{y}$. This yields $x - y = \frac{\alpha}{1-\alpha}(\tilde{y} - \tilde{x})$ and thus:

$$\|x - y\| = \frac{\alpha}{1 - \alpha}\|\tilde{y} - \tilde{x}\| \leq \frac{\alpha}{1 - \alpha}(\|\tilde{y}\| + \|\tilde{x}\|) = \frac{2\alpha}{1 - \alpha}$$

Hence, $\|x - y\| \leq 2d(x, y)$.

For the other direction, first consider the case that $\|x - y\| > 0$. Let $\epsilon > 0$ be fixed and $z := \frac{1-\epsilon}{\|x-y\|}(x - y)$. Since $\|z\| = 1 - \epsilon$, there are $\tilde{x}, \tilde{y} \in B$ and $\beta \in [0, 1]$ such that $z = (1 - \beta)\tilde{y} - \beta\tilde{x}$. Since x and y are in the affine hull of $B_1 = \{\frac{1}{2}(1 + 1)u - \frac{1}{2}(1 - 1)v : u, v \in B\}$, according to lemma 1.2.18, z is in the affine hull of B_0 and hence $\beta = 1 - \beta$. Thus $\frac{1-\epsilon}{\|x-y\|}(x - y) = \frac{1}{2}\tilde{y} - \frac{1}{2}\tilde{x}$. Let $\alpha := \frac{\|x-y\|}{2-2\epsilon+\|x-y\|}$, then $\frac{\|x-y\|}{2(1-\epsilon)} = \frac{\alpha}{1-\alpha}$ and thus $(1 - \alpha)x + \alpha\tilde{x} = (1 - \alpha)y + \alpha\tilde{y}$. Since ϵ was arbitrary, $\|x - y\| = 2d(x, y)$ follows.

In case that $\|x - y\| = 0$, for all $\lambda \in \mathbb{R}_{++}$ there is a $z \in B$ such that $z = \lambda(x - y)$. Again, since z is in the affine subspace generated by B_0, there are $\tilde{x}, \tilde{y} \in B$, such that $\lambda(x - y) = \frac{1}{2}\tilde{y} - \frac{1}{2}\tilde{x}$. Let $\alpha := \frac{1}{1+2\lambda}$, so that $\frac{1}{2\lambda} = \frac{\alpha}{1-\alpha}$. Thereby $(1 - \alpha)x + \alpha\tilde{x} = (1 - \alpha)y + \alpha\tilde{y}$ and since λ was arbitrary, $d(x, y) = 0$ follows. □

Corollary 1.2.20. *The convex semimetric of a convex module X is a metric if and only if ρ is injective and conv$(\rho(X) \cup (-\rho(X)))$ is linearly bounded.*

Proof. If ρ is injective, it is an isomorphism and thereby an isometry. According to 1.2.17 and 1.2.19 it is metric if and only if $\mathrm{conv}(\rho(X) \cup (-\rho(X)))$ is linearly bounded. In case X is metric, X is preseparated according to 1.2.6 and hence ρ is an isomorphism. □

Proposition 1.2.21. *Let X be a base ordered vector space with base B. Let $B_0 := \frac{1}{2}(B-B)$ be linearly bounded, then $B_\lambda := \{\frac{1}{2}(1+\lambda)x - \frac{1}{2}(1-\lambda)y : x, y \in B\}$ is linearly bounded for any $\lambda \in \mathbb{R}$.*

Proof. Since $B_{-\lambda} = -B_\lambda$, we can assume that $\lambda \geq 0$. First consider the case $\lambda \geq 1$. Then, $B_\lambda = \lambda B$, hence it suffices to show that B is linearly bounded. Let $b \in B$ be fixed let $f : \mathbb{R}_{++} \to B$ be an affine mapping and let $g : B \to B_0$ be defined as $g(z) := \frac{1}{2}z - \frac{1}{2}b$. Clearly, g is affine and injective. Since B_0 is linearly bounded $g \circ f$ must be constant and because g is injective, f must be constant too. Thus, B is linearly bounded.

Next, let $\lambda \in [0, 1)$. Again, let $b \in B$ be fixed and let $f : \mathbb{R}_{++} \to B_\lambda$ be an affine mapping. Define $g : B_\lambda \to B_0$ by: $g(z) := \frac{1}{1+\lambda}z - \frac{\lambda}{1+\lambda}b$. To assure that the image of g is contained in B_0, let $z = \frac{1}{2}(1+\lambda)x - \frac{1}{2}(1-\lambda)y$, with $x, y \in B$. Then:

$$g(z) = \frac{1}{1+\lambda}(\frac{1}{2}(1+\lambda)x - \frac{1}{2}(1-\lambda)y) - \frac{\lambda}{1+\lambda}b = \frac{1}{2}x - \frac{1}{2}(\frac{1-\lambda}{1+\lambda}y + \frac{2\lambda}{1+\lambda}b)$$

Indeed g maps B_λ into B_0 and clearly g is affine and injective. Since B_0 is linearly bounded $g \circ f$ must be constant and because g is injective, f must be constant, too. Hence B_λ is linearly bounded. □

Proposition 1.2.22. *Let B be the base of a base ordered vector space X. The following statements are equivalent:*

- *$\frac{1}{2}(B-B)$ is linearly bounded.*
- *conv$(B \cup (-B))$ is linearly bounded.*

Proof. First let $\mathrm{conv}(B \cup (-B))$ be linearly bounded. Since $\frac{1}{2}(B-B)$ is a subset of $\mathrm{conv}(B \cup (-B))$, it is linearly bounded too. For the other direction let $f : \mathbb{R}_{++} \to \mathrm{conv}(B \cup (-B))$ be affine. Let π be the linear functional of lemma 1.2.18. Since $\pi \circ f$ is affine and $\pi(\mathrm{conv}(B \cup (-B))) = [-1, 1]$, $\pi \circ f$ must be constant. Thereby there exist $\lambda \in [-1, 1]$, such that $f(\mathbb{R}_{++}) \subseteq B_\lambda := \{(\lambda + \frac{1}{2})x - (\lambda - \frac{1}{2})y : x, y \in B\}$. According to the preceding proposition B_λ is linearly bounded and thereby f is constant. □

Corollary 1.2.23. *The semimetric of a convex module X is a metric if and only if ρ is injective and $\rho(X) - \rho(X)$ is linearly bounded.*

Proof. Follows directly from the preceding proposition and corollary 1.2.20. □

The following example shows that the linearly boundedness of a convex module B does not imply the linearly boundedness of $B - B$ and hence the condition of $\rho(X) - \rho(X)$ being linearly bounded cannot be reduced to a condition of $\rho(X)$ being bounded.

Example 1.2.24. Let $\mathbb{R}^{<\mathbb{N}>}$ denote the real vector space freely generated by a countably infinte set, i.e. all $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ such that $x_n = 0$ for all but finitely many $n \in \mathbb{N}$. Let $B := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{<\mathbb{N}>} : |x_n| \leq n \ \forall n > 1; |x_0| \leq \sum_{j=1}^{\infty} |x_j|\}$. To show that B is convex it is sufficient to show convexity for all finite dimensional subspaces, since for each convex combination of two vectors, all but finitely many components are zero already. Define $B_m := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{<\mathbb{N}>} : |x_m| \leq m; |x_0| \leq |x_m|; x_n = 0 \ \forall n > 1, n \neq m\}$. Clearly B_m is convex for each positive integer m. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in B$ and $p \in \mathbb{N}$, such that $x_j = 0$ and $y_j = 0$ for all $j > p$. Since $|x_0| \leq \sum_{j=1}^{p} |x_j|$ and $|y_0| \leq \sum_{j=1}^{p} |y_j|$, $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \sum_{j=1}^{p} B_j$. The sets B_j are convex for all j, hence $\sum_{j=1}^{p} B_j$ is convex. Therefore there exists a convex combination of $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\sum_{j=1}^{p} B_j \subseteq B$, hence B is convex.

To show that B is linearly bounded let $f : \mathbb{R}_{++} \to B$. Let $n \in \mathbb{N}$ be such that the n-th component of $f(2) - f(1)$ is not equal to zero. Let this difference be denoted by δ. In case $n > 0$, $f(1 + \frac{2n}{|\delta|}) = f(1) + \frac{2n}{|\delta|}(f(2) - f(1)) \notin B$, because the absolute value of the n-th component is larger than n. In case $n = 0$, let p be such that all k-th components with $k > p$ of $f(1)$ and $f(2)$ vanish. Then $= f(1 + \frac{2p(p+1)}{|\delta|}) = f(1) + \frac{2p(p+1)}{\delta}(f(2) - f(1))$ cannot be in B, since the absolute value of the 0-th component is larger than $\frac{p(p+1)}{2} = \sum_{j=1}^{p} j$. Hence $f(2) - f(1) = 0$, which implies that f is constant.

Now consider $B - B$. Let e_n denote the n-th unit vector and let $\lceil \cdot \rceil$ denote the ceiling function, i.e. rounding up to the next integer. For $\lambda \in \mathbb{R}$ define $g_1(\lambda) := \frac{\lambda}{2} e_0 + \frac{\lambda}{2} e_{\lceil |\frac{\lambda}{2}| \rceil} \in B$ and $g_2(\lambda) := -\frac{\lambda}{2} e_0 + \frac{\lambda}{2} e_{\lceil |\frac{\lambda}{2}| \rceil} \in B$. Thus; $g_1(\lambda) - g_2(\lambda) = \lambda e_0$ and $\lambda e_0 \in B - B$ for all $\lambda \in \mathbb{R}$. Now define the function $f(\lambda) := \lambda e_0$, which is clearly affine and nonconstant. Hence $B - B$ cannot be linearly bounded.

1.3 Superconvex Modules and the Completion

Definition 1.3.1. A family $(\lambda_n)_{n\in\mathbb{N}}$ of countably infinite many real numbers is called a superconvex combination if $0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \lambda_n = 1$. Let Ω_{sc} denote the set of all superconvex combinations.

Definition 1.3.2. A set X and a function $c : \Omega_{sc} \times X^{\mathbb{N}} \to X$, which is written as $c((\lambda_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}) =: \sum_{n=1}^{\infty} \lambda_n x_n$, are called a superconvex module and c a superconvex combination if they satisfy the following conditions:

1. $\sum_{n=1}^{\infty} \lambda_n x_n = \sum_{n=1}^{\infty} \lambda_{\sigma(n)} x_{\sigma(n)}$, for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$
2. $\sum_{n=1}^{\infty} \mu_n (\sum_{m=1}^{\infty} \lambda_{n,m} x_m) = \sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} \mu_n \lambda_{n,m}) x_m$ for all $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, and $(\lambda_n, m)_{m\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}} \in \Omega_{sc}$.
3. $\sum_{n=1}^{\infty} \lambda_n x_n = x_i$ if $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \neq i$, for $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$.

Remark 1.3.3. Clearly the latter term in the second condition is well-defined since $(\sum_{n=1}^{\infty} \mu_n \lambda_{n,m})_{m\in\mathbb{N}}$ is a superconvex combination.

Every superconvex module is as a convex module too: Let $\sum_{i=1}^{N} \lambda_i = 1$ be arbitrary. Let $\tilde{\lambda}_i := \lambda_i$ for $i \leq N$ and $\tilde{\lambda}_i := 0$ for $i > N$. Define the operation of a convex module on X by letting $\sum_{i=1}^{N} \lambda_i := \sum_{i=1}^{\infty} \tilde{\lambda}_i$. One easily sees that the first condition of definition 1.3.2, the commutativity of the formal sum, implies the first condition in 1.1. The second condition, the associativity of the formal sum, implies the second and the fourth condition of 1.1 and the third condition in 1.3.2 implies the third condition in 1.1. From now on, when a superconvex module is considered, define this as its canonical underlying structure as a convex module. A superconvex module is said to be preseparated or linearly bounded if the underlying convex module has that property. Let the semimetric on a superconvex module be defined as the semimetric on its underlying convex module.

Definition 1.3.4. Let X and Y be superconvex modules. A mapping $f : X \to Y$ is called superaffine, if for all $x_j \in X$ and $(\lambda_j)_{j\in\mathbb{N}} \in \Omega_{sc}$:

$$f(\sum_{j=1}^{\infty} \lambda_j x_j) = \sum_{j=1}^{\infty} \lambda_j f(x_j)$$

Let **SConv** denote the category of superaffine modules with superaffine mappings as morphisms. Since every superaffine mapping is affine too, **SConv** is a subcategory of **Conv**.

Lemma 1.3.5. *Every preseparated superconvex module is linearly bounded.*

Let X be a preseparated convex module and let $f : \mathbb{R}_{++} \to X$ be an affine function. Then:

$$\frac{1}{3} f(1) + \sum_{j=0}^{\infty} \frac{1}{3 \cdot 2^j} f(2^j + 1) = \sum_{j=1}^{\infty} \frac{1}{3 \cdot 2^j} f(1) + \frac{1}{3} f(2) + \sum_{j=1}^{\infty} \frac{1}{3 \cdot 2^j} f(2^j + 1) =$$

$$= \frac{1}{3}f(2) + \sum_{j=1}^{\infty} \frac{1}{3\cdot 2^{j-1}}(\frac{1}{2}f(2^j+1) + \frac{1}{2}f(1)) = \frac{1}{3}f(2) + \sum_{j=1}^{\infty} \frac{1}{3\cdot 2^{j-1}} f(2^{j-1}+1) =$$

$$= \frac{1}{3}f(2) + \sum_{j=0}^{\infty} \frac{1}{3\cdot 2^{j}} f(2^{j}+1)$$

Since X is preseparated, this implies $f(1) = f(2)$. According to lemma 1.2.9, f is constant.

Definition 1.3.6. Let X be a normed real vector space. A subset $C \subseteq X$ is called superconvex if there exists a superconvex structure on C such that the underlying convex module coincides with the linear structure on C.

Lemma 1.3.7. *Let X be a normed real vector space and let $B \subseteq X$ be bounded and superconvex. Then, the superconvex structure on B extending the linear structure of B is unique and satisfies:*

$$\sum_{j=1}^{\infty} \alpha_j x_j = \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j x_j$$

If X is a Banach space and $B \subseteq X$ is convex, bounded and closed, then B is a superconvex set.

Proof. To show uniqueness, it satisfies to show that the equation holds true. Let $\sum_{j=1}^{\infty} \alpha_j x_j$ be an arbitrary superconvex combination and let $c \in \mathbb{R}_+$ such that $\|x\| \leq c$ for all $x \in X$. Then:

$$\sum_{\ell=1}^{\infty} \alpha_\ell x_\ell = (\frac{\sum_{j=1}^{N}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k}) \sum_{\ell=1}^{N} \frac{\alpha_\ell \sum_{m=1}^{\infty}\alpha_m}{\sum_{n=1}^{N}\alpha_n} x_\ell + (\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k}) \sum_{\ell=N+1}^{\infty} \frac{\alpha_\ell \sum_{m=1}^{\infty}\alpha_m}{\sum_{n=N+1}^{\infty}\alpha_n} x_\ell$$

$$\|\sum_{\ell=1}^{\infty} \alpha_\ell x_\ell - \sum_{\ell=1}^{N} \alpha_\ell x_\ell\| \leq (\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k})\| \sum_{\ell=N+1}^{\infty} \frac{\alpha_\ell \sum_{m=1}^{\infty}\alpha_m}{\sum_{n=N+1}^{\infty}\alpha_n} x_\ell\| \leq (\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k}) c \to 0$$

Thus, the superconvex combination has to coincide with the limit of the series. For the second part, the boundedness implies $\|\sum_{j=N+1}^{\infty}\alpha_j x_j\| \leq \sum_{j=N+1}^{\infty}\alpha_j\|x_j\| \to 0$. The series converges and hence is well-defined. Since the series converges absolutely this definition does not depend on the order of the summands and $\sum_{j=1}^{\infty}\beta_j(\sum_{k=1}^{\infty}\alpha_{j,k}x_k) = \sum_{k=1}^{\infty}(\sum_{j=1}^{\infty}\beta_j\alpha_{j,k}x_k)$. Hence, this indeed defines a superconvex module. □

Definition 1.3.8. Let X be a Banach space and let $B \subseteq X$. Let superconv(B) denote the superconvex closure of B, i.e. the intersection of all superconvex sets containing B.

Lemma 1.3.9. *Let X be a Banach space and let $B \subseteq X$ be a bounded subset. Then, superconv$(B) = \{\lim_{N\to\infty}\sum_{j=1}^{N}\alpha_j x_j : \sum_{j=1}^{\infty}\alpha_j = 1; x_j \in X$ for all $j \in \mathbb{N}\} \subseteq \overline{conv(B)}$.*

Proof. Since B is bounded the series does converge absolutely, hence the set is well-defined. Clearly, $D := \{\lim_{N\to\infty}\sum_{j=1}^{N}\alpha_j x_j : \sum_{j=1}^{\infty}\alpha_j = 1; x_j \in X \text{ for all } j \in \mathbb{N}\}$ contains $\operatorname{conv}(B)$. Since any element of D is the limit of a sequence in $\operatorname{conv}(B)$, it is contained in $\overline{\operatorname{conv}(B)}$ and hence still bounded. Thus, for all $\beta_j, \alpha_{j,k} \in [0,1]$, such that $\sum_{j=1}^{\infty}\beta_j = 1$ and $\sum_{k=1}^{\infty}\alpha_{j,k} = 1$ for all $j \in \mathbb{N}$, and for all $x_{j,k} \in X$:

$$\lim_{N\to\infty}\sum_{j=1}^{N}\beta_j(\lim_{M\to\infty}\sum_{j=1}^{M}\alpha_{j,k}x_{j,k}) = \lim_{M\to\infty}(\lim_{N\to\infty}\sum_{j=1}^{N}\sum_{j=1}^{M}\beta_j\alpha_{j,k}x_{j,k})$$

is well-defined. Since $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\beta_j\alpha_{j,k} = 1$, it is also in D. □

Lemma 1.3.10. *Let X be a superconvex module, let Y be a bounded, closed, convex subset of a Banach sapce and let $f : X \to Y$ be an affine mapping. Then f is superaffine.*

Proof. Let $\sum_{\ell=1}^{\infty}\alpha_j x_j$ be an arbitrary convex combination in X and let $c \in \mathbb{R}_+$, such that $\|y\| \leq c$ for all $y \in Y$. Since f is affine the following equations holds true:

$$f(\sum_{\ell=1}^{\infty}\alpha_\ell x_\ell) = (\frac{\sum_{j=1}^{N}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k})f(\sum_{\ell=1}^{N}\frac{\alpha_\ell\sum_{m=1}^{\infty}\alpha_m}{\sum_{n=1}^{N}\alpha_n}x_\ell)+$$

$$+(\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k})f(\sum_{\ell=N+1}^{\infty}\frac{\alpha_\ell\sum_{m=1}^{\infty}\alpha_m}{\sum_{n=N+1}^{\infty}\alpha_n}x_\ell) =$$

$$= \sum_{\ell=1}^{N}\alpha_j f(x_j) + (\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k})f(\sum_{\ell=N+1}^{\infty}\frac{\alpha_\ell\sum_{m=1}^{\infty}\alpha_m}{\sum_{n=N+1}^{\infty}\alpha_n}x_\ell)$$

$$\|f(\sum_{\ell=1}^{\infty}\alpha_\ell x_\ell) - \sum_{\ell=1}^{N}\alpha_j f(x_j)\| = \|(\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k})f(\sum_{\ell=N+1}^{\infty}\frac{\alpha_\ell\sum_{m=1}^{\infty}\alpha_m}{\sum_{n=N+1}^{\infty}\alpha_n}x_\ell)\| =$$

$$= (\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k})\|f(\sum_{\ell=N+1}^{\infty}\frac{\alpha_\ell\sum_{m=1}^{\infty}\alpha_m}{\sum_{n=N+1}^{\infty}\alpha_n}x_\ell)\| \leq (\frac{\sum_{j=N+1}^{\infty}\alpha_j}{\sum_{k=1}^{\infty}\alpha_k})c \to 0$$

Hence $f(\sum_{\ell=1}^{\infty}\alpha_\ell x_\ell) = \lim_{N\to\infty}\sum_{\ell=1}^{N}\alpha_\ell f(x_\ell) = \sum_{\ell=1}^{\infty}\alpha_\ell f(x_\ell)$. Thus, f is superaffine. □

Definition 1.3.11. A base normed vector space X with base B is called base ordered Banach space, if X is a Banach space in regard to its base norm.

Proposition 1.3.12. *Let X be a base ordered vector space with superconvex base B. Then X is a base ordered Banach space with base B.*

Proof. Since B is superconvex, $B - B$ is superconvex too and hence linearly bounded. According to proposition 1.2.22 and corollary 1.2.17, X is a base normed vector space.

To show completeness, let $(x_j)_{j\in\mathbb{N}}$ be an arbitrary Cauchy sequence. Let $(y_j)_{j\in\mathbb{N}}$ be a subsequence such that $\|y_j - y_k\| < 2^{-j}$ for all $k \geq j$. Let $z_1 := y_1$ and let $z_j := y_j - y_{j-1}$ for all $j > 1$, then $y_j = \sum_{k=1}^{j} z_k$. Since $\|z_k\| < 2^{-k+1}$, $z_k \in \mathrm{conv}((2^{-k+1}B) \cup (-2^{-k+1}B)) \subseteq [0, 2^{-k+1}]B - [0, 2^{-k+1}]B$. Hence there are $\alpha_j, \beta_j \in [0, 2^{-k+1}]$ and $u_j, v_j \in B$, such that $z_j = \alpha_j u_j - \beta_j v_j$. To show that the sequence converges, it is sufficient to show that the series $\sum_{j=1}^{N} \alpha_j u_j$ and $\sum_{j=1}^{N} \beta_j v_j$ do. Since $|\alpha_j| \leq 2^{-j+1}$ the series $\sum_{j=1}^{N} \alpha_j$ converges absolutely and $\sum_{j=1}^{\infty} \alpha_j$ is well-defined. In case that all but finitely many α_j vanish the series clearly converges. In case this does not happen, the rules for superconvex combinations yield the following equation:

$$(\sum_{k=1}^{\infty} \alpha_k) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{l=1}^{\infty} \alpha_l} u_j = (\sum_{k=1}^{N} \alpha_k) \sum_{j=1}^{N} \frac{\alpha_j}{\sum_{l=1}^{N} \alpha_l} u_j + (\sum_{k=N+1}^{\infty} \alpha_k) \sum_{j=N+1}^{\infty} \frac{\alpha_j}{\sum_{l=N+1}^{\infty} \alpha_l} u_j$$

Since $\sum_{j=N+1}^{\infty} \frac{\alpha_j}{\sum_{l=N+1}^{\infty} \alpha_l} u_j \in B$, its norm is not greater than 1. Hence:

$$\lim_{N\to\infty} \|(\sum_{k=1}^{\infty} \alpha_k) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{l=1}^{\infty} \alpha_l} u_j - (\sum_{k=1}^{N} \alpha_k) \sum_{j=1}^{N} \frac{\alpha_j}{\sum_{l=1}^{N} \alpha_l} u_j\| = \lim_{N\to\infty} \| \sum_{k=N+1}^{\infty} \alpha_k \| = 0$$

The proof for $\sum_{j=1}^{N} \beta_j v_j$ is identical. By that,

$$\lim_{m\to\infty} z_m = (\sum_{k=1}^{\infty} \alpha_k) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{l=1}^{\infty} \alpha_l} u_j - (\sum_{k=1}^{\infty} \beta_k) \sum_{j=1}^{\infty} \frac{\beta_j}{\sum_{l=1}^{\infty} \beta_l} v_j$$

□

Definition 1.3.13. Let X be an ordered vector space. An element $e \in X$ is called order unit, if for any $x \in X$ there is a $\lambda \in \mathbb{R}_+$, such that $x \leq \lambda e$.

Definition 1.3.14. Let X be an ordered vector space with order unit e. Let the order unit (semi-)norm of e be definied as:

$$\|x\|_e := \inf\{\lambda \in \mathbb{R} : -\lambda e \leq x \leq \lambda e\}$$

Remark 1.3.15. The infimum is well-defined, because the set is non-empty, since e is an order unit. The homogeneity and the triangle equation follow directly from the corresponding properties of the partial ordering.

Definition 1.3.16. Let X be an ordered vector space. A function $f : X \to \mathbb{R}$ is called a positive linear functional, if it is linear and if for all $x \in X$, $0 \leq x$ implies $0 \leq f(x)$.

Lemma 1.3.17. *Let X be an ordered vector space with order unit e and order unit (semi-)norm $\|\cdot\|_e$. For any positive linear functional f the following equation holds:*

$$|f(x)| \leq \|x\|_e f(e)$$

Proof. Let $\epsilon > 0$ be arbitrary. Then, $x \leq (\|x\|_e + \epsilon)e$ implies that $(\|x\|_e + \epsilon)e - x \geq 0$. Hence $f((\|x\|_e + \epsilon)e - x) \geq 0$ and, since f is linear, $f(x) \leq (\|x\|_e + \epsilon)f(e)$. Since ϵ was arbitrary this shows the inequality for $f(x) \geq 0$. For $f(x) \leq 0$ insert $-x$ instead of x. □

Theorem 1.3.18. *Let X be an ordered vector space with order unit e. Let Y be a subspace of X, such that $e \in Y$ and equipped with the inherited order. Let f be a positive linear functional on Y. Then there exists a positive linear functional F on X, such that $f(y) = F(y)$ for all $y \in Y$.*

Proof. Let $p(x) := \inf\{f(y) : y \in Y \text{ and } x \leq y\}$. Clearly, $p(\lambda x) = \lambda p(x)$ for all $\lambda \in \mathbb{R}_+$ for all $x \in X$ and $f(x+z) \leq f(x)+f(z)$ for all $x, z \in X$. Also, $p(y) = f(y)$ for all $y \in Y$, since for any $z \in Y$ with $y \leq z$, $f(z) = f(z-y) + f(y) \geq f(y)$. Hence the Hahn-Banach theorem can be applied and yields a linear extension F of f. To show that F is positive let $x \geq 0$. Thus, $f(-x) \leq p(-x) \leq 0$ and hence $f(x) \geq 0$. □

Theorem 1.3.19. *Let X be an ordered vector space with order unit e and order unit norm $\|\cdot\|_e$. Let f be in the topological dual of X, denoted by X'. Then there exist positive linear functionals $g, h \in X'$, such that $f = g - h$ and $\|f\| = \|g\| + \|h\|$, whereat $\|\cdot\|$ is the operator norm.*

Proof. Let $Y := X \times X$ and define a partial order on Y by $(u, v) \leq (w, x)$, whenever $u \leq w$ and $v \leq x$. Clearly (e, e) is a order unit of Y. Now define $Z := \{(\lambda e - x, \lambda e + x) : \lambda \in \mathbb{R}, x \in X\}$. Let $f \in X'$ be arbitrary. Define $\hat{f} : Z \to \mathbb{R}$ as $\hat{f}((\lambda e - x, \lambda e + x)) := \lambda\|f\| - f(x)$. Obviously $\hat{f}$ is linear. To show that $\hat{f}$ is positive let $(\lambda e - x, \lambda e + x) \geq 0$. Since $-\lambda e \leq x \leq \lambda e$, $\|x\|_e \leq \lambda$ and thereby $\lambda\|f\| - f(x) \leq 0$. Thus we can apply theorem 1.3.18 and get a linear positive extension of $\hat{f}$ to Y called F. Let $h(x) := F(x, 0)$ and let $g(x) := F(0, x)$. Then g and h are positive linear functionals and $g(x) - h(x) = F(x, -x) = \hat{f}(x, -x) = f(x)$ for all $x \in X$. According to lemma 1.3.17, g and h are continuous and the equation $\|g\| + \|h\| = g(e) + h(e) = F(e, e) = \hat{f}(e, e) = \|f\|$ holds. □

Definition 1.3.20. Let C be a convex module. Define $\mathrm{Aff}_b(C)$ as the set of all bounded affine functions from C to the reals, i.e. all affine $f \in \mathrm{Conv}(C, \mathbb{R})$ such that there is a $c \in \mathbb{R}_+$ with $|f(x)| < c$ for all $x \in C$.

Lemma 1.3.21. *Let C be a convex module. With the usual addition of functions and multiplications with reals, $\mathit{Aff}_b(C)$ is an ordered Banach space with order unit the constant 1-function.*

Proof. Clearly, $\mathrm{Aff}_b(C)$ is a real vector space and the supremum norm $\|\cdot\|_\infty$ makes it a Banach space. Let 1_C denote the constant 1-function. Define a partial order on $\mathrm{Aff}_b(C)$

by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in C$. Clearly, 1_C is an order unit. Since $-1 \leq f(x) \leq 1$ for all $x \in C$ if and only if $\|f\|_\infty \leq 1$, the order unit (semi-)norm coincides with the supremum norm. □

Proposition 1.3.22. *Let C be a convex module and let $\mathrm{Aff}_b(C)'$ denote the topological dual of $\mathrm{Aff}_b(C)$. Then, $\mathrm{Aff}_b(C)'$ equipped with the operator norm is a base ordered Banach space with base $B := \{f \in \mathrm{Aff}_b(C)' : f(1_C) = 1$ and $\|f\| \leq 1\}$.*

Proof. As the topological dual of a Banach space, $\mathrm{Aff}_b(C)'$ equipped with the operator norm clearly is a Banach space. To show that B is a base, let $b_1, b_2 \in B$ and $\alpha \in [0,1]$. Then, $\|(1-\alpha)b_1 + \alpha b_2\| \leq (1-\alpha)\|b_1\| + \alpha\|b_2\| \leq 1$ and $((1-\alpha)b_1 + \alpha b_2)(1_C) = (1-\alpha)b_1(1_C) + \alpha b_2(1_C) = 1$, hence B is convex. Next, let $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $b_1, b_2 \in B$ and $\alpha_1 b_1 = \alpha_2 b_2$, then $\alpha_1 = \alpha_1 b_1(1_C) = \alpha_2 b_2(1_C) = \alpha_1$. Let $f \in \mathrm{Aff}_b(C)'$ be an arbitrary positive linear functional. Then, according to lemma 1.3.17, $\|f\| = f(1_C)$ and hence $\frac{1}{f(1_C)} f \in B$. According to theorem 1.3.19, any $f \in \mathrm{Aff}_b(C)'$ can be decomposed into the difference of two positive linear functionals and hence $\mathbb{R}_+B - \mathbb{R}_+B = \mathrm{Aff}_b(C)'$. Thus, B is a base. To show that the base norm coincides with the operator norm, let the base norm be denoted by $\|\cdot\|_B$. Any element $f \in \mathrm{Aff}_b(C)'$ with $\|f\|_B < 1$ can be written as $f = \alpha_1 b_1 - \alpha_2 b_2$ with $\alpha_1, \alpha_2 \in [0,1)$ and $b_1, b_2 \in B$, such that $\alpha_1 + \alpha_2 < 1$. Since $\|f\| \leq \alpha_1\|b_1\| + \alpha_2\|b_2\| = \alpha_1 + \alpha_2 < 1$ and q was arbitrary, $\|\cdot\| \leq \|\cdot\|_B$. On the other hand let $f < 1$, then, according to theorem 1.3.19, there are positive linear functionals g and h, such that $f = g - h$ and $\|f\| = \|g\| + \|h\|$. Since $\frac{1}{\|g\|} g \in B$ and $\frac{1}{\|h\|} h \in B$, the following equation holds: $f = \|g\| \frac{1}{\|g\|} g - \|h\| \frac{1}{\|h\|} h$. Since $\|g\| + \|h\| \leq 1$, this shows $\|f\|_B < 1$ and thus $\|\cdot\| \geq \|\cdot\|_B$. □

Definition 1.3.23. Let X be a convex module. Let $\tilde{\tau} : X \to \mathrm{Aff}_b(X)'$ be defined as $(\tilde{\tau}(x))(f) := f(x)$. Let $\tilde{B} := \mathrm{superconv}(\tilde{\tau}(X))$, i.e. the intersection of all superconvex sets containing $\tilde{\tau}(X)$. Let $T(X) := \mathbb{R}_+\tilde{B} - \mathbb{R}_+\tilde{B}$ and let τ be defined as the corestriction of $\tilde{\tau}$ to $T(X)$.

Lemma 1.3.24. *In the above definition $\tilde{\tau}$ and τ are well-defined, affine functions. $T(X)$ is a base ordered Banach space with base $\tilde{B}$.*

Proof. Let $x, y \in X$, let $\alpha \in [0,1]$ and let $f \in \mathrm{Aff}_b(X)$, then $(\tilde{\tau}((1-\alpha)x + \alpha y))(f) = f((1-\alpha)x + \alpha y) = (1-\alpha)f(x) + \alpha f(y) = (1-\alpha)(\tilde{\tau}(x))(f) + \alpha(\tilde{\tau}(y))(f)$, hence $\tilde{\tau}$ is affine. Since $\|\tilde{\tau}(x)\| \leq 1$, $\tilde{\tau}(x)$ is continuous and hence in $\mathrm{Aff}_b(X)'$, for all $x \in X$. Since $(\tilde{\tau}(x))(1_X) = 1$, the image of $\tilde{\tau}$ is a convex subset of the base $B := \{f \in \mathrm{Aff}_b(C)' : f(1_C) = 1$ and $\|f\| \leq 1\}$. Since B is the base of the norm it is bounded and thus $\tilde{\tau}(X)$ is bounded. Let H be the affine proper hyperplane generated by B. According to lemma 1.3.9, $\tilde{B}$ is contained in $\overline{\tilde{\tau}(X)}$ and hence contained in H. Since $\tilde{B}$ is contained in the proper affine hyperplane $H \cap T(X)$ it satisfies that for all $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $b_1, b_2 \in \tilde{B}$, $\alpha_1 b_1 = \alpha_2 b_2$ implies $\alpha_1 = \alpha_2$. Thereby $\tilde{B}$ is a base of $T(X) = \mathbb{R}_+\tilde{B} - \mathbb{R}_+\tilde{B}$. Since $\tilde{B}$ is superconvex, $T(X)$ is a base ordered Banach space. Obviously the corestriction of $\tilde{\tau}$ to the subspace $T(X)$ is affine. □

Theorem 1.3.25. *Let X be a convex module and let Y be a base normed Banach space with base B. Let $f : X \to B$ be affine. Then there exists a unique linear continuous function $F : T(X) \to Y$, such that $F \circ \tau = f$. This function satisfies $\|F\| \leq 1$.*

Proof. First show that there is a unique affine mapping $\tilde{F} : \tau(X) \to B$ satisfying $\tilde{F} \circ \tau = f$. For such a function to be well-defined $f(x) \neq f(y)$ has to imply $\tau(x) \neq \tau(y)$, for all $x, y \in X$. Let x and y be fixed, such that $f(x) \neq f(y)$. Since Y is a Banach space there is a continuous linear functional $g : Y \to \mathbb{R}$, such that $g(f(x)) \neq g(f(y))$. Since B is bounded, $g \circ f$ is a bounded, affine function to the reals, thus $\tau(x) \neq \tau(y)$. In order to show that $\tilde{F}$ is affine, let $x_j \in \tau(X)$ and let $y_j \in X$ such that $\tau(y_j) = x_j$ for all $j \in \mathbb{N}$. Thereby, $\tilde{F}(\sum_{j=1}^{\infty} \alpha_j x_j) = \tilde{F}(\tau(\sum_{j=1}^{\infty} \alpha_j y_j)) = f(\sum_{j=1}^{\infty} \alpha_j y_j) = \sum_{j=1}^{\infty} \alpha_j f(y_j) = \sum_{j=1}^{\infty} \alpha_j \tilde{F}(x_j)$. According to lemma 1.1.28, there is a unique linear extension $\hat{F} : \mathbb{R}_+\tau(X) - \mathbb{R}_+\tau(X)$ of $\tilde{F}$. Let $x \in \tau(X) - \tau(X)$ with $\|x\| \leq 1$. Since $\mathrm{conv}(\tau(X) \cup (-\tau(X)))$ is dense in $\mathrm{conv}(\mathrm{superconv}(\tau(X)) \cup (-\mathrm{superconv}(\tau(X))))$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{conv}(\tau_{X_1}(X_1) \cup (-\tau_{X_1}(X_1)))$ converging towards x. There are $y_n, z_n \in \tau(X)$ and $\alpha_n, \beta_n \in [0, 1]$, such that $\alpha_n y_n - \beta_n z_n = x_n$ and $\alpha_n + \beta_n = 1$ for all $n \in \mathbb{N}$. Then:

$$\|\hat{F}(x_n)\| = \|\alpha_n \hat{F}(y_n) - \beta_n \hat{F}(z_n)\| \leq \alpha_n \|\hat{F}(y_n)\| + \beta_n \|\hat{F}(z_n)\| \leq \alpha_n + \beta_n = 1$$

Hence $\|\hat{F}(x)\| \leq 1$. Now, $\hat{F}$ is a continuous linear function, defined on a dense subspace of $T(X)$. Hence there exists a unique linear continuous extension $F : T(X) \to Y$. □

Lemma 1.3.26. *Let X be a convex module and let $T(X)$ and $\tilde{B}$ be as above. Then $\overline{B}$ is superconvex and is a base of $T(X)$.*

Proof. As the closure of a convex and bounded set, $\overline{B}$ is superconvex, according to lemma 1.3.9. The closure of B is contained in the same affine hyperplane as B, hence it fulfills that for all $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $b_1, b_2 \in \tilde{B}$, $\alpha_1 b_1 = \alpha_2 b_2$ implies $\alpha_1 = \alpha_2$. Thus $T(X) = \mathbb{R}_+\tilde{B} - \mathbb{R}_+\tilde{B}$ and $\tilde{B} \subseteq \overline{B}$, imply that $\overline{B}$ is a base. Since $\mathrm{conv}(\tilde{B} \cup (-\tilde{B})) \subseteq \mathrm{conv}(\overline{B} \cup (-\overline{B})) \subseteq \overline{\mathrm{conv}(\tilde{B} \cup (-\tilde{B}))}$, the Minkowski-functionals and hence the norms coincide. □

Corollary 1.3.27. *Let X be a superaffine module, then τ is superaffine.*

Proof. Since $\overline{B}$ is a bounded and convex subset of a Banach space, this follows directly from lemma 1.3.10. □

Definition 1.3.28. Let **BNBan** denote the category of base normed Banach space with closed bases and with morphisms linear functions that map the domain's base into the codomain's base. Clearly, **BNBan** is a full subcategory of **BOVec**. Let $\widehat{\mathbf{Bs}}$ denote the restriction of **Bs** to **BNBan**. Let $\mathbf{T}$: **Conv** → **BNBan** denote the functor along τ, which assigns to each convex module X the base normed Banach space $T(X)$ with base $\overline{\tau(X)} = \overline{\tilde{B}}$ and to each affine mapping f, the unique linear contnous extension of $\rho \circ f$ according to theorem 1.3.25.

Remark 1.3.29. Note that, since the extension in continuous ant the bases are closed, $\mathbf{T}(f)$ maps the domain's base into the codomain's base.

Remark 1.3.30. Note that for any convex module that is isomorphic to a closed, bounded, convex subset of a Banach space, $\mathrm{Aff}_b(X)$ is point separating and $\overline{\tau(X)} = \tau(X)$, hence $\widehat{\mathbf{Bs}} \circ \mathbf{T}$ is an isomorphism.

Definition 1.3.31. Let $\hat{\tau}$ denote the corestriction of $\tilde{\tau}$ to $\overline{\tilde{\tau}(X)}$.

Corollary 1.3.32. *$\boldsymbol{T}$: **Conv** → **BNBan** is left adjoint to $\widehat{\boldsymbol{Bs}}$: **BNBan** → **Conv**, i.e. for each convex module X, for each base normed Banach space Y and for each affine $f : X \to \widehat{\boldsymbol{Bs}}(Y)$, there is a unique linear continuous function $g : \boldsymbol{T}(X) \to Y$ that maps $\overline{\tau(X)}$ into the base of Y, such that $\widehat{\boldsymbol{Bs}}(g) \circ \hat{\tau} = f$.*

Proof. Let ι denote the inclusion of $\widehat{\mathbf{Bs}}(Y)$ into Y. According to theorem 1.3.25, there is a unique linear continuous function $g : \mathbf{T}(X) \to Y$, such that $\overline{\tau(X)}$ is mapped into the base of Y and such that $g \circ \tau = \iota \circ f$. Let $x \in X$, then $\mathbf{Bs}(g)(\hat{\tau}(x)) = g(\hat{\tau}(x)) = g(\tau(x)) = \iota(f(x)) = f(x)$. According to lemma 1.1.28, g is the unique function with this property. □

Definition 1.3.33. A convex module X is called separated, if for any $x, y \in X$ there is an affine function $f : X \to [0, 1]$, such that $f(x) \neq f(y)$.

Theorem 1.3.34. *For a convex module X the following statements are equivalent:*

1. *X is metric.*
2. *ρ is injective and conv$(\rho(X) \cup (-\rho(X)))$ is linearly bounded.*
3. *ρ is injective and conv$(B \cup (-B))$ is linearly bounded.*
4. *X is separated.*
5. *τ is injective.*

Proof. The equality of the first three statements has been shown in corollaries 1.2.20, 1.2.22 and 1.2.23.

To show that the third statement implies the fourth, let $x, y \in X$ be arbitrary. Let V be the one dimensional subspace of $R(X)$ generated by $\rho(y) - \rho(x)$ and let $g : V \to \mathbb{R}$ be an arbitrary nonconstant linear function such that $|g(v)| \leq \|v\|$ for all $v \in V$, in which $\|\cdot\|$ denotes the base norm. According to the Hahn-Banach theorem, there exists a linear function G, which extends g to $R(X)$, that satisfies $|G(z)| \leq \|z\|$ for all $z \in Z$. Now, $G(\rho(x)) \subseteq [-1, 1]$, hence $f : X \to [0, 1]$ defined by $f(w) := \frac{1}{2}G(\rho(w)) + \frac{1}{2}$ is an affine function which separates x and y.

To show that the fourth statement implies the first, let $x, y \in X$ be arbitrary, such that $d(x, y) = 0$. Hence there are sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (\alpha_n)_{n\in\mathbb{N}}$, such that $x_n, y_n \in X$, $\alpha_n \in [0, 1]$, $(1-\alpha_n)x+\alpha_n x_n = (1-\alpha_n)y+\alpha_n y_n$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n = 0$. Now, let $f : X \to [0, 1]$ be an affine function. Then, $(1 - \alpha_n)f(x) + \alpha_n f(x_n) = (1 - \alpha_n)f(y) + \alpha_n f(y_n)$ for all $n \in \mathbb{N}$ implies that $(1 - \alpha_n)|f(x) - f(y)| \leq \alpha|f(x_n) - f(y_n)| \leq 2\alpha$ for all $n \in \mathbb{N}$. Hence $f(x) = f(y)$ and since f was arbitrary, $x = y$. Thus, X is metric.

To show that the fourth statement implies the fifth, assume there are $x, y \in X$, such that $x \neq y$ and let $f : X \to [0, 1]$ be an affine function separating x and y. Then, $(\tilde{\tau}(x))(f) = f(x) \neq f(y) = (\tilde{\tau}(y))(f)$. Since τ is the corestriction of $\tilde{\tau}$, it is injective.

To show that, on the other hand, that the fifth implies the fourth statement, let $x, y \in X$, such that $x \neq y$. Since τ is injective, $\tilde{\tau}$ is injective aswell, hence the exists an $g \in \mathrm{Aff}_b(X)$ such that $g(x) = (\tilde{\tau}(x))(g) \neq (\tilde{\tau}(y))(g) = g(y)$. Let $c \in \mathbb{R}_{++}$ such that $\|g\|_\infty \leq c$. Let $f(z) := \frac{1}{2c}g(z) + \frac{1}{2}$. Clearly f is affine and separates x and y. □

Lemma 1.3.35. *Let X be a convex module. If τ in injective, τ is isometric.*

Proof. Consider $\mathbb{R}_+\tau(X)-\mathbb{R}_+\tau(X)$ as a subspace of $T(X)$. Since $\tau(X)$ is a convex subset of the base $\overline{B}$ and it generates $\mathbb{R}_+\tau(X)-\mathbb{R}_+\tau(X)$, $\tau(X)$ is a base of $\mathbb{R}_+\tau(X)-\mathbb{R}_+\tau(X)$. Since $\mathrm{conv}(\tau(X)\cup(-\tau(X)))$ is a dense subset of $\mathrm{conv}(\overline{B}\cup(-\overline{B}))$, the base norms coincide. Since τ is injective, $\tau(X)$ is isomorphic to X. According to proposition 1.2.19, the metric on $\overline{B}$ coincides on $\tau(X)$ with the convex metric on $\tau(X)$, hence τ is isometric. □

Remark 1.3.36. Clearly, the corestriction of an affine isometry to its image is an isomorphism.

Theorem 1.3.37. *Any convex module X is metric and complete, if and only if $\hat{\tau}$ is isomorphic.*

Proof. If $\hat{\tau}$ is isomorphic, X is isomorphic to a closed and bounded subset of a Banach space and hence metric and complete. If X is metric and complete, $\tau(X)$ is a closed, bounded, convex subset of the Banach space $T(X)$, since $\tau(X)$ is isomorphic to X and therefore complete. Thereby, $\tau(X) = \text{superconv}(\tau(X)) = \overline{\text{superconv}(\tau(X))}$. Thus, X is surjective and hence isomorphic. □

Corollary 1.3.38. *Any complete metric convex module is superaffine.*

Proof. According to the preceding theorem, any complete metric convex module is isomorphic to a closed, bounded, convex subset of a Banach space. □

Corollary 1.3.39. *Let X and Y be complete metric convex modules and let $f : X \to Y$ be affine. Then f is superaffine.*

Proof. Any affine function between closed, bounded convex subsetes of Banach spaces can be regarded as a superconvex function. □

Definition 1.3.40. Let **ComplConv** denote the category of complete metric convex modules with morphisms affine (and therefore superaffine) functions.

Remark 1.3.41. Clearly, **ComplConv** is a full and faithful subcategory of **Conv**.

Theorem 1.3.42. ***ComplConv*** *is a reflective subcategory of* ***Conv*** *with reflection* $\widehat{\boldsymbol{Bs}} \circ \boldsymbol{T}$*, i.e. for any* $X \in$ ***Conv****, any* $Y \in$ ***ComplConv*** *and any affine function* $f : X \to Y$*, there is a unique affine function* $g : \widehat{\boldsymbol{Bs}} \circ \boldsymbol{T}(X) \to Y$*, such that* $g \circ \hat{\tau} = f$*.*

Proof. Let $\tilde{f} := \hat{\tau} \circ f$. According to theorem 1.3.32, there is a unique affine function $\tilde{g} : \mathbf{T}(X) \to Y$, such that $\tilde{g} \circ \hat{\tau} = \tilde{f}$. Since Y is complete and hence $\hat{\tau}$ is an isomorphism, $g := \hat{\tau}^{-1} \circ \tilde{g}$ uniquely satisfies $g \circ \hat{\tau} = \hat{\tau}^{-1} \circ \tilde{f} = f$. □

Chapter 2

Positively Convex Modules

In this chapter positively convex modules, i.e. convex modules with a distinguished element called 0, are discussed. In the first part basic definition are given and some results about convex modules are transferred.

In the second section a semimetric is introduced. Metric positively convex modules are characterized and a metrization functor is constructed. This construction coincides with the construction in [12].

In the final section the metrization functor is extended to a completion functor. This part of the construction is different from the construction in [12], since one of the proofs in [12] is faulty.

2.1 Positively Convex Modules and Positively Superconvex Modules

Definition 2.1.1. A pair consisting of a convex module X and an element $0 \in X$ is called a positively convex module. A pair of a superconvex module X and an element $0 \in X$ is called a positively superconvex module. In both cases 0 is called the zero element of X.

For the remainder of this paper, that zero element will always be denoted by 0.

Definition 2.1.2. Let X be a positively convex or positively superconvex module. Let $(\alpha_j)_{j\in\mathbb{N}}$, such that $\alpha_j \in [0,1]$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} \alpha_j < 1$. Let $(x_j)_{j\in\mathbb{N}}$, such that

$x_j \in X$ for all $j \in \mathbb{N}$. In case $\sum_{j=1}^{\infty} \alpha_j \neq 0$, define:

$$\sum_{j=1}^{\infty} \alpha_j x_j := (\sum_{j=1}^{\infty} \alpha_j) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{j=1}^{\infty} \alpha_j} x_j + (1 - \sum_{j=1}^{\infty} \alpha_j) 0$$

In case $\sum_{j=1}^{\infty} \alpha_j = 0$, define:

$$\sum_{j=1}^{\infty} \alpha_j x_j := 0$$

Remark 2.1.3. Some authors do define positively convex modules as a pair of a set X and a function $c : \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha + \beta \leq 1\} \times X \times X \to X$, such that $c(\alpha, \beta, x, y) = \alpha x + \beta y$ satisfying analogous axioms as the convex combination in 1.1.1. To show that these definitions are equivalent, one just has to repeat the steps at the beginning of the previous chapter.

Example 2.1.4. Any real vector space with 0 being the zero vector and the usual addition clearly is a positively convex module.

From now on whenever a positively convex structure on real vector space is considered, the zero element is assumed to be the zero vector.

Definition 2.1.5. Any affine function is called positively affine if it maps its domain's zero element to its codomain's zero element. Let **PosConv** denote the category of positively convex modules with morphisms positively affine functions.

Lemma 2.1.6. *Let X_1, X_2 be positively convex modules and let $f : X_1 \to X_2$. The function f is positively affine if and only if for all sequences $(\alpha_j)_{j \in \mathbb{N}}$, such that $\alpha_j \in [0, 1]$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} \alpha_j < 1$ and all $(x_j)_{j \in \mathbb{N}}$, such that $x_j \in X_1$ for all $j \in \mathbb{N}$ the following equation holds:*

$$f(\sum_{j=1}^{\infty} \alpha_j x_j) = \sum_{j=1}^{\infty} \alpha_j f(x_j)$$

Proof. Let all $\alpha_j = 0$, then the equation reads $f(0) = 0$, hence it maps the zero element to the zero element. Since the equation clearly implies the affinity of the f, it is positively convex. To show the other direction, first assume $\sum_{j=1}^{\infty} \alpha_j \neq 0$, then:

$$f(\sum_{j=1}^{\infty} \alpha_j x_j) = (\sum_{j=1}^{\infty} \alpha_j) f(\sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{j=1}^{\infty} \alpha_j} x_j) + (1 - \sum_{j=1}^{\infty} \alpha_j) f(0) =$$

$$= (\sum_{j=1}^{\infty} \alpha_j) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{j=1}^{\infty} \alpha_j} f(x_j) + 0 = \sum_{j=1}^{\infty} \alpha_j f(x_j)$$

The case $\sum_{j=1}^{\infty} \alpha_j \neq 0$ reads $f(0) = 0$, which was the premise. $\square$

Definition 2.1.7. A real vector space X is called preordered vector space with order $\leq$, if $\leq$ is a preorder that fulfills:

1. $x \leq y$ implies $\alpha x \leq \alpha y$, for all $x, y \in X$, $\alpha \in \mathbb{R}_+$.
2. $x \leq y$ implies $x + z \leq y + z$, for all $x, y, z \in X$.

Remark 2.1.8. For any preordered vector space X, $C := \{x \in X : 0 \leq x\}$ defines a cone. On the other hand, for any given cone C, $x \leq y \leftrightarrow (y - x) \in C$ defines a preorder satisfying the condition for a preordered vector space. Hence a preordered vector space could equally be defined by giving a cone.

Next, theorems similar to 1.1.29 and 1.1.32 are proved.

Definition 2.1.9. Let **POVec** denote the category with objects preordered vector spaces and with morphisms linear functions, such that the image of the domain's cone is contained in the codomain's cone.

Definition 2.1.10. Let X be a positively convex module. Then, $\mathrm{PosConv}(X, \mathbb{R})$, i.e. the set of positively affine functions to the reals, is a real vector space. Let $\mathrm{PosConv}(X, \mathbb{R})^*$ denote its algebraic dual. Define $\tilde{\psi} : X \to \mathrm{Conv}(X, \mathbb{R})^*$ as $(\tilde{\psi}(x))(f) = f(x)$. Let $P(X)$ denote the subspace of $\mathrm{Conv}(X, \mathbb{R})^*$ generated by $\tilde{\psi}(X)$. Let ψ be defined as the corestriction of $\tilde{\psi}$ to $P(X)$ and let $\hat{\psi}$ be defined as the corestriction of $\tilde{\psi}$ to its image.

Lemma 2.1.11. *ψ is positively affine and $P(X)$ is a preordered vector space with cone $\mathbb{R}_+\psi(X)$ and $\mathbb{R}_+\psi(X) - \mathbb{R}_+\psi(X) = P(X)$.*

Proof. $\psi(x)$ is defined by acting on affine functions f therefore,

$$\psi(\sum_{j=1}^{\infty} \alpha_j x_j)(f) = f(\sum_{j=1}^{\infty} \alpha_j x_j) = \sum_{j=1}^{\infty} \alpha_j f(x_j) = (\sum_{j=1}^{\infty} \alpha_j \psi(x_j))(f)$$

Thus ψ is positively affine and $\psi(X)$ is convex. Let $x, y \in X$ and $\lambda, \mu \in \mathbb{R}_{++}$, then $\lambda x + \mu y = (\lambda + \mu)(\frac{\lambda}{\lambda+\mu} x + \frac{\mu}{\lambda+\mu} y) \in \mathbb{R}_+\psi(X)$. Thus $\mathbb{R}_+\psi(X)$ is a cone and hence its span coincides with $\mathbb{R}_+\psi(X) - \mathbb{R}_+\psi(X)$. □

Lemma 2.1.12. *X is preseparated if and only if ψ is injective.*

Proof. Assume that $x \neq y$ and $\alpha x + (1 - \alpha)z = \alpha y + (1 - \alpha)z$ with $\alpha \in (0, 1]$, then

$$\alpha\psi(x) + (1 - \alpha)\psi(z) = \psi(\alpha x + (1 - \alpha)z) = \psi(\alpha y + (1 - \alpha)) = \alpha\psi(x) + (1 - \alpha)\psi(z)$$

Therefore $\psi(x) = \psi(y)$ and consequently ψ is not injective. On the other hand if X is preseparated, then X is isomorphic to a convex subset of some real vector space and thus for any $x \neq y$ with $x, y \in X$ there exists a function $\tilde{f} \in \mathrm{Conv}(X, \mathbb{R})$ such that $\tilde{f}(x) \neq \tilde{f}(y)$. Then, $f(z) := \tilde{f}(z) - \tilde{f}(0)$ is positively affine. Hence $\psi(x)(f) \neq \psi(y)(f)$. □

Corollary 2.1.13. *X is preseparated if and only if the corestriction of ψ to its image is an isomorphism.*

Proof. This follows directly from lemmata 2.1.12 and 1.1.10. □

Lemma 2.1.14. *Let X, Y be a real vector space and let $P \subseteq X$ and $Q \subseteq Y$ be positively convex sets. Let $f : P \to Q$ be positively affine. Then there is a unique linear extension $F : span(P) \to Y$, such that $F(x) = f(x)$ for all $x \in P$.*

Proof. Let $(\lambda_j)_{j\in\mathbb{N}}, (\mu_j)_{j\in\mathbb{N}} \in \mathbb{R}_+$, such that $\sum_{j=0}^{\infty} \lambda_j < \infty$, that $\sum_{j=0}^{\infty} \mu_j < \infty$ and that all but finitely many summands are equal to zero in both cases. Let $(p_j)_{j\in\mathbb{N}}, (q_j)_{j\in\mathbb{N}} \in P$, then define $F : \mathrm{span}(P) \to Y$ as:

$$F(\sum_{j=0}^{\infty} \lambda_j p_j - \sum_{j=0}^{\infty} \mu_j q_j) := (\sum_{j=0}^{\infty} \lambda_j) f(\sum_{j=0}^{\infty} \frac{\lambda_j}{\sum_{j=0}^{\infty} \lambda_j} p_j) - (\sum_{j=0}^{\infty} \mu_j) f(\sum_{j=0}^{\infty} \frac{\mu_j}{\sum_{j=0}^{\infty} \mu_j} q_j)$$

Since f is additive and homogeneous, F is linear. Thereby, for F to be well-defined it suffices to show that $F(0)$ is independent of its representation. Let $\sum_{j=0}^{\infty} \lambda_j p_j - \sum_{j=0}^{\infty} \mu_j q_j$ be a representation for 0, then:

$$F(\sum_{j=0}^{\infty} \lambda_j p_j - \sum_{j=0}^{\infty} \mu_j q_j) = (\sum_{i=0}^{\infty} \lambda_i) f(\sum_{j=0}^{\infty} \frac{\lambda_j}{\sum_{k=0}^{\infty} \lambda_k} p_j) - (\sum_{i=0}^{\infty} \mu_i) f(\sum_{j=0}^{\infty} \frac{\mu_j}{\sum_{k=0}^{\infty} \mu_k} q_j) =$$

$$= (\sum_{i=0}^{\infty} \lambda_i) f\left(\frac{\sum_{k=0}^{\infty} \lambda_k + \sum_{k=0}^{\infty} \mu_k}{\sum_{m=0}^{\infty} \lambda_m} \sum_{j=0}^{\infty} \frac{\lambda_j}{\sum_{n=0}^{\infty} \lambda_n + \sum_{n=0}^{\infty} \mu_n} p_j\right) -$$

$$(\sum_{i=0}^{\infty} \mu_i) f\left(\frac{\sum_{k=0}^{\infty} \lambda_k + \sum_{k=0}^{\infty} \mu_k}{\sum_{m=0}^{\infty} \mu_m} \sum_{j=0}^{\infty} \frac{\mu_j}{\sum_{n=0}^{\infty} \lambda_n + \sum_{n=0}^{\infty} \mu_n} q_j\right) =$$

$$= (\sum_{k=0}^{\infty} \lambda_k + \sum_{k=0}^{\infty} \mu_k) f(\sum_{j=0}^{\infty} \frac{\lambda_j}{\sum_{n=0}^{\infty} \lambda_n + \sum_{n=0}^{\infty} \mu_n} p_j) -$$

$$(\sum_{k=0}^{\infty} \lambda_k + \sum_{k=0}^{\infty} \mu_k) f(\sum_{j=0}^{\infty} \frac{\mu_j}{\sum_{n=0}^{\infty} \lambda_n + \sum_{n=0}^{\infty} \mu_n} q_j)$$

Since $\sum_{j=0}^{\infty} \frac{\lambda_j}{\sum_{n=0}^{\infty} \lambda_n + \sum_{n=0}^{\infty} \mu_n} p_j = \sum_{j=0}^{\infty} \frac{\mu_j}{\sum_{n=0}^{\infty} \lambda_n + \sum_{n=0}^{\infty} \mu_n} q_j$, this verifies that $F(0) = 0$. Since any different extension has to satisfy the defining equation of F, this extension is unique. □

Theorem 2.1.15. *Let X be a positively convex module, let Y be a preordered vector space with cone C and let $f : X \to C$ be affine. Then there is a unique linear mapping $F : P(X) \to Y$, such that $F \circ \psi = f$ and $F(\mathbb{R}_+\psi(X)) \subseteq C$.*

Proof. First, show that there is a unique affine mapping $\tilde{F} : \psi(X) \to Y$ satisfying $\tilde{F} \circ \psi = f$. For such a function to be well-defined $f(x) \neq f(y)$ has to imply $\psi(x) \neq \psi(y)$, for all $x, y \in X$. Let x and y be, such that $f(x) \neq f(y)$. Since Y is a real linear space there is a linear functional $g : Y \to \mathbb{R}$, such that $g(f(x)) \neq g(f(y))$. Since $g \circ f$ is a positively affine function to the reals, $\psi(x) \neq \psi(y)$. In order to show that $\tilde{F}$ is affine, let $x_j \in \psi(X)$ and let $y \in X$ such that $\psi(y_j) = x_j$ for all $j \in \mathbb{N}$. Thereby, $\tilde{F}(\sum_{j=1}^{\infty} \alpha_j x_j) = \tilde{F}(\psi(\sum_{j=1}^{\infty} \alpha_j y_j)) = f(\sum_{j=1}^{\infty} \alpha_j y_j) = \sum_{j=1}^{\infty} \alpha_j f(y_j) = \sum_{j=1}^{\infty} \alpha_j \tilde{F}(x_j)$. Since $P(X)$ is a preordered vector space with generating cone $\mathbb{R}_+\psi(X)$, there is a unique linear extension F of $\tilde{F}$, according to the preceding lemma. Clearly, $\psi(X) \subseteq C$ implies $\mathbb{R}_+\psi(X) \subseteq C$. $\square$

Definition 2.1.16. Let $\mathbf{P} : \mathbf{Conv} \to \mathbf{POVec}$ denote the functor along ψ, which assigns to each convex module X the preordered vector space $P(X)$ and to each affine mapping $f : X \to Y$, the unique linear mapping $F : P(X) \to P(Y)$ uniquely satisfying $F \circ \psi_X = \psi_Y \circ f$ according to the preceding theorem. Let $\mathbf{Cone} : \mathbf{POVec} \to \mathbf{PosConv}$ denote the functor that maps a preordered vector spaces to its cone and morphisms to their corresponding restrictions.

Remark 2.1.17. Note that $\hat{\psi} : X \to \mathbf{Cone} \circ \mathbf{P}(X)$ constitutes an affine function.

Corollary 2.1.18. *$\boldsymbol{P} : \boldsymbol{Conv} \to \boldsymbol{POVec}$ is left adjoint to $\boldsymbol{Cone} : \boldsymbol{POVec} \to \boldsymbol{Conv}$, i.e. for each convex module X, for each preordered vector space Y and for each affine $f : X \to \boldsymbol{Cone}(Y)$, there is a unique linear function $g : \boldsymbol{P}(X) \to Y$ that maps $\mathbb{R}_+\psi(X)$ into the cone of Y, such that $\boldsymbol{Cone}(g) \circ \hat{\psi} = f$.*

Proof. Let ι denote the inclusion of $\mathbf{Cone}(Y)$ into Y. According to the preceding theorem, there is a unique linear function $g : \mathbf{P}(X) \to Y$, such that $\mathbb{R}\psi(X)$ is mapped into the cone of Y and such that $g \circ \psi = \iota \circ f$. Let $x \in X$, then $\mathbf{Cone}(g)(\hat{\psi}(x)) = g(\hat{\psi}(x)) = g(\psi(x)) = \iota(f(x)) = f(x)$, hence g uniquely satisfies this equation. $\square$

2.2 A Semimetric on Positively Convex Modules

It is possible to use the same semimetric for positively convex modules as we did in the previous chapter for convex modules, but instead a new semimetric is introduced which has some properties in common with a norm, such that 0 is the zero element in regard to the norm.

Definition 2.2.1. Let X be a positively convex module. Define $d_P : X \times X \to \mathbb{R}_+$ as $d_P(x,y) := \inf\{\frac{\alpha}{2\beta} : \alpha, \beta \in \mathbb{R}_{++}; \mu, \nu \in \mathbb{R}_+; u, v, w \in X; \alpha + \beta \leq 1; \mu + \beta \leq 1; \nu + \beta \leq 1; \beta x + \alpha w = \beta y + \nu v; \beta y + \alpha w = \beta x + \mu u\}$ and call it the positively convex semimetric on X.

Theorem 2.2.2. *d_P is a semimetric.*

Proof. Clearly d_P is symmetric. For the triangle inequality let X be a positively convex module, let $x_1, x_2, y \in X$ and let $\epsilon > 0$ be fixed. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1]$, $\mu_1, \mu_2, \nu_1, \nu_2 \in [0, 1]$, $u_1, u_2, v_1, v_2, w_1, w_2 \in X$, such that $\frac{\alpha_1}{2\beta_1} \leq d_P(x_1, y) + \epsilon$, $\alpha_1 + \beta_1 \leq 1$, $\mu_1 + \beta_1 \leq 1$, $\nu_1 + \beta_1 \leq 1$, $\beta_1 x_1 + \alpha_1 w_1 = \beta_1 y + \nu_1 v_1$, $\beta_1 y + \alpha_1 w_1 = \beta_1 x_1 + \mu_1 u_1$, $\frac{\alpha_2}{2\beta_2} \leq d_P(x_2, y) + \epsilon$, $\alpha_2 + \beta_2 \leq 1$, $\mu_2 + \beta_2 \leq 1$, $\nu_2 + \beta_2 \leq 1$, $\beta_2 x_2 + \alpha_2 w_2 = \beta_2 y + \nu_2 v_2$ and $\beta_2 y + \alpha_2 w_2 = \beta_2 x_2 + \mu_2 u_2$. The following inequalities hold:

$$\beta_1\beta_2 + \alpha_1\beta_2 + \alpha_2\beta_1 = (\beta_1 + \alpha_1)\beta_2 + \alpha_2\beta_1 \leq \beta_2 + \alpha_2\beta_1 \leq \beta_2 + \alpha_2 \leq 1$$

$$\beta_1\beta_2 + \nu_1\beta_2 + \mu_2\beta_1 = (\beta_1 + \nu_1)\beta_2 + \mu_2\beta_1 \leq \beta_2 + \mu_2\beta_1 \leq \beta_2 + \mu_2 \leq 1$$

$$\beta_1\beta_2 + \mu_1\beta_2 + \nu_2\beta_1 = (\beta_1 + \mu_1)\beta_2 + \nu_2\beta_1 \leq \beta_2 + \nu_2\beta_1 \leq \beta_2 + \nu_2 \leq 1$$

Next, let $u_3 := \frac{\nu_1\beta_2}{\nu_1\beta_2+\mu_2\beta_1} v_1 + \frac{\mu_2\beta_1}{\nu_1\beta_2+\mu_2\beta_1} u_2$, let $u_4 := \frac{\mu_1\beta_2}{\mu_1\beta_2+\nu_2\beta_1} u_1 + \frac{\nu_2\beta_1}{\mu_1\beta_2+\nu_2\beta_1} v_2$ and let $w_3 := \frac{\alpha_1\beta_2}{\alpha_1\beta_2+\alpha_2\beta_1} w_1 + \frac{\alpha_2\beta_1}{\alpha_1\beta_2+\alpha_2\beta_1} w_2$. Then, the following equations hold true, since the sum of coefficients of any occurring positively convex combination is less than 1:

$$\beta_1\beta_2 x_1 + (\alpha_1\beta_2 + \alpha_2\beta_1) w_3 = \beta_1\beta_2 x_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)(\frac{\alpha_1\beta_2}{\alpha_1\beta_2 + \alpha_2\beta_1} w_1 + \frac{\alpha_2\beta_1}{\alpha_1\beta_2 + \alpha_2\beta_1} w_2) =$$

$$= \beta_1\beta_2 x_1 + \alpha_1\beta_2 w_1 + \alpha_2\beta_1 w_2 = \beta_2(\beta_1 x_1 + \alpha_1 w_1) + \alpha_2\beta_1 w_2 = \beta_2(\beta_1 y + \nu_1 v_1) + \alpha_2\beta_1 w_2 =$$

$$= \beta_1\beta_2 y + \beta_2\nu_1 v_1 + \alpha_2\beta_1 w_2 = \beta_1(\beta_2 y + \alpha_2 w_2) + \beta_2\nu_1 v_1 = \beta_1(\beta_2 x_2 + \mu_2 u_2) + \beta_2\nu_1 v_1 =$$

$$= \beta_1\beta_2 x_2 + (\nu_1\beta_2 + \mu_2\beta_1)(\frac{\nu_1\beta_2}{\nu_1\beta_2 + \mu_2\beta_1} v_1 + \frac{\mu_2\beta_1}{\nu_1\beta_2 + \mu_2\beta_1} u_2) = \beta_1\beta_2 x_2 + (\nu_1\beta_2 + \mu_2\beta_1) u_3$$

$$\beta_1\beta_2 x_2 + (\alpha_1\beta_2 + \alpha_2\beta_1) w_3 = \beta_1\beta_2 x_2 + (\alpha_1\beta_2 + \alpha_2\beta_1)(\frac{\alpha_1\beta_2}{\alpha_1\beta_2 + \alpha_2\beta_1} w_1 + \frac{\alpha_2\beta_1}{\alpha_1\beta_2 + \alpha_2\beta_1} w_2) =$$

$$= \beta_1\beta_2 x_2 + \alpha_1\beta_2 w_1 + \alpha_2\beta_1 w_2 = \beta_1(\beta_2 x_2 + \alpha_2 w_2) + \alpha_1\beta_2 w_1 = \beta_1(\beta_2 y + \nu_2 v_2) + \alpha_1\beta_2 w_1 =$$

$$= \beta_1\beta_2 y + \beta_1\nu_2 v_2 + \alpha_1\beta_2 w_1 = \beta_2(\beta_1 y + \alpha_1 w_1) + \beta_1\nu_2 v_2 = \beta_2(\beta_1 x_1 + \mu_1 u_1) + \beta_1\nu_2 v_2 =$$

$$= \beta_1\beta_2 x_1 + (\mu_1\beta_2 + \nu_2\beta_1)(\frac{\mu_1\beta_2}{\mu_1\beta_2 + \nu_2\beta_1} u_1 + \frac{\nu_2\beta_1}{\mu_1\beta_2 + \nu_2\beta_1} v_2) = \beta_1\beta_2 x_1 + (\mu_1\beta_2 + \nu_2\beta_1) u_4$$

Hence, with $\alpha_3 := \alpha_1\beta_2 + \alpha_2\beta_1$, $\beta_3 := \beta_1\beta_2$, $\mu_3 := \nu_1\beta_2 + \mu_2\beta_1$ and $\mu_4 := \mu_1\beta_2 + \nu_2\beta_1$, one gets $\beta_3 x_1 + \alpha_3 w_3 = \beta_3 x_2 + \mu_3 u_3$ and $\beta_3 x_2 + \alpha_3 w_3 = \beta_3 x_1 + \mu_4 u_4$. Thus, $d_P(x_1, x_2) \leq \frac{\alpha_3}{2\beta_3} = \frac{\alpha_1\beta_2+\alpha_2\beta_1}{2\beta_1\beta_2} = \frac{\alpha_1}{2\beta_1} + \frac{\alpha_2}{2\beta_2} \leq d_P(x_1, y) + d_P(x_2, y) + 2\epsilon$. Since ϵ was arbitrary, this asserts the inequality. □

Remark 2.2.3. Let $u := y$, let $v := x$, let $w := \frac{1}{2}x + \frac{1}{2}y$, let $\alpha = \mu = \nu = \frac{2}{3}$ and let $\beta = \frac{1}{3}$. Then $\alpha+\beta = 1$, $\mu+\beta = 1$, $\nu+\beta = 1$, $\beta x+\alpha w = \frac{1}{3}x+\frac{2}{3}(\frac{1}{2}x+\frac{1}{2}y) = \frac{1}{3}y+\frac{2}{3}x = \beta y+\nu v$, $\beta y + \alpha w = \frac{1}{3}y + \frac{2}{3}(\frac{1}{2}x + \frac{1}{2}y) = \frac{1}{3}x + \frac{2}{3}y = \beta x + \mu u$. Hence $d(x, y) \leq 1$ for all $x, y \in X$.

Lemma 2.2.4. *Let X be a positively convex module. For all $x, y \in X$, $\lambda \in [0, 1]$, $d_P(\lambda x, \lambda y) = \lambda d_P(x, y)$.*

Proof. Let $\epsilon > 0$. Let $\alpha, \beta \in (0, 1]$, $\mu, \nu \in [0, 1]$ and $u, v, w \in X$, such that $\frac{\alpha}{2\beta} \leq d_P(x, y)+\epsilon$, $\alpha+\beta \leq 1$, $\mu+\beta \leq 1$, $\nu+\beta \leq 1$, $\beta x+\alpha w = \beta y+\nu v$ and $\beta y+\alpha w = \beta x+\mu u$. Then, $\beta\lambda x + \alpha\lambda w = \beta\lambda y + \nu\lambda v$ and $\beta y + \alpha w = \beta x + \mu u$. Also, $\alpha\lambda + \beta \leq 1$, $\mu\lambda + \beta \leq 1$ and $\nu\lambda+\beta \leq 1$, hence $d_P(\lambda x, \lambda y) \leq \frac{\alpha\lambda}{2\beta} \leq \lambda d_P(x, y)+\lambda\epsilon$. Thus, $d_P(\lambda x, \lambda y) \leq \lambda d_P(x, y)$.

For the other direction, first consider the case $\lambda \neq 0$. Let $\epsilon > 0$. Let $\alpha, \beta \in (0, 1]$, $\mu, \nu \in [0, 1]$ and $u, v, w \in X$, such that $\frac{\alpha}{2\beta} \leq d_P(\lambda x, \lambda y) + \epsilon$, $\alpha + \beta \leq 1$, $\mu + \beta \leq 1$, $\nu + \beta \leq 1$, $\beta\lambda x + \alpha w = \beta\lambda y + \nu v$ and $\beta\lambda y + \alpha w = \beta\lambda x + \mu u$. Then $\alpha + \beta\lambda \leq 1$, $\mu + \beta\lambda \leq 1$ and $\nu + \beta\lambda \leq 1$, hence $d_P(x, y) \leq \frac{\alpha}{2\beta\lambda} \leq \frac{1}{\lambda}d_P(\lambda x, \lambda y) + \frac{\epsilon}{\lambda}$. Thereby, $\lambda d_P(x, y) \leq d_P(\lambda x, \lambda y)$. The case $\lambda = 0$ is trivial, since d_P is a semimetric. □

Proposition 2.2.5. *Let X_1, X_2 be positively convex modules. Let $f : X_1 \to X_2$ be a positively affine function. Then f satisfies for all $x, y \in X_1$:*

$$d_{P,2}(f(x), f(y)) \leq d_{P,1}(x, y)$$

Proof. Let $\epsilon > 0$. Let $\alpha, \beta \in (0, 1]$, $\mu, \nu \in [0, 1]$ and $u, v, w \in X_1$, such that $\frac{\alpha}{2\beta} \leq d_{P,1}(x, y)+\epsilon$, $\alpha+\beta \leq 1$, $\mu+\beta \leq 1$, $\nu+\beta \leq 1$, $\beta x+\alpha w = \beta y+\nu v$ and $\beta y+\alpha w = \beta x+\mu u$. Then, $\beta f(x)+\alpha f(w) = f(\beta x+\alpha w) = f(\beta y+\nu v) = \beta f(y)+\nu f(v)$ and $\beta f(y)+\alpha f(w) = f(\beta y+\alpha w) = f(\beta x+\mu u) = \beta f(x)+\mu f(u)$, hence $d_{P,2}(f(x), f(y)) \leq \frac{\alpha}{2\beta} \leq d_{P,1}(x, y)+\epsilon$. Since ϵ was arbitrary, this meets the assertion. □

Proposition 2.2.6. *Let X be a positively convex module. All $x, y \in X$ fulfill the following equation:*

$$d_P(x, y) \leq d(x, y)$$

Proof. Let $\epsilon > 0$ be fixed and let $x, y \in X$. Let $\alpha \in [0, 1]$, $\tilde{x}, \tilde{y} \in X$, such that $\frac{\alpha}{1-\alpha} \leq d(x, y) + \epsilon$ and that $(1 - \alpha)x + \alpha\tilde{x} = (1 - \alpha)y + \alpha\tilde{y}$. Let $u := \tilde{x}$, let $v := \tilde{y}$ and let $w := \frac{1}{2}\tilde{x} + \frac{1}{2}\tilde{y}$. Let $\tilde{\alpha} = \mu = \nu := \frac{2\alpha}{1+\alpha}$ and let $\beta := \frac{1-\alpha}{1+\alpha}$. Then the following equations hold true:

$$\beta x + \tilde{\alpha} w = \frac{1-\alpha}{1+\alpha}x + \frac{2\alpha}{1+\alpha}(\frac{1}{2}\tilde{x} + \frac{1}{2}\tilde{y}) = \frac{1}{1+\alpha}((1-\alpha)x + \alpha\tilde{x}) + \frac{\alpha}{1+\alpha}\tilde{y} =$$

$$= \frac{1}{1+\alpha}((1-\alpha)y + \alpha\tilde{y}) + \frac{\alpha}{1+\alpha}\tilde{y} = \frac{1-\alpha}{1+\alpha}y + \frac{2\alpha}{1+\alpha}\tilde{y} = \beta y + \nu v$$

$$\beta y + \tilde{\alpha} w = \frac{1-\alpha}{1+\alpha} y + \frac{2\alpha}{1+\alpha}(\frac{1}{2}\tilde{x} + \frac{1}{2}\tilde{y}) = \frac{1}{1+\alpha}((1-\alpha)y + \alpha\tilde{y}) + \frac{\alpha}{1+\alpha}\tilde{x} =$$
$$= \frac{1}{1+\alpha}((1-\alpha)x + \alpha\tilde{x}) + \frac{\alpha}{1+\alpha}\tilde{x} = \frac{1-\alpha}{1+\alpha}x + \frac{2\alpha}{1+\alpha}\tilde{x} = \beta x + \mu u$$

Since $\tilde{\alpha} + \beta = \mu + \beta = \nu + \beta = \frac{2\alpha}{1+\alpha} + \frac{1-\alpha}{1+\alpha} = 1$, this implies that $d_P(x, y) \leq \frac{\tilde{\alpha}}{2\beta} = \frac{\alpha}{1-\alpha} \leq d(x, y) + \epsilon$. Hence $d_P(x, y) \leq d(x, y)$. □

The following example shows that the inverse inequation does not hold true in general.

Example 2.2.7. Let $X := \{(x, y) \in [-1, 1] \times [0, 1] : x^2 \leq y\}$. As a convex subset of real linear space that includes 0, this obviously defines a positively convex module. Let $t \in [0, 1]$ be arbitrary, let $x := (-t, t^2)$, let $y := (t, t^2)$ and let $w := (0, t)$. Let $\mu = \nu = \frac{1}{2}$. For all $\epsilon \in (0, 1]$, let $u_\epsilon := (\epsilon t, \epsilon^2 t^2)$, let $v_\epsilon := (-\epsilon t, \epsilon^2 t^2)$, let $\beta_\epsilon := \frac{\epsilon}{4}$, let $\alpha_\epsilon := \frac{\epsilon^2}{2}$. Then:

$$\beta_\epsilon x + \alpha_\epsilon w = \frac{\epsilon}{4}(-t, t^2) + \frac{\epsilon^2}{2}(0, t) = (-\frac{\epsilon t}{4}, \frac{\epsilon t^2}{4} + \frac{\epsilon^2 t}{2}) = \frac{\epsilon}{4}(t, t^2) + \frac{1}{2}(-\epsilon t, \epsilon^2 t^2) = \beta_\epsilon y + \nu_\epsilon v_\epsilon$$

$$\beta_\epsilon y + \alpha_\epsilon w = \frac{\epsilon}{4}(t, t^2) + \frac{\epsilon^2}{2}(0, t) = (\frac{\epsilon t}{4}, -\frac{\epsilon t^2}{4} + \frac{\epsilon^2 t}{2}) = \frac{\epsilon}{4}(-t, t^2) + \frac{1}{2}(\epsilon t, \epsilon^2 t^2) = \beta_\epsilon x + \mu_\epsilon u_\epsilon$$

Since $\alpha_\epsilon + \beta_\epsilon = \frac{\epsilon^2}{2} + \frac{\epsilon}{4} \leq 1$, $\mu + \beta_\epsilon = \frac{1}{2} + \frac{\epsilon}{4} \leq 1$ and $\nu + \beta_\epsilon = \frac{1}{2} + \frac{\epsilon}{4} \leq 1$, $d_P(x, y) \leq \frac{\alpha_\epsilon}{2\beta_\epsilon} = \epsilon$. Since ϵ was arbitrary, $d_P(x, y) = 0$. On the other hand, $\{1\} \times X$ is a base of $\mathbb{R}^3$ and according to proposition 1.2.19, the convex metric coincides with the base norm up to a factor 2. In particular $x \neq y$ implies $d(x, y) \neq 0$. (Furthermore, one can calculate that $d_P(x, y) = t$.) Hence the inverse of the inequation of the preceding proposition does not hold true in general.

Corollary 2.2.8. *Let X be a positively convex module. If d_P is a metric, then X is preseparated.*

Proof. According to the preceding proposition, d_P being a metric implies that d is a metric and thus X is preseparated. □

Definition 2.2.9. An normed and ordered vector space X with cone C and norm $\|\cdot\|$ is called regularly ordered if the following equation is fulfilled for all $x \in X$:

$$\|x\| = \inf\{\|c\| : c \in C; -c \leq x \leq c\}$$

Remark 2.2.10. Note that this implies that C is generating.

Proposition 2.2.11. *Let X be a regularly ordered vector space with cone C and norm $\|\cdot\|$. Let P be positively affine, such that $\{c \in C : \|c\| < 1\} \subseteq P \subseteq \{c \in C : \|c\| \leq 1\}$. Then the positively convex metric d_P on P and the norm fulfill for all $x, y \in P$:*

$$d_P(x, y) = \frac{1}{2}\|x - y\|$$

Proof. Let $\epsilon > 0$. Let $\alpha, \beta \in (0,1]$, $\mu, \nu \in [0,1]$ and $u, v, w \in P$, such that $\frac{\alpha}{2\beta} \leq d_P(x,y) + \epsilon$, $\alpha + \beta \leq 1$, $\mu + \beta \leq 1$, $\nu + \beta \leq 1$, $\beta x + \alpha w = \beta y + \nu v$ and $\beta y + \alpha w = \beta x + \mu u$. Then:

$$-\frac{\alpha}{2\beta} w \leq -\frac{\alpha}{2\beta} w + \frac{\nu}{2\beta} v = \frac{1}{2}(x-y) = \frac{\alpha}{2\beta} w - \frac{\mu}{2\beta} u \leq \frac{\alpha}{2\beta} w$$

Since $\|w\| \leq 1$, this means that $\frac{1}{2}\|x-y\| \leq \frac{\alpha}{2\beta} = d_P(x,y) + \epsilon$. Hence $\frac{1}{2}\|x-y\| \leq d_P(x,y)$.

For the other direction, let $\epsilon > 0$ and let $\tilde{w} \in C$, such that $-\tilde{w} \leq \frac{1}{2}(x-y) \leq \tilde{w}$ and $\|\tilde{w}\| \leq \|\frac{1}{2}(x-y)\| + \epsilon$. The lefthand inequality implies, that there is a $\tilde{v} \in C$, such that $\frac{1}{2}y + \tilde{v} = \frac{1}{2}x + \tilde{w}$. The righthand inequality implies, that there is a $\tilde{u} \in C$, such that $\frac{1}{2}x + \tilde{u} = \frac{1}{2}y + \tilde{w}$. Now, let $D := 2\max\{(1+\epsilon)\|\tilde{w}\|, (1+\epsilon)\|\tilde{u}\|, (1+\epsilon)\|\tilde{v}\|, 1\}$. Let $w := \frac{1}{(1+\epsilon)\|\tilde{w}\|}\tilde{w}$, let $u := \frac{1}{(1+\epsilon)\|\tilde{u}\|}\tilde{u}$, let $v := \frac{1}{(1+\epsilon)\|\tilde{v}\|}\tilde{v}$. Then, $\frac{1}{2D}y + \frac{(1+\epsilon)\|\tilde{v}\|}{D}v = \frac{1}{2D}x + \frac{(1+\epsilon)\|\tilde{w}\|}{D}w$ and $\frac{1}{2D}x + \frac{(1+\epsilon)\|\tilde{u}\|}{D}u = \frac{1}{2D}y + \frac{(1+\epsilon)\|\tilde{w}\|}{D}w$. Hence $d_P(x,y) \leq (1+\epsilon)\|\tilde{w}\| \leq (1+\epsilon)(\|\frac{1}{2}(x-y)\| + \epsilon)$. Since ϵ was arbitrary, $d_P(x,y) \leq \frac{1}{2}\|x-y\|$. □

Definition 2.2.12. Let X be a preordered vector space with cone C. A set $E \subseteq X$ is called absolutely dominated, if for any $x \in E$, there is a $y \in E$, such that $-y \leq x \leq y$. A set $E \subseteq X$ is called absolutely order convex, if for any $x \in E$ the interval $[-x, x] := \{y \in X : -x \leq y \leq x\}$ is contained in E. A set is called solid if it is both absolutely dominated and absolutely order convex. Let the solid hull of E be defined as $\mathrm{sol}(E) := \bigcup_{x \in E}[-x, x]$.

Remark 2.2.13. Since $-x \leq x$ is equivalent to $0 \leq 2x$, the set $[-x, x]$ is nonempty if and only if $x \in C$. Hence, $\mathrm{sol}(E) = \bigcup_{x \in E \cap C}[-x, x]$.

Lemma 2.2.14. *Let X be a preordered linear space with cone C and let $E \subseteq X$. Then sol(E) is solid. If in addition E is absolutely dominated, then sol(E) is the smallest solid set containing E.*

Proof. Let $x \in \mathrm{sol}(E)$ be arbitrary. Then there exists a $y \in E$, such that $-y \leq x \leq y$. Hence $\mathrm{sol}(E)$ is absolutely dominated. Let $x \in \mathrm{sol}(E)$ be arbitrary. Then there exists a $y \in E$, such that $-y \leq x \leq y$. Thus, $[-x, x] = \{z \in X : -x \leq z \leq x\} \subseteq \{z \in X : -y \leq z \leq y\} = [-y, y] \subseteq \mathrm{sol}(E)$. Hence $\mathrm{sol}(E)$ is absolutely order convex, and thus solid.

Let E be absolutely dominated. To show that $\mathrm{sol}(E)$ is the smallest solid set containing E, let F be an arbitrary solid set containing E. For any $x \in E$ there is a $y \in E \cap C$, such that $-y \leq x \leq y$. Since F is absolutely order convex, $[-y, y] \subseteq F$. Hence, $[-x, x] \subseteq F$. Since $x \in E$ was arbitrary, this shows that $\mathrm{sol}(E) \subseteq F$. □

Lemma 2.2.15. *Let X be a regularly ordered vector space with cone C and norm $\|\cdot\|$. Then the open unit ball and the closed unit ball are both solid.*

Proof. Let B denote either the open or the closed unit ball. Let $b \in B$ and let $x \in [-b, b]$. Then, $\|x\| \leq \|b\|$, hence $x \in B$. □

Proposition 2.2.16. *Let X be an ordered linear space with cone C. Let $P \subseteq C$ be positively convex, such that $\mathbb{R}_+ P = C$ and let $Q := C \cap sol(P)$. Then the semimetric $d_{p,P}$ of P coincides with the semimetric $d_{p,Q}$ of Q for any pair of points in P.*

Proof. Let $x, y \in P$ be arbitrary. Since $P \subseteq Q$, $d_{p,Q}(x,y) \leq d_{p,P}(x,y)$. For the other direction let $\epsilon > 0$. Let $\alpha, \beta \in (0,1]$, $\mu, \nu \in [0,1]$ and $u, v, w \in Q$, such that $\frac{\alpha}{2\beta} \leq d_{p,Q}(x,y) + \epsilon$, $\alpha + \beta \leq 1$, $\mu + \beta \leq 1$, $\nu + \beta \leq 1$, $\beta x + \alpha w = \beta y + \nu v$ and $\beta y + \alpha w = \beta x + \mu u$. Since $w \in Q$, there is a $\tilde{w} \in P$, such that $w \leq \tilde{w}$, i.e. $\tilde{w} - w \in C$. Hence $\mu u + \alpha(\tilde{w} - w) \in C$ and $\nu v + \alpha(\tilde{w} - w) \in C$. Since $\mathbb{R}_+ P = C$, there is a $\lambda \in (0,1]$, such that $\lambda(\mu u + \alpha(\tilde{w} - w)) \in P$ and $\lambda(\nu v + \alpha(\tilde{w} - w)) \in P$. Then:

$$\frac{\lambda\beta}{2}x + \frac{\lambda\alpha}{2}\tilde{w} = \frac{\lambda\beta}{2}y + \frac{\lambda\nu}{2}v + \frac{\lambda\alpha}{2}(\tilde{w} - w) = \frac{\lambda\beta}{2}y + \frac{1}{2}(\lambda(\nu v + \alpha(\tilde{w} - w)))$$

$$\frac{\lambda\beta}{2}y + \frac{\lambda\alpha}{2}\tilde{w} = \frac{\lambda\beta}{2}x + \frac{\lambda\mu}{2}u + \frac{\lambda\alpha}{2}(\tilde{w} - w) = \frac{\lambda\beta}{2}x + \frac{1}{2}(\lambda(\mu u + \alpha(\tilde{w} - w)))$$

Also, $\frac{\lambda\beta}{2} + \frac{\lambda\alpha}{2} \leq 1$ and $\frac{\lambda\beta}{2}x + \frac{1}{2} \leq 1$. Thus, $d_{p,P}(x,y) \leq \frac{\frac{\lambda\alpha}{2}}{2\frac{\lambda\beta}{2}} = \frac{\alpha}{2\beta} \leq d_{p,Q}(x,y) + \epsilon$. Hence the semimetrics coincide. □

Lemma 2.2.17. *Let X be a preordered linear space with cone C. Let $E \subseteq X$ be convex, solid and absorbing. Then the Minkowski functional $\|\cdot\|_E$ of E is a seminorm and fulfills for all $x \in X$:*

$$\|x\|_E = \inf\{\|c\|_E : c \in C; -c \leq x \leq c\}$$

If in addition E is linearly bounded, then $\|\cdot\|_E$ is a norm and X is a regularly ordered vector space with cone C.

Proof. Since $E = \bigcup_{x \in E}[-x, x]$ is symmetric, convex and absorbing, $\|\cdot\|_E$ is a seminorm. Let $\|x\|_{\text{inf}} := \inf\{\|c\|_E : c \in C; -c \leq x \leq c\}$. Let $\mu := \|x\|_E$ and let $\epsilon > 0$ be arbitrary. Since E is absolutely dominated, there is a $y \in E \cap C$, such that $-y \leq \frac{1}{\mu+\epsilon}x \leq y$. Hence $-\mu y \leq x \leq \mu y$ and $\|x\|_{\text{inf}} \leq \|x\|_E$.

For the other direction let $\mu := \|x\|_{\text{inf}}$ and let $\epsilon > 0$. Let $y \in C$ such that $-y \leq x \leq y$ and $\|y\|_E \leq \mu + \epsilon$. Then, $\frac{1}{\mu+2\epsilon}y \in E \cap C$ and since E is absolutely order convex, $\frac{1}{\mu+2\epsilon}x \in E$. Hence $\|x\|_E \leq \mu + 2\epsilon$ and thus $\|x\|_E \leq \|x\|_{\text{inf}}$.

For the additional part let $x \in X$, such that $\|x\|_E = 0$. Let $f : \mathbb{R}_{++} \to E$ defined as $f(\lambda) := \lambda x$. Since f is constant, this implies $x = 0$. Let $c \in C \cap (-C)$, then $0 \leq c \leq 0$, hence $\|c\|_E = 0$. Thereby, $c = 0$ and thus C is proper and X is a regularly ordered vector space. □

Lemma 2.2.18. *Let X be a preordered vector space with cone C, such that $X = C - C$ and let $P \subseteq C$ be positively convex, such that $\mathbb{R}_+ P = C$. Then sol(P) is positively convex, solid and absorbing. The Minkowski functional of sol(P) is a seminorm that fulfills:*

$$\|x\| = \inf\{\|c\| : c \in C; -c \leq x \leq c\}$$

If in addition sol(P) is linearly bounded, then X is a regularly ordered vector space.

Proof. Since $P \subseteq C$, it is absolutely dominated and thereby $\mathrm{sol}(P)$ is solid. To show that $\mathrm{sol}(P)$ is positively convex, let $x_1, x_2 \in \mathrm{sol}(P)$ be arbitrary, let $p_1, p_2 \in P$ and $c_1, c_2, d_1, d_2 \in C$, such that $x_1 = p_1 - c_1 = -p_1 + d_1$ and $x_2 = p_2 - c_2 = -p_2 + d_2$. Let $\lambda \in [0, 1]$ be arbitrary, then $(1-\lambda)p_1 + \lambda p_2 \in P$, $(1-\lambda)c_1 + \lambda c_2 \in C$ and $(1-\lambda)d_1 + \lambda d_2 \in C$. Hence, $(1-\lambda)x_1 + \lambda x_2 = ((1-\lambda)p_1 + \lambda p_2) - ((1-\lambda)c_1 + \lambda c_2) = -((1-\lambda)p_1 + \lambda p_2) + ((1-\lambda)d_1 + \lambda d_2)$. Clearly, $0 \in \mathrm{sol}(P)$, hence $\mathrm{sol}(P)$ is positively convex.

To show that $\mathrm{sol}(P)$ is absorbing, let $x \in \frac{1}{2}P - \frac{1}{2}P$ and let $x_1, x_2 \in P$, such that $x = \frac{1}{2}x_1 - \frac{1}{2}x_2$. Then, $x \in [-(\frac{1}{2}x_1 + \frac{1}{2}x_2), \frac{1}{2}x_1 + \frac{1}{2}x_2]$, hence $x \in \mathrm{sol}(P)$ and thus $\frac{1}{2}P - \frac{1}{2}P \subseteq \mathrm{sol}(P)$. Since $\frac{1}{2}P - \frac{1}{2}P$ is absorbing, $\mathrm{sol}(P)$ is absorbing too. According to the preceding lemma the Minkowski functional of $\mathrm{sol}(P)$ is a seminorm, the equation for the seminorm is verified and the additional statements hold true, if $\mathrm{sol}(P)$ is linearly bounded. □

Definition 2.2.19. Let P be a positively convex module. Let $\mathrm{Aff}_b^+(P)$ denote the Banach space of bounded, positively affine function $f : P \to \mathbb{R}$ with point-wise addition and multiplication, equipped with the supremum norm. Let $C_0(P) \subseteq \mathrm{Aff}_b^+(P)$ denote the set of positive functions, i.e. $f(x) \geq 0$ for all $x \in P$. Let $Q_0(P) := C_0(P) - C_0(P)$.

Lemma 2.2.20. *Let P be a positively convex module. Then, $Q_0(P)$ is a regularly ordered vector space with cone $C_0(P)$ and with norm*

$$\|f\| := \{\|g\|_\infty \in C_0(P) : -g \leq f \leq g\}$$

Proof. Let $f_1, f_2 \in Q_0(P)$, then $\|f_1 + f_2\| = \{\|g\|_\infty \in C_0(P) : -g \leq f_1 + f_2 \leq g\} \leq \{\|g\|_\infty \in C_0(P) : -g \leq f_1 \leq g\} + \{\|g\|_\infty \in C_0(P) : -g \leq f_2 \leq g\} = \|f_1\| + \|f_2\|$. Let $f \in Q_0(P)$ and $\lambda \in \mathbb{R}\backslash\{0\}$, then $\|\lambda f\| = \{\|g\|_\infty \in C_0(P) : -g \leq \lambda f \leq g\} = \{\|g\|_\infty \in C_0(P) : -\frac{1}{\lambda}g \leq f \leq \frac{1}{\lambda}g\} = \{\|\lambda g\|_\infty \in C_0(P) : -g \leq f \leq g\} = |\lambda|\|f\|$. Hence this defines a seminorm. Let $f \in Q_0(P)$ and $g \in C_0(P)$, such that $-g \leq f \leq g$. Let $h_1, h_2 \in C_0(P)$, such that $f = g - h_1 = -g + h_2$. Since h_1 and h_2 are positive, this implies $f \geq g$ and $f \leq -g$. Hence, $\|f\|_\infty \leq \|g\|_\infty$. Thus, $\|f\|_\infty \leq \|f\|$, which implies that $\|\cdot\|$ indeed is a norm. Let $f \in C_0(P) \cap (-C_0(P))$ and $p \in P$, then $f(p) \geq (0)$ and $f(p) \leq 0$, hence $f = 0$. Thus, $C_0(P)$ is proper. □

Definition 2.2.21. Let $Q(P)$ denote the topological dual of $Q_0(P)$, in regard to the norm defined in the preceding lemma, equipped with the dual norm. Let $\sigma : P \to Q(P)$ denote the canonical identification, i.e. $(\sigma(x))(f) = f(x)$. Let $C(P)$ be defined as $C(P) := \mathbb{R}_+\sigma(P)$ and let $S(P) := C(P) - C(P)$.

Lemma 2.2.22. *σ is positively affine and $S(P)$ equipped with the Minkowski functional of sol($\sigma(P)$) and with the cone $C(P)$ is a regularly ordered vector space.*

Proof. Let $f \in Q_0(P)$, then $(\sigma(0))(f) = f(0) = 0$, let $x, y \in P$ and $\lambda \in [0, 1]$, then $((1-\lambda)\sigma(x)+\lambda\sigma(y))(f) = (1-\lambda)f(x)+\lambda f(y) = f((1-\lambda)x+\lambda y) = (\sigma((1-\lambda)x+\lambda y))(f)$. Since $\sigma(P)$ is the image of a positively convex module under a positively convex mapping, it is positively convex according to lemma 1.1.11.

To show that $\text{sol}(\sigma(P))$ is linearly bounded, it suffices to show that it is bounded in regard to the norm on $Q(P)$. Let $\epsilon > 0$ be arbitrary. Let $x \in \text{sol}(\sigma(P))$ and let $p \in P$ and $c_1, c_2 \in C(P)$, such that $x = \sigma(p) - c_1 = -\sigma(p) + c_2$. Let $f \in Q_0(P)$, such that $\|f\| \leq 1$, be arbitrary. Because of the definition of the norm on $Q_0(P)$, there are $g, h_1, h_2 \in C_0(P)$, such that $\|g\|_\infty \leq 1 + \epsilon$ and $f = g - h_1 = -g + h_2$. This implies, that $\|h_1\| \leq 2 + \epsilon$ and $\|h_2\| \leq 2 + \epsilon$. Let $k_1, k_2 \in C_0(P)$, such that $h_1 \leq k_1$ and $\|k_1\|_\infty \leq 2 + \epsilon$ and that $h_1 \leq k_1$ and $\|k_1\|_\infty \leq 2 + \epsilon$. Then:

$$x(f) = x(-g) + x(h_2) = -(\sigma(p))(-g) + c_2(-g) + (\sigma(p))(h_2) - c_1(h_2) =$$

$$= g(p) - c_2(g) + h_2(p) - c_1(h_2) \leq g(p) + h_2(p) \leq g(p) + k_2(p) \leq \|g\|_\infty + \|k_2\|_\infty \leq 3 + 3\epsilon$$

$$x(f) = x(g) + x(-h_1) = -(\sigma(p))(g) + c_2(g) + (\sigma(p))(-h_1) - c_1(-h_1) =$$

$$= -g(p) + c_2(g) - h_1(p) + c_1(h_1) \geq -g(p) - h_1(p) \geq -g(p) - k_1(p) \geq -\|g\|_\infty - \|k_1\|_\infty \geq$$

$$\geq -3 - 3\epsilon$$

Thus, $|x(f)| \leq 3 + 3\epsilon$ and since f and ϵ were arbitrary, $\|x\| \leq 3$. According to lemma 2.2.18, $S(P)$ is a regularly ordered vector space. □

Lemma 2.2.23. *Let Y be a normed vector space with norm $\|\cdot\|$ and let $X \subseteq Y$ be a regularly ordered vector space with cone C and norm $\|\cdot\|$ restricted to X. Then $\overline{C}$ is a proper cone. If in addition $X = Y$, then X equipped with the cone $\overline{C}$ and with $\|\cdot\|$ is a regularly ordered vector space.*

Proof. Since addition and scalar multiplication are continuous, $\overline{C}$ indeed is a cone. To show that $\overline{C} \cap (-\overline{C}) = \{0\}$, let $z \in \overline{C} \cap (-\overline{C})$ and let $\epsilon > 0$ be arbitrary. Let $x, y \in C$, such that $\|x - z\| \leq \epsilon$ and $\| - y - z\| \leq \epsilon$. Since $\|x + y\| \leq 2\epsilon$ and since X is regularly ordered, there are $p \in C$ and $c \in C$, such that $\|p\| \leq 1$ and that $x + y = 4\epsilon p - c$. Thereby,

$x = 4\epsilon p - (c + y)$. Let $d := x + 4\epsilon p$, then $x = d - 4\epsilon p$, hence $\|x\| \leq 4\epsilon$. Therefore, $\|z\| \leq \|x\| + \|x - z\| \leq 5\epsilon$. Since ϵ was arbitrary, $z = 0$.

For the second part it suffices to show that if $x \in X$ and $c, c_1, c_2 \in \overline{C}$ satisfy $x = c - c_1 = -c + c_2$, then $\|x\| \leq \|c\|$ follows. Let $\epsilon > 0$ be arbitrary and let $\tilde{c_1}, \tilde{c_2} \in C$, such that $\|c_1 - \tilde{c_1}\| \leq \epsilon$ and $\|c_2 - \tilde{c_2}\| \leq \epsilon$. There are $d_1, \tilde{d_1} \in C$, such that $\|\tilde{d_1}\| \leq \epsilon$ and $c_1 - \tilde{c_1} = d_1 - \tilde{d_1}$, and there are $d_2, \tilde{d_2} \in C$, such that $\|\tilde{d_2}\| \leq \epsilon$ and $c_2 - \tilde{c_2} = d_2 - \tilde{d_2}$. Thereby, $x = (c + \tilde{d_1} + \tilde{d_2}) - (\tilde{c_1} + d_1 + \tilde{d_2}) = -(c + \tilde{d_1} + \tilde{d_2}) + (\tilde{c_2} + \tilde{d_1} + d_2)$. Since $(\tilde{c_1} + d_1 + \tilde{d_2}) \in C$ and $(\tilde{c_2} + \tilde{d_1} + d_2) \in C$, $\|x\| \leq \|c + \tilde{d_1} + \tilde{d_2}\| \leq \|c\| + \|\tilde{d_1}\| + \|\tilde{d_2}\| \leq \|c\| + 2\epsilon$. Since ϵ was arbitrary, this verifies the assertion. □

Lemma 2.2.24. *Let X be a regularly ordered vector space with cone C and norm $\|\cdot\|$. Let $P \subseteq C$ be positively convex, such that $\mathbb{R}_+ P = C$ and that $\|\cdot\|$ is the Minkowski functional of sol(P). Let $x, y \in P$. If $x \neq y$, then exists a continous positive linear functional $f : X \to \mathbb{R}$, such that $f(x) \neq f(y)$.*

Let V denote the subspace spanned by x, y and 0. In case V is 0-dimensional, let $\tilde{f} : V \to \mathbb{R}$ be defined as $\tilde{f}(z) := 0$. In case $x = 0$ and $y \neq 0$, let $\tilde{f}$ be such that $\tilde{f}(y) = 1$ and in case $y = 0$ and $x \neq 0$, let $\tilde{f}$ be such that $\tilde{f}(x) = 1$. In case $x \neq 0$, $y \neq 0$, $x = \mu y$ with $\mu \in \mathbb{R}_+$, let $\tilde{f}(\lambda x) := \lambda$. In case V is 2-dimensional, there exists a hyperplane H, such that $\overline{C} \cap H = \{0\}$, because $\overline{C}$ is proper, according to the preceding lemma. Since $\overline{C} \cap V$ is closed in V, there exist $h_1 \in H$ and $\epsilon > 0$, such that $(h_1 + \epsilon(\mathrm{sol}(P) \cap V)) \cap \overline{C} = \emptyset$. Since $\overline{C} \cap V$ is a cone, all $h_2 \in h_1 + \epsilon(\mathrm{sol}(P) \cap V)$ satisfy $\mathbb{R}\tilde{h}_2 \cap \overline{C} = \{0\}$. Hence, there are at least two different hyperplanes the intersect $\overline{C}$ only at 0. Let W be a hyperplane satisfying this condition, such that $y - x \notin W$. Let $\tilde{f}$ be such that $\ker(\tilde{f}) = W$ and that $\tilde{f}(X) > 0$.

In all cases, $\tilde{f}$ satisfies $\tilde{f}(C \cap V) \subseteq \mathbb{R}_+$ and $f(x) \neq f(y)$. Since $V \cap \mathrm{sol}(P)$ is linearly bounded and convex and since V is finite-dimensional $V \cap \mathrm{sol}(P)$ is bounded by the euclidean norm on V. Let $D \in \mathbb{R}_{++}$ be such that $\tilde{f}(V \cap \mathrm{sol}(P)) \subseteq [-D, D]$. Hence, $\frac{1}{D}\tilde{f}(V \cap (\mathrm{sol}(P) - C)) \subseteq (-\infty, 1]$. Let g denote the Minkowski functional of the convex and absorbing set $\mathrm{sol}(P) - C$. Then $\frac{1}{D}\tilde{f}(z) \leq g(z)$, for all $z \in V$. Thus, there exists a linear extension $f : X \to \mathbb{R}$, such that $f(z) \leq g(z)$, for all $z \in X$. In particular, $f(z) \leq 1$, for $z \in \mathrm{sol}(P)$. Thus f is continuous. Since C is a cone and f is linear, $f(z) \geq -1$, for all $z \in C$ implies that $f(z) \geq 0$, for all $z \in C$. Hence f is positive.

Definition 2.2.25. A positively convex module P is called positively separated, if for any $x, y \in X$ there is a positively affine function $f : P \to [0, 1]$, such that $f(x) \neq f(y)$.

Theorem 2.2.26. *For a positively convex module P the following statements are equivalent:*

1. *P is metric.*

2. *ψ is injective and sol($\psi(P)$) is linearly bounded.*

3. *P is positively separated.*

4. *σ is injective.*

Proof. Let P be metric, then P is preseparated, according to lemma 2.2.8 and hence ψ is injective. Then $\mathbf{P}(P) = \mathbb{R}_+\psi(P) - \mathbb{R}_+\psi(P)$ implies that $\mathrm{sol}(\psi(P))$ is positively convex, according to lemma 2.2.18. Assume that $\mathrm{sol}(\psi(P))$ is not linearly bounded, then there exists an $x \in \mathrm{sol}(\psi(P))$, such that $x \neq 0$ and $\mathbb{R}_+x \subseteq \mathrm{sol}(\psi(P))$. There are $p_1, p_2 \in P$ and $\lambda \in \mathbb{R}_+$, such that $x = \psi(p_1) - \lambda\psi(p_2)$. By substituting p_1 with λp_1 if $\lambda < 1$ and substituting x with λx and p_2 with $\frac{1}{\lambda}p_2$ if $\lambda > 1$, we can assume that $x = \psi(p_1) - \psi(p_2)$. Let $\epsilon > 0$ be arbitrary. There are $q, c_1, c_2 \in P$ and $\eta_1, \eta_2 \in \mathbb{R}_+$, such that $\frac{1}{\epsilon}x = \psi(q) - \eta_1\psi(c_1) = -\psi(q) + \eta_2\psi(c_2)$. Choose $\delta \in \mathbb{R}_{++}$, such that $\delta \leq \min\{\frac{\epsilon}{1+\epsilon}, \frac{\epsilon}{1+\epsilon\eta_1}, \frac{\epsilon}{1+\epsilon\eta_2}\}$. Thus:

$$\psi(\frac{\delta}{\epsilon}p_1 + \delta\eta_1 c_1) = \frac{\delta}{\epsilon}\psi(p_1) + \delta\eta_1\psi(c_1) = \frac{\delta}{\epsilon}\psi(p_2) + \delta\psi(q) = \psi(\frac{\delta}{\epsilon}p_2 + \delta q)$$

$$\psi(\frac{\delta}{\epsilon}p_2 + \delta\eta_2 c_2) = \frac{\delta}{\epsilon}\psi(p_2) + \delta\eta_2\psi(c_2) = \frac{\delta}{\epsilon}\psi(p_1) + \delta\psi(q) = \psi(\frac{\delta}{\epsilon}p_1 + \delta q)$$

Since P is preseparated, ψ is injective and thus $\frac{\delta}{\epsilon}p_1 + \delta\eta_1 c_1 = \frac{\delta}{\epsilon}p_2 + \delta q$ and $\frac{\delta}{\epsilon}p_2 + \delta\eta_2 c_2 = \frac{\delta}{\epsilon}p_1 + \delta q$. Hence, $d_P(p_1, p_2) \leq \frac{\epsilon}{2}$ and since ϵ was arbitrary $d_P(p_1, p_2) = 0$. Thereby, $p_1 = p_2$ and $x = p_1 - p_2 = 0$ which contradicts the assumption.

Let ψ be injective and $\mathrm{sol}(P)$ be linearly bounded. According to lemma 2.2.18, $\mathbf{P}(P)$ is a regularly ordered vector space with cone $\mathbb{R}_+\psi(P)$ and the Minkowski functional of $\mathrm{sol}(\psi(P))$ as is a norm. Ler $x, y \in P$, such that $x \neq y$. Since ψ is injective, $\psi(x) \neq \psi(y)$ follows. According to lemma 2.2.24, there is a continuous positive linear functional f, such that $f(\psi(x)) \neq f(\psi(y))$ and $\|f\| \leq 1$. Then, since $f(\psi(P)) \subseteq [0, 1]$ and since x and y were arbitrary, P is positively separated.

Let P be positively separated. Let $x, y \in P$, such that $x \neq y$ and let $f : P \to [0, 1]$ be positively affine such that $f(x) \neq f(y)$. Then, $f \in C_0(P)$ and $(\sigma(x))(f) = f(x) \neq f(y) = (\sigma(y))(f)$. Hence σ is injective.

Let σ be injective. Then $\sigma : P \to \sigma(P)$ is isomorphic as a positively affine mapping. Let $Q := \mathrm{sol}(\sigma(P)) \cap C(P)$ and let $x, y \in P$ be arbitrary. According to propositions 2.2.11 and 2.2.16, $d_P(x, y) = d_{\sigma(P)}(\sigma(x), \sigma(y) = d_Q(\sigma(x), \sigma(y) = \frac{1}{2}\|\sigma(x) - \sigma(y)\|$. Thereby, d_P is metric. □

Theorem 2.2.27. *Let P be a convex module and let Y be a regularly ordered vector space with cone C and norm $\|\cdot\|$. Let $f : P \to C$ be positively affine, such that $\|f(x)\| \leq 1$, for all $x \in P$. Then there exists a unique positive linear continuous function $F : S(X) \to Y$, such that $F \circ \sigma = f$. This function satisfies $\|F\| \leq 1$.*

Proof. Let B denote the closed unit ball of Y. First show that there is a unique positively affine mapping $\tilde{F} : \sigma(P) \to B \cap C$ satisfying $\tilde{F} \circ \sigma = f$. For such a function to be well-defined $f(x) \neq f(y)$ has to imply $\sigma(x) \neq \sigma(y)$, for all $x, y \in P$. Let x and y be fixed, such that $f(x) \neq f(y)$. Since Y is a regularly ordered vector space there is a continuous positive linear functional $g : Y \to \mathbb{R}$, such that $g(f(x))) \neq g(f(y))$, according to lemma 2.2.24. Then, $g \circ f$ is a bounded, positively affine function to the interval $[0, 1]$, thus $\sigma(x) \neq \sigma(y)$. In order to show that $\tilde{F}$ is positively affine, let $x_j \in \sigma(P)$ and let $y_j \in P$ such that $\sigma(y_j) = x_j$ for all $j \in \mathbb{N}$. Thereby, $\tilde{F}(\sum_{j=1}^{\infty} \alpha_j x_j) = \tilde{F}(\sigma(\sum_{j=1}^{\infty} \alpha_j y_j)) = f(\sum_{j=1}^{\infty} \alpha_j y_j) = \sum_{j=1}^{\infty} \alpha_j f(y_j) = \sum_{j=1}^{\infty} \alpha_j \tilde{F}(x_j)$. According to the lemma 2.1.14, there is a unique linear extension $F : S(P) \to Y$. Since $f(C(P)) = F(\mathbb{R}_+(\sigma)(P)) \subseteq \mathbb{R}_+ C = C$, this extension is positive. To show that $\|F\| \leq 1$, let $x \in \mathrm{sol}(\sigma(P))$, let $p \in P$ and let $c_1, c_2 \in C(P)$, such that $x = \sigma(p) - c_1 = -\sigma(p) + c_2$. Then, $F(x) = F(\sigma(p)) - F(c_1) = -F(\sigma(p)) + F(c_2)$ is contained in $\mathrm{sol}(B) = B$, since $F(\sigma(p)) = f(p) \in B$ and $F(c_1), F(c_2) \in C$. Hence F is continuous and satisfies $\|F\| \leq 1$. □

Definition 2.2.28. Let **ROVec** denote the category with objects regularly ordered vector space and morphisms positive linear contractions, i.e. the domain's cone is mapped into the codomain's cone and the domain's closed unit ball is mapped into the codomain's close unit ball. Let $\mathbf{S} : \mathbf{PosConv} \to \mathbf{ROVec}$ denote the functor along σ mapping P to $S(P)$ and mapping positively affine mappings $f : P \to Q$ to the unique linear extension of $\sigma \circ f$, according to the preceding theorem. Let $\mathbf{U} : \mathbf{ROVec} \to \mathbf{PosConv}$ denote the functor assigning to a regularly ordered vector space with cone C the positively convex set $\{c \in C : \|c\| \leq 1\}$.

2.3 The Completion

Definition 2.3.1. A regularly ordered vector space is called a regularly ordered Banach space if it is a Banach space in regard to its norm. The category of regularly ordered Banach spaces with positive linear contractions, i.e. the domain's cone is mapped into the codomain's cone and the domain's closed unit ball is mapped into the codomain's close unit ball, is denoted by **ROBan**.

Remark 2.3.2. Clearly, **ROBan** is a full and faithful subcategory of **ROVec**.

Proposition 2.3.3. *Let X be an ordered vector space with cone C, such that $X = C - C$. Let $P \subseteq C$ be a positively superconvex set, such that $\mathbb{R}_+P = C$ and that sol$(P) \cap C = P$. Then X equipped with the Minkowski functional of sol(P) is a regularly ordered Banach space.*

Proof. The set $\frac{1}{2}P - \frac{1}{2}P$ is superconvex and absorbing. Since it is linearly bounded, its Minkowski functional defines a norm. Let this norm be denoted by $\|\cdot\|$. Let $x \in \frac{1}{2}P - \frac{1}{2}P$ and let $x_1, x_2 \in P$, such that $x = \frac{1}{2}x_1 - \frac{1}{2}x_2$. Then, $x \in [-(\frac{1}{2}x_1 + \frac{1}{2}x_2), \frac{1}{2}x_1 + \frac{1}{2}x_2]$, hence $x \in \mathrm{sol}(P)$ and thus $\frac{1}{2}P - \frac{1}{2}P \subseteq \mathrm{sol}(P)$. Hence the Minkowski functional of sol(P) indeed induces a seminorm that fulfills: $\|x\|_{\mathrm{sol}(P)} = \inf\{\|c\|_{\mathrm{sol}(P)} : c \in C; -c \leq x \leq c\}$ for all $x \in X$. On the other hand let $x \in \mathrm{sol}(P)$. Let x_0, such that $x_0 \in P$ and $x \in [-x_0, x_0]$. Hence there is an $x_1 \in C$, such that $x + x_1 = x_0$. Since $\|x_1\|_{\mathrm{sol}(P)} = \|x_0 - x\|_{\mathrm{sol}(P)} \leq 2$, $x_1 \in 2\mathrm{sol}(P) \cap C = 2P$ follows. Hence $\mathrm{sol}(P) \subseteq 2P - 2P$.

Thus, it suffices to show that $\|\cdot\|$ is the norm of a Banach space. To show completeness, let $(x_j)_{j\in\mathbb{N}}$ be an arbitrary Cauchy sequence. Let $(y_j)_{j\in\mathbb{N}}$ be a subsequence such that $\|y_j - y_k\| < 2^{-j}$ for all $k \geq j$. Let $z_1 := y_1$ and let $z_j := y_j - y_{j-1}$ for all $j > 1$, then $y_j = \sum_{k=1}^{j} z_k$. Since $\|z_k\| < 2^{-k+1}$, $z_k \in 2^{-k}P - 2^{-k}P$. Hence there are $\alpha_k, \beta_k \in [0, 2^{-k}]$ and $u_k, v_k \in P$, such that $z_k = \alpha_k u_k - \beta_k v_k$. To show that the sequence converges, it is sufficient to show that the series $\sum_{j=1}^{N} \alpha_j u_j$ and $\sum_{j=1}^{N} \beta_j v_j$ do. Since $|\alpha_j| \leq 2^{-j}$ the series $\sum_{j=1}^{N} \alpha_j$ converges absolutely and $\sum_{j=1}^{\infty} \alpha_j$ is well-defined. In case that all but finitely many α_j vanish the series clearly converges. In case this does not happen, the rules for superconvex combinations yield the following equation:

$$(\sum_{k=1}^{\infty} \alpha_k) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{l=1}^{\infty} \alpha_l} u_j = (\sum_{k=1}^{N} \alpha_k) \sum_{j=1}^{N} \frac{\alpha_j}{\sum_{l=1}^{N} \alpha_l} u_j + (\sum_{k=N+1}^{\infty} \alpha_k) \sum_{j=N+1}^{\infty} \frac{\alpha_j}{\sum_{l=N+1}^{\infty} \alpha_l} u_j$$

Since $\sum_{j=N+1}^{\infty} \frac{\alpha_j}{\sum_{l=N+1}^{\infty} \alpha_l} u_j \in P$, its norm is equal or less than 1. Hence:

$$\lim_{N\to\infty} \|(\sum_{k=1}^{\infty} \alpha_k) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{l=1}^{\infty} \alpha_l} u_j - (\sum_{k=1}^{N} \alpha_k) \sum_{j=1}^{N} \frac{\alpha_j}{\sum_{l=1}^{N} \alpha_l} u_j\| = \lim_{N\to\infty} \| \sum_{k=N+1}^{\infty} \alpha_k \| = 0$$

The proof for $\sum_{j=1}^{N} \beta_j v_j$ is identical. Hence,

$$\lim_{m\to\infty} z_m = (\sum_{k=1}^{\infty} \alpha_k) \sum_{j=1}^{\infty} \frac{\alpha_j}{\sum_{l=1}^{\infty} \alpha_l} u_j - (\sum_{k=1}^{\infty} \beta_k) \sum_{j=1}^{\infty} \frac{\beta_j}{\sum_{l=1}^{\infty} \beta_l} v_j$$

□

Theorem 2.3.4. *Let X be a regularly ordered vector space with cone C and norm $\|\cdot\|$. Let $P \subseteq C$ be positively convex, such that $\mathbb{R}_+P = C$ and that $\|\cdot\|$ is the Minkowski* *functional of sol(P). Let $\overline{X}$ be the completion of X. Then $\overline{X}$ equipped with the cone $\overline{C}$ and the completion of $\|\cdot\|$ is a regularly ordered Banach space.*

Proof. Since addition and scalar multiplication are continuous, $\overline{C}$ indeed is a cone. Let $Y := \overline{C} - \overline{C}$. According to the lemma 2.2.23, $\overline{C}$ is proper and hence Y is an ordered linear space. Let sol_X denote the solid closure in X in regard to C and let sol_Y denote the solid closure in Y in regard to $\overline{C}$. Let B denote the closed unit ball in $\overline{X}$. Next, show that $\mathrm{sol}_Y(B \cap \overline{C}) \cap \overline{C} = B \cap \overline{C}$. Clearly, $\mathrm{sol}_Y(B \cap \overline{C}) \cap \overline{C} \supseteq B \cap \overline{C}$, since $B \cap \overline{C}$ is absolutely dominated. To show the other inclusion, it suffices to show that $B \cap Y$ is solid in regard to $\overline{C}$. Therefore, let $x \in \mathrm{sol}(B \cap Y)$ be arbitrary, let $p \in B \cap \overline{C}$ and let $c_1, c_2 \in \overline{C}$, such that $x = p - c_1 = -p + c_2$. Let $(c_{1,n})_{n\in\mathbb{N}}, (c_{2,n})_{n\in\mathbb{N}} \in C$, such that $\lim_{n\to\infty} c_{1,n} = c_1$ and $\lim_{n\to\infty} c_{2,n} = c_2$. Since addition is continuous, this implies $\lim_{n\to\infty} \frac{1}{2}(c_{1,n} + c_{2,n}) = \frac{1}{2}(c_1 + c_2) = p$. Thus, $p_n := \frac{1}{2}(c_{1,n} + c_{2,n})$ satisfies $\lim_{n\to\infty} p_n = p$. Let $\epsilon > 0$ be arbitrary. Let $N_\epsilon \in \mathbb{N}$ be, such that $\|p_n - p\| \leq \epsilon$ for all $n \geq N_\epsilon$. In particular $\|p_n\| \leq (1+\epsilon)$ for all $n \geq N_\epsilon$. Let $x_n := p_n - c_{1,n} = c_{2,n} - p_n$. Then for all $n \geq N_\epsilon$,

$$x_n \in \mathrm{sol}_X((1+\epsilon)(B \cap X)) \subseteq \mathrm{sol}_X((1+2\epsilon)\mathrm{sol}_X(P)) = (1+2\epsilon)\mathrm{sol}_X(\mathrm{sol}_X(P)) =$$

$$= (1+2\epsilon)\mathrm{sol}_X(P) \subseteq (1+2\epsilon)B$$

Since ϵ was arbitrary, $x = \lim_{n\to\infty} x_n \in B$. According to proposition 2.3.3, Y equipped with the cone $\overline{C}$ and with the Minkowski functional of $\mathrm{sol}(B \cap \overline{C})$ is a regularly ordered Banach space. Since $\mathrm{sol}(B \cap \overline{C}) = \mathrm{sol}(B \cap Y) = B \cap Y$, the Minkowski functional coincides with the norm on $\overline{X}$. By uniqueness of the completion, $Y = \overline{X}$ follows. □

Definition 2.3.5. Let $\mathbf{J} : \mathbf{ROVec} \to \mathbf{ROBan}$ the functor assigning to a regularly ordered vector space its completion ordered by the closure of its cone and assigning to a positive linear contraction its unique extension. Let j denote the canonical embedding.

Remark 2.3.6. Since the extension is continuous the closure of the domain's cone (unit ball) is mapped into the closure of the codomain's cone (unit ball), hence the extension is a positive linear contraction.

Corollary 2.3.7. *Let X be a convex module and let Y be a regularly ordered Banach space with closed cone C and norm $\|\cdot\|$. Let $f : X \to C$ be positively affine, such that $\|f(x)\| \leq 1$, for all $x \in X$. Then there exists a unique positive linear continuous function $F : \overline{S(X)} \to Y$, such that $F \circ j \circ \sigma = f$. This function satisfies $\|F\| \leq 1$.*

Proof. According to theorem 2.2.27, there exists a unique positive linear continuous contraction $\tilde{F} : S(X) \to Y$, such that $\tilde{F} \circ \sigma = f$ and that $\|F\| \leq 1$. Let $F : \overline{S(X)} \to Y$

be its completion. Since C is closed and $C(X)$ is dense in the cone of $\overline{S(X)}$, this defines a positive linear continuous function that satisfies $F \circ j \circ \sigma = f$ and $\|F\| \leq 1$. Let G be an arbitrary function satisfying the conditions of the theorem. Then, its restriction to $S(X)$ is a positive linear continuous function such that $\tilde{F} \circ \sigma = f$ and that $\|F\| \leq 1$. By uniqueness in theorem 2.2.27, $G = F$. □

Definition 2.3.8. Let $\mathbf{W} : \mathbf{PosConv} \to \mathbf{ROBan}$ be defined as $\mathbf{W} := \mathbf{J} \circ \mathbf{S}$ and let $\mathbf{O} : \mathbf{ROBan} \to \mathbf{PosConv}$ be the restriction of $\mathbf{U}$ to $\mathbf{ROBan}$.

Corollary 2.3.9. *$\boldsymbol{W} : \boldsymbol{PosConv} \to \boldsymbol{ROBan}$ is left adjoint to $\boldsymbol{O} : \boldsymbol{ROBan} \to \boldsymbol{PosConv}$, i.e. for each positively convex module X, for each regularly ordered Banach space Y and for each positively affine $f : X \to \boldsymbol{O}(Y)$, there is a unique positive linear contraction $g : \boldsymbol{W}(X) \to Y$, such that $\boldsymbol{O}(g) \circ j \circ \sigma = f$.*

Proof. Let ι denote the inclusion of $\mathbf{O}(Y)$ into Y. According to corollary 2.3.7, there exists a unique positive linear contraction $g : \mathbf{W}(X) \to Y$, such that $g \circ j \circ \sigma = \iota \circ f$. Let $x \in X$, then $\mathbf{O}(g)(j(\sigma(x)) = g(j(\sigma(x))) = \iota(f(x)) = f(x)$. Hence, g uniquely satisfies this equation. □

Definition 2.3.10. Let **ComplPosConv** denote the category of complete metric positively convex modules with morphisms positively affine functions. Let $\mathbf{V}$ denote the functor that assigns to a positively convex module X the complete metric positively convex module $\overline{j_X(\sigma_X(X))}$ and to each positively affine function $f : X \to Y$ the restriction of $\mathbf{W}(f)$ to $\overline{j_X(\sigma_X(X))}$.

Remark 2.3.11. Since $\mathbf{W}(f)$ is continuous and $(\mathbf{W}(f))(j_X(\sigma(X_X))) \subseteq j_Y(\sigma_Y(Y))$, the image of $\overline{j_X(\sigma_X(X))}$ is contained in $\overline{j_Y(\sigma_Y(Y))}$.

Theorem 2.3.12. *$\boldsymbol{ComplPosConv}$ is a reflective subcategory of $\boldsymbol{PosConv}$ with reflection functor $\boldsymbol{V}$, i.e. for all $X \in \boldsymbol{PosConv}$, all $Y \in \boldsymbol{ComplPosConv}$ and all affine function $f : X \to Y$, there is a unique positively affine function $g : \boldsymbol{V}(X) \to Y$, such that $g \circ j_X \circ \sigma_X = f$.*

Proof. According to corollary 2.3.7, $\mathbf{W}(f) : \mathbf{W}(X) \to \mathbf{W}(Y)$ is the unique positive linear contraction satisfying $\mathbf{W}(f) \circ j_X \circ \sigma_X = j_Y \circ \sigma_Y \circ f$. Since Y is metric, $j_Y \circ \sigma_Y$ is isometric up to a factor 2 and hence injective. Ssince Y is complete, $j_Y \circ \sigma_Y : Y \to \overline{j_Y(\sigma_Y(Y))}$ is bijective and hence an isomorphism in **ComplPosConv**. Let k denote its inverse. Then $g := k \circ \mathbf{W}(f)$ uniquely satisfies $g \circ j_X \circ \sigma_X = f$. □

Appendix A

Appendix

A.1 A Flaw in the Construction of the Completion Functor in "Positively Convex Modules and Ordered Linear Spaces"

In "Positively convex modules and ordered linear spaces" ([12]) a different construction of **W** is given. But the construction is incomplete, since the proof of the following proposition, similar to 2.3.3, is faulty:

Proposition A.1.1. *Let X be an ordered vector space with cone C, such that $X = C - C$. Let $P \subseteq C$ be a positively superconvex set, such that $\mathbb{R}_+ P = C$. Then X equipped with the Minkowski functional of sol(P) is a regularly ordered Banach space.*

To show that sol(P) is linearly bounded, the proof attempts to show that sol(P) $\subseteq 2P - 2P$. But this inclusion does not hold true in general as the following example shows:

Example A.1.2. Let $X := \mathbb{R} \times \mathbb{R}$, let $C := \{(x, y) : x \leq y; -x \leq y\}$ and let $P := \{(x, y) : x \leq y; -x \leq y; 8x + y \leq 1; -8x + y \leq 1\}$. ($P$ is the closed polygon with vertices $(0, 0)$,$(\frac{1}{9}, \frac{1}{9})$,$(1, 0)$ and $(-\frac{1}{9}, \frac{1}{9})$.) Clearly, C defines a cone. Let $(x, y) \in C \cap (-C)$, then $x \leq y$ and $-x \leq -y$ imply $x = y$, and $-x \leq y$ implies $(x, y) = (0, 0)$. Thus, C is proper. Let $(x, y) \in X$, then $(x, y) = (x, y + |x| + |y|) - (0, |x| + |y|) \in C \cap (-C)$. Thus C is generating. Clearly, P is a subset of C. To show that $\mathbb{R}_+ P = C$, let $(x, y) \in C$ and assume that $(x, y) \neq (0, 0)$. Let $(u, v) := (\frac{x}{8|x|+|y|}, \frac{y}{8|x|+|y|})$. Since $(8|x| + |y|)(u, v) = (x, y)$, it suffices to show $(u, v) \in P$. The condition $x \leq y$ implies $\frac{x}{8|x|+|y|} \leq \frac{y}{8|x|+|y|}$ and $-x \leq y$ implies $-\frac{x}{8|x|+|y|} \leq \frac{y}{8|x|+|y|}$. Next, $8u + v = 8\frac{x}{8|x|+|y|} + \frac{y}{8|x|+|y|} \leq 8\frac{|x|}{8|x|+|y|} + \frac{|y|}{8|x|+|y|} = 1$ and $-8u + v = -8\frac{x}{8|x|+|y|} + \frac{y}{8|x|+|y|} \leq 8\frac{|x|}{8|x|+|y|} + \frac{|y|}{8|x|+|y|} = 1$. Thus, $(u, v) \in P$.

The case $(x, y) = (0, 0)$ is obvious, since $(0, 0) \in P$. Hence $\mathbb{R}_+ P = C$. P is a closed, bounded and convex subset of the Banach space $\mathbb{R} \times \mathbb{R}$ and contains $(0, 0)$, hence it is positively superconvex. Thereby E, C and P fulfill all conditions in the preceding lemma.

The solid hull of P can easily be calculated as follows: Let $(x, y) \in P$ be arbitrary, then $-x \leq \frac{1}{8} - \frac{1}{8}y \leq 1 - y$ and $x \leq \frac{1}{8} - \frac{1}{8}y \leq 1 - y$, in which the second inequalities hold because $y = \frac{1}{2}y + \frac{1}{2}y \leq \frac{1}{2} - 4x + \frac{1}{2} + 4x = 1$. Hence $(0, -1) \leq -(x, y) \leq (0, 0) \leq (x, y) \leq (0, 1)$. Then, $[-(x, y), (x, y)] = \{p \in P : -(x, y) \leq p \leq (x, y)\} \subseteq \{p \in P : (0, -1) \leq p \leq (0, 1)\} = [(0, -1), (0, 1)]$. Thus, $\mathrm{sol}(P) = \bigcup_{p \in P} [-p, p] = [(0, -1), (0, 1)] = \{(x, y) : y \leq 1 + x; y \leq 1 - x; -y \leq 1 + x; -y \leq 1 - x\}$. In particular, $(1, 0) \in \mathrm{sol}(P)$. Any $(x, y) \in P$ satisfies $|x| \leq y \leq 1 - 8|x|$, hence $|x| \leq \frac{1}{9}$. Thus, any $(x, y) \in 2P - 2P$ satisfies $|x| \leq \frac{4}{9}$, hence $(1, 0)$ cannot be an element of $2P - 2P$ and the inclusion $\mathrm{sol}(P) \subseteq 2P - 2P$ cannot hold true.

Similar examples show that the inclusion $\mathrm{sol}(P) \subseteq \lambda P - \lambda P$ does not hold true in general for any $\lambda \in \mathbb{R}_+$.

A.2 A List of Categories

- **Conv**
  1. Objects: Convex modules
  2. Morphisms: Affine mappings
- **PresepConv**
  1. Objects: Preseparated convex modules
  2. Morphisms: Affine mappings
- **SConv**
  1. Objects: Superconvex modules
  2. Morphisms: Superaffine mappings
- **CompConv**
  1. Objects: Complete metric (regarding the convex semimetric) convex modules
  2. Morphisms: Affine mappings
- **PosConv**
  1. Objects: Positively convex modules

2. Morphisms: Positively affine mappings

- **CompPosConv**

1. Objects: Complete metric (regarding the positively convex semimetric) positively convex modules
2. Morphisms: Positively affine mappings

- **BOVec**

1. Objects: Base ordered vector spaces, i.e. the base $B \subseteq X$ is convex, for all $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $b_1, b_2 \in B$, $\alpha_1 b_1 = \alpha_2 b_2$ implies $\alpha_1 = \alpha_2$ and $\mathbb{R}_+ B - \mathbb{R}_+ B = X$.
2. Morphisms: Linear mappings, that map bases into bases

- **BNBan**

1. Objects: Base normed Banach spaces with closed bases, i.e. the base $B \subseteq X$ is convex and closed, for all $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $b_1, b_2 \in B$, $\alpha_1 b_1 = \alpha_2 b_2$ implies $\alpha_1 = \alpha_2$, $\mathbb{R}_+ B - \mathbb{R}_+ B = X$ and the Minkowski functional of $\text{conv}(B \cup (-B))$ is a complete norm.
2. Morphisms: Linear mappings, that map bases into bases

- **POVec**

1. Objects: Preordered vector spaces
2. Morphisms: Linear mappings, that map cones into cones

- **ROVec**

1. Objects: Regularly ordered vector spaces, i.e. the cone C is proper and $\|x\| = \inf\{\|c\| : c \in C; -c \leq x \leq c\}$.
2. Morphisms: Linear mappings, that map cones into cones and unit balls into unit balls

- **ROBan**

1. Objects: Regularly ordered Banach spaces, i.e. the cone C is proper and $\|x\| = \inf\{\|c\| : c \in C; -c \leq x \leq c\}$.
2. Morphisms: Linear mappings, that map cones into cones and unit balls into unit balls

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