Construction of a Hilbert L^2-analogue for Differential Forms on a Compact Riemannian Manifold

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December 6, 2016

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Contents

1 Abstract

This bachelor thesis shall treat the construction of a particular Hilbert space, namely the space of differentiable forms on a smooth compact Hausdorff Riemannian manifold of finite dimension.

This Hilbert space is used in the construction of a *spectral triple* for said manifold, which is a triple (A, H, D) consisting of a C^*-algebra A that is represented on the Hilbert space H and a Dirac operator D that fulfils a certain set of properties. Such *spectral triples* are a concept developed in the field of non-commutative geometry and, put simply, allow for some geometrical objects to be retrieved from them. A prominent, exemplary result is that any closed Riemannian manifold endowed with a spin^c-structure can be reconstructed from its algebra of C^∞-functions on the manifold, using the *reconstruction theorem* of the french mathematician A.Connes.

The Hilbert space of differentiable forms on a manifold - the construction of which is outlined in this thesis - constitutes the second component of the spectral triple dual to a manifold with above-mentioned properties.

The content of the courses taught in the bachelors program at the Vienna University of Technology and the first two chapters of [1], which was used in the course on differential geometry in spring 2016, constitute the theoretical ground this work is based upon.

2 A Concise Review of Differential Geometry

In this first section a short review of some important results and some useful (notational) conventions regarding differential geometry are outlined.

Definition 2.1 (manifold)**.** *Let M be a topological space with the topology $\mathfrak{T}$. M is called a n-dimensional topological manifold if it satisfies the following conditions:*

1. *M is a Hausdorff space*
2. *the topology $\mathfrak{T}$ of M has a countable basis*
3. *M is locally homeomorphic to $\mathbb{R}^n$, or equivalently, for all $p \in M$ there exists some open set $U \subset M, p \in U$, an open set $V \subset \mathbb{R}^n$ and a homeomorphism $x : U \longrightarrow V$. Such a homeomorphism x will here be referred to as chart.*

Definition 2.2. *The manifold M is called compact if it is compact as a topological space.*

Since the term *smooth* is not uniformly used, given different mathematical contexts, a definition is given in (2.4).

Definition 2.3 (atlas)**.** *A set of charts $x_\alpha : U_\alpha \longrightarrow V_\alpha, \alpha \in A$, is called an atlas of M, if every point in the manifold is included in the domain of some chart, that is*

$$\bigcup_{\alpha \in A} U_\alpha = M. \tag{1}$$

Definition 2.4 (smoothness)**.** *An atlas $\mathfrak{A}$ is called smooth (or C^∞), if all the coordinate changing functions are C^∞-diffeomorphisms. That is, for two charts x_α, x_β the mapping*

$$x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cup U_\beta) \longrightarrow x_\beta(U_\alpha \cup U_\beta)$$

is a $C^\infty(\mathbb{R}^n$-function.
Two such atlases are called equivalent, if their union also gives a smooth atlas.
An atlas $\mathfrak{A}_{max}$ is called maximal if it already contains all charts that can be added safely to $\mathfrak{A}$ without destroying its smoothness.
A manifold paired with a maximal atlas $(M, \mathfrak{A}_{max})$ is called a smooth manifold.

Definition 2.5 (orientation)**.** *Two charts $x_\alpha, x_\beta \in \mathfrak{A}$ are said to be orientation preserving, if the Jacobian of the transition mapping $J(x_\beta \circ x_\alpha^{-1})$ is everywhere positive on $U_\alpha \cap U_\beta$. A manifold together with an atlas that contains only orientation preserving charts is called an orientable manifold.*

Remark 2.6. *There are several ways to introduce the tangent space at a point $p \in M$, denoted with T_pM. We will make use of the conventions set in [1], where the tangent space is defined as the set of equivalence classes on the set of (at least once) differentiable curves $\{\gamma : \mathbb{R} \longrightarrow M | \gamma(0) = p\}$, with the equivalence relation*

$$a \sim b \quad :\Leftrightarrow \quad \frac{d}{dt}(x \circ a) \restriction_{t=0} = \frac{d}{dt}(x \circ b) \restriction_{t=0},$$

where x is a chart. The equivalence class of a curve c will be denoted as $\dot{c}(0)$.
For a chart x and a point $p \in M$ we define the differential of that x at p as

$$\begin{aligned} dx\restriction_p : T_pM &\longrightarrow \mathbb{R}^n \\ \dot{c}(0) &\mapsto \frac{d}{dt}(x \circ c) \restriction_{t=0} . \end{aligned}$$

For a basis $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$ we can define $c_i := x^{-1}(x(p) + tb_i)$ and hence

$$\frac{\partial}{\partial x_i}(p) := \dot{c}_i(0)$$

give the basis elements for T_pM.

Definition 2.7 (Riemannian)**.** *A pair (M, g) is called a riemannian manifold, if M is a manifold g is a function that to each $p \in M$ assigns a inner product g_p on the respective tangent space*

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

that is differentiable as a function of p.

Throughout this thesis M will denote the manifold, T_pM the tangent space at the point $p \in M$ and the symbol $*$ used in the exponent will denote the dual of the respective vector space.

Remark 2.8. *We assume throughout that the manifold M is*
· *smooth*
· *n-dimensional*
· *compact*
· *orientable*
· *Riemannian*

To construct an inner product space for the smooth sections of the exterior algebra we first need to gain some ground in tensor field theory. We will start with elaborating basic properties of the Grassmann algebra $\Lambda_\bullet(T_p^*M)$ and then justify the integration of n-forms over a manifold M with the properties stated in (2.8). With that, we can define a inner product by making use of the Hodge-$*$-operator and obtain a Hilbert space by completion.

3 Construction of the Hodge-$*$-Operator

To begin with, we have to establish some properties and definitions of the tensor algebras that the Hodge-$*$-operator acts on.

Definition 3.1. *A q-multilinear function*

$$\omega: \quad \underbrace{V \times V \times \cdots \times V}_{q\ times} \longrightarrow \mathbb{R}$$

is called a q-form, or equivalently, a covariant tensor of degree q. Let

$$\mathcal{T}_q(V) := \{\omega | \omega \text{ is q-form over } V\}$$

denote the set of q-forms over V, with the identifications

$$\mathcal{T}_0(V) = \mathbb{R} \quad \text{and} \quad \mathcal{T}_1(V) = V^*,$$

where V^* denotes the vector space dual to V.

Definition 3.2. *An $\omega \in \mathcal{T}_q(V)$ is called skew-symmetric (or antisymmetric) if the equality*

$$\omega(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_q) = -\omega(x_1, \ldots, x_j. \ldots, x_i, \ldots, x_q) \quad \forall i, j \in \{1, \ldots, q\}$$

holds. Let

$$\Lambda_q(V) := \{w \in \mathcal{T}_q(V) | \omega \text{ is skew-symmetric}\}$$

denote the set of skew-symmetric forms of degree q over V.

Proposition 3.3. *With the relations*

$$(\omega_1 + \omega_2)(x) := \omega_1(x) + \omega_2(x)$$
$$(\lambda\omega)(x) := \lambda\omega(x),$$

where ω_1 and ω_2 are q-degree tensors and $\lambda \in \mathbb{R}$, both the spaces $\mathcal{T}_q(V)$ and $\Lambda_q(V)$ become vector spaces over $\mathbb{R}$.

Definition 3.4 (tensor product)**.** *The tensor product is defined as*

$$\otimes: \quad \mathcal{T}_q(V) \times \mathcal{T}_k(V) \longrightarrow \mathcal{T}_{q+k}(V)$$
$$(\omega_q \otimes \omega_k)(x_1, \ldots, x_q, x_{q+1} \ldots, x_{q+k}) := \omega_q(x_1, \ldots, x_q)\omega_k(x_{q+1}, \ldots, x_{q+k}),$$

Proposition 3.5. *The following relations are clear:*

$$(\lambda\omega_q) \otimes \omega_k = \lambda(\omega_q \otimes \omega_k),$$
$$(\omega_q + \gamma_q) \otimes \omega_k = \omega_q \otimes \omega_k + \gamma_q \otimes \omega_k,$$
$$\omega_q \otimes (\omega_k + \gamma_k) = \omega_q \otimes \omega_k + \omega_1 \otimes \gamma_k,$$
$$(\omega_q \otimes \omega_k) \otimes \omega_l = \omega_q \otimes (\omega_k \otimes \omega_l),$$

where $\omega_q, \gamma_q \in \mathcal{T}_q(V), \omega_k, \gamma_k \in \mathcal{T}_k(V), \omega_l \in \mathcal{T}_l(V)$ and $\lambda \in \mathbb{R}$.

The tensor product of two skew-symmetric forms is not necessarily skew-symmetric. For example, take the product of
$\omega_1 = (1,1), \omega_2 = (1,-1) \in \mathcal{T}_1(\mathbb{R}^2) = (\mathbb{R}^2)^*$. Therefore we define another operation for skew-symmetric multilinear forms. For that purpose, let $A : \mathcal{T}_q(V) \longrightarrow \Lambda_q(V)$ denote the *alteration* operator, given by the relation

$$A\omega_q(x_1, \dots, x_q) = \frac{1}{q!} \sum_{\sigma \in S_q} \omega_q(x_{\sigma(i)}, \dots, x_{\sigma(q)}) \cdot sgn(\sigma),$$

where S_q is the permutation group on q elements. Obviously, $A\omega_q$ is indeed skew-symmetric.

Definition 3.6. *For $\omega_q \in \Lambda_q(V), \omega_k \in \Lambda_k(V)$ we define the exterior product (or wedge product) as*

$$\begin{aligned} \wedge : \quad \Lambda_q(V) \times \Lambda_k(V) &\longrightarrow \Lambda_{q+k}(V) \\ \omega_q \wedge \omega_k &:= \frac{(q+k)!}{q!k!} A(\omega_q \otimes \omega_k). \end{aligned}$$

Moreover, denote by

$$\Lambda_\bullet(V) := \bigoplus_{i \in \mathbb{N}} \Lambda_i(V)$$

the direct sum of all i-degree skew-symmetric tensors, i.e., the set of all skew-symmetric forms on V.

Example 3.7. *Let us compute the exterior product for a basis of the vector space. Let $e_k^*, e_l^* \in V$ be two vectors of the basis dual to the one in V and $x_1 = x^{i_1} e_{i_1}, x_2 = x^{i_2} e_{i_2} \in V$[1]. We can compute that*

$$\begin{aligned} (e_k^* \wedge e_l^*)(x_1, x_2) &= \frac{2!}{1!1!} A(e_k^* \otimes e_l^*)(x_1, x_2) \\ = \begin{vmatrix} e_k^*(x_1) & e_k^*(x_2) \\ e_l^*(x_1) & e_l^*(x_2) \end{vmatrix} &= \begin{vmatrix} x_{k_1} & x_{k_2} \\ x_{l_1} & x_{l_2} \end{vmatrix}. \end{aligned}$$

By induction we can compute for m dual basis vectors

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_m}^*)(x_1, \dots, x_m) = \begin{vmatrix} e_{i_1}^*(x_1) & \cdots & e_{i_1}^*(x_m) \\ \vdots & & \vdots \\ e_{i_m}^*(x_1) & \cdots & e_{i_m}^*(x_m) \end{vmatrix}. \tag{2}$$

[1]Here the "Einstein convention" is used to denote the coordinate form of the vectors: the superscript denotes the coordinate values, the subscript belongs to the basis vectors and if an index appears both above and below of a term, this index is to be summed over.

Proposition 3.8. *The following computational rules hold for the exterior product:*

$$(\omega_q + \gamma_q) \wedge \omega_k = \omega_q \wedge \omega_k + \gamma_q \wedge \omega_k \tag{3}$$
$$(\lambda \omega_q) \wedge \omega_k = \lambda(\omega_q \wedge \omega_k) \tag{4}$$
$$\omega_q \wedge \omega_k = (-1)^{qk} \omega_k \wedge \omega_q \tag{5}$$
$$(\omega_q \wedge \omega_k) \wedge \omega_l = \omega_q \wedge (\omega_k \wedge \omega_l), \tag{6}$$

where $\omega_q, \gamma_q \in \Lambda_q(V)$,$\omega_k \in \Lambda_k(V)$ *and* $\omega_l \in \Lambda_l(V)$.

Proof. We will make use of the relations we already have for the tensor product.
(i) A straightforward computation yields

$$(\omega_q + \gamma_q) \wedge \omega_k = \frac{(q+k)!}{q!k!} A((\omega_q + \gamma_q) \otimes \omega_k)$$
$$= \frac{(q+k)!}{q!k!} A(\omega_q \otimes \omega_k + \gamma_q \otimes \omega_k) = \omega_q \wedge \omega_k + \gamma_q \wedge \omega_k$$

(ii) Follows directly from the definition of the exterior product.
(iii) Let $e_1, \dots, e_n \in V$ be a basis of V and let $x^{i_1} e_{i_1}, \dots, x^{i_m} e_{i_m} \in V$ be a set of m vectors. We first note that for every m-tensor β on V, since it is m-multilinear, we can compute

$$\beta(x^{i_1} e_{i_1}, \dots, x^{i_m} e_{i_m}) = \beta(e_{i_1}, \dots, e_{i_m}) x^{i_1} \cdots x^{i_m} = a_{i_1,\dots,i_m} x^{i_1} \cdots x^{i_m},$$

so that we can identify each m-degree tensor with a set of numbers $a_{i_1,\dots,i_m}$ (note that for the readers convenience the Einstein convention is used again). Thus it is enough to check equation (5) on the basis of V. When we recall the computation of the exterior product for m basis vectors as in (2) we obtain (5) by the computational rules for determinants for matrices with exchanged columns.
(iv) Follows again directly from the properties of the tensor product. $\square$

Proposition 3.9. *With the definition of the exterior product and the vector space operations on* $\Lambda_i(V), \forall i \in \mathbb{N}$ *the set* $\Lambda_\bullet(V)$ *becomes a graded algebra.*

Proof. Clearly the sum of two skew-symmetric tensors yields again a skew-symmetric form, as does multiplication by a scalar. The exterior product is the bilinear relation that satisfies the axioms for an algebra, as was shown in Proposition (3.8).
Finally we recall that an algebra A over a ring R is called *graded*, if it is graded as a ring. More explicitly, a *graded ring* allows a decomposition of its Abelian groups $(R, +)$ into a direct sum of Abelian groups $(R_g, +)$ with an index set G

$$R = \bigoplus_{g \in G} R_g, \tag{7}$$

such that the second binary operation $\cdot$ of the algebra satisfies

$$R_i \cdot R_j \subset R_{i+j}.$$

This definition of graduation was taken from [6]. The algebra A over R is called graded, if it satisfies

$$R_i \cdot A_j \subset A_{i+j} \tag{8}$$
$$A_i \cdot R_j \subset A_{i+j}. \tag{9}$$

In our case, the operation $+$ is the usual addition and the multiplication is the exterior product and hence it is clear that conditions (7),(8) and are fulfilled, since

$$\Lambda_k(V) \wedge \Lambda_l(V) \subset \Lambda_{k+l}(V).$$

□

To define further properties that shall lead to the definition of an inner product on $\Lambda_\bullet(V)$, we need to give some definition of a basis and an *orientation.*

Remark 3.10. *For $k > n$ the space $\Lambda_k(V)$ equals the trivial space. This can easily be seen, as any such form in $\Lambda_k(V)$ would have to contain at least one basis element at least twice. But then, recalling the antisymmetry property, this form must be the zero-form.*

Theorem 3.11. *Let V be a n-dimensional vector space and $0 \leq k \leq n$. Let further $e_i^*, i \in \{1, \ldots, n\}$ denote the unit basis vectors of V^* ($\cong V$) dual to $e_1, \ldots, e_n \in V$. Then the dimension of $\Lambda_k(V)$ is $dim(\Lambda_k(V)) = \binom{n}{k}$ and a basis is given by*

$$\tilde{E} = \{e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* | \{i_1, \cdots, i_k\} \subset \{1, \cdots, n\}\}, \tag{10}$$

where $i_1 \leq i_2 \leq \cdots \leq i_k$ is an increasing index set.

Proof. To prove that the set of all $\binom{n}{k}$ exterior products of k unit vectors (of the dual basis) give a basis for $\Lambda_k(V)$ we have to show that (i) they are linearly independent and (ii) they span the whole space.
(i) Suppose the set defined in (10) is linearly dependent. Of course no set of Indices $i_1 \leq i_2 \leq \cdots \leq i_k$ is used twice in that set. Then we have

$$0 = \left(\sum_{\substack{\text{I increasing index set} \\ |I|=k}} e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \right) (x_1, \ldots, x_k) \quad \forall x_1, \ldots, x_k \in V. \tag{11}$$

We remember, that the vectors e_i^* belong to the basis dual to the one we choose in V, meaning that $e_i^*(e_j) = \delta_{i,j}$, where δ is the Kronecker-Delta operation. Thus when we specify $(x_{i_1}, \dots, x_{i_k}) = (e_{\tilde{i}_1}, \dots, e_{\tilde{i}_k})$ for a fixed set of indices $\tilde{I} = \{\tilde{i}_1, \dots, \tilde{i}_k\}$, we get that

$$\left(\bigwedge_{\tilde{i}\in\tilde{I}} e^*_{\tilde{i}}\right)((e_{\tilde{i}})_{\tilde{i}\in\tilde{I}}) = \prod_{\tilde{i}\in\tilde{I}} \delta_{\tilde{i},\tilde{i}} = 1$$

$$\left(\bigwedge_{j\in I\neq\tilde{I}} e^*_{j}\right)((e_{\tilde{i}})_{\tilde{i}\in\tilde{I}}) = 0 \quad \forall I \neq \tilde{I}$$

So for these set of vectors equation (11) cannot be fulfilled and our set of k-degree tensors is linearly independent.
(ii) Let $T \in \Lambda_k(V)$ be arbitrary. For each set of indices $I_j, |I_j| = k$ with $I_j \neq I_l \Leftrightarrow j \neq l$ we define a number by

$$T_{I_j} = T((e_i)_{i\in I_j})$$

Along the proof for the linear independence we got that

$$\left(\bigwedge_{\tilde{j}\in\tilde{J}} e^*_{\tilde{j}}\right)((e_{\tilde{i}})_{\tilde{i}\in\tilde{I}}) = \delta_{\tilde{J},\tilde{I}} \quad \tilde{J}, \tilde{I} \text{ increasing index sets,}$$

so that

$$\left(\sum_{\substack{|I_j|=k\\ I_j \text{ increasing index set}}} T_{I_j} \bigwedge_{i\in I_j} e^*_i\right)((e_n)_{n\in I_n}) = \sum_{|I_j|=k} T_{I_j}\delta_{I_j,I_n} = T_{I_n}$$

and we have thus defined a k-form as linear combination of elements of $\tilde{E}$ that coincides with T on all basis elements in V. Thus we have that

$$dim(\Lambda_k(V)) = \binom{n}{k} \tag{12}$$

□

Remark 3.12. *The proof for the space $\Lambda_{n-k}(V)$ can be done analogously and hence we obtain that $dim(\Lambda_{n-k}(V)) = dim(\Lambda_k(V)))$ since there are equally many basis vectors.*

We will now define an *orientation* for a vector space and will see, that we can equivalently define it via tensors over the vector space.

Definition 3.13. *Let $\mathfrak{B}$ denote the set of all ordered bases on V. For $C, B \in \mathfrak{B}$ we say that they are consistently oriented($:\Leftrightarrow B \sim C$), if the mapping T between them, that satisfies $B = TC$ has positive determinant $det(T) > 0$.*

Proposition 3.14. *The relation $\sim$ is an equivalence relation and there are exactly two equivalence classes on $[\mathfrak{B}]/\sim$.*

Definition 3.15. *For a fixed $B \in \mathfrak{B}$ the equivalence class $[B]/\sim$ defines an orientation, as we say that a basis C is positively oriented, if $C \in [B]/\sim$ and negatively oriented otherwise. The pair $(V, [B]/\sim)$ is called an oriented vector space.*

We will now see, that we can get an equivalent definition of an orientation via tensors on V.

Theorem 3.16. *For any fixed $\omega \in \Lambda_n(V)$ the set*

$$\mathfrak{O}_\omega := \{B \in \mathfrak{B} | \omega(B) > 0\}$$

is an orientation, and conversely, for every orientation $[B]$ there exists $\omega \in \Lambda_n(V)$, such that $\mathfrak{O}_\omega = [B]$

Proof. To prove the first assertion, we will simply make use of the multilinearity of the n-degree tensor ω. If we have $B \sim C$ and the transition T that satisfies $B = TC$ and $det(T) > 0$, we also have that

$$\omega(B) = \omega(TC) = \omega(Tc_1, \dots, Tc_n) = det(T)\omega(C),$$

which can easily be seen by recalling the computation (2) and making use of the multiplication rule for determinants. We conclude that $\omega(C) > 0$ and therefore $C \in \mathfrak{O}_\omega$ if and only if $C \in [B]$
For the second assertion we fix any positively oriented $(b_1, \dots, b_n) = B \in [B]$ and choose a tensor $\omega \in \Lambda_n(V)$, such that

$$\omega(b_1, \dots, b_n) > 0.$$

Since we know that $V \cong V^*$ we can always find such a ω. The first step in our proof shows us that for any other $C \in [B]$ we also get $\omega(C) > 0$. Conversely, if we only know that $\omega(C) > 0$ can compute for the transition mapping T

$$\det(T) = \frac{\omega(b_1, \dots, b_n)}{\omega(c_1, \dots, c_n)} > 0$$

and therefore $C \in [B]$ if and only if $C \in \mathfrak{O}_\omega$. $\square$

For the following sections let $< \cdot, \cdot >$ denote an inner product.

Lemma 3.17. *Let $(V, < \cdot, \cdot >, [B])$ be an oriented inner product space. There exists exactly one $dV \in \Lambda_n(V)$ that satisfies*

$$dV(b_1, \dots, b_n) = 1$$

for any positively oriented ONB $(b_1, \dots, b_n) = B \in [B]$ and it has the form

$$dV = b_1^* \wedge \dots \wedge b_n^* \tag{13}$$

Proof. By the Gram-Schmidt theorem there exists an ONB for V and (if necessary) by switching two basis elements, we get a positively oriented ONB $B \in [B]$. Now it is also clear that dV as defined in (13) fulfils $dV(B) = 1$. For any other $C \in [B]$ there exists T such that $C = TB$ and $det(T) = 1$. Then it follows that

$$dV(c_1, \dots, c_n) = \det(T) dV(b_1, \dots, b_n) = 1,$$

which proves the first part of the lemma. To obtain uniqueness note that $\dim(\Lambda_n(V)) = 1$ and therefore for any other multi-linear form $dV' \in \Lambda_n(V)$ there exists a $\lambda \in \mathbb{R}$, such that $dV = \lambda dV'$, but then we get for any positively oriented basis $B \in \mathfrak{B}$

$$1 = dV(B) = \lambda dV'(B) = \lambda$$

and we have uniqueness. $\square$

Definition 3.18. *The n-degree tensor dV defined above is called a volume form.*

We are now able to define a inner product on $\Lambda_k(V)$.

Theorem 3.19 (inner product for tensors)**.** *Let $(V, < \cdot, \cdot >)$ be an inner product space and $0 \leq k \leq n$. There exists a unique inner product*

$$< \cdot, \cdot >_{\Lambda_k} : \quad \Lambda_k(V) \times \Lambda_k(V) \longrightarrow \mathbb{R}$$

such that for any ONB basis $C \in \mathfrak{B}$ the set defined via C as

$$C_{\Lambda_k} := \{c^*_{i_1} \wedge \cdots \wedge c^*_{i_k} | 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

is an ONB of $\Lambda_k(V)$ with respect to $< \cdot, \cdot >_{\lambda_k}$. This inner product on $\Lambda_k(V)$ has the form

$$< v^*_1 \wedge \cdots \wedge v^*_k, w^*_1 \wedge \cdots \wedge w^*_k >= det(< v^*_i, w^*_j >).$$

Proof. Note that here the inner product in V^* referred to above in the expression $< v^*_i, w^*_j >$ is well defined via the pullback $< \psi^{-1}(\cdot), \psi^{-1}(\cdot) >$, where $\psi : V \longrightarrow V^*$ is the unique Riesz-isomorphism between V and its dual space.
We know already, that for a basis $B \in \mathfrak{B}$ the equivalently defined set B_{Λ_n} is a basis for $\Lambda_n(V)$. If we now define

$$< b_{i_1} \wedge \cdots \wedge b_{i_k}, b_{j_1} \wedge \cdots \wedge b_{j_k} >:= det(< b^*_{i_r}, b^*_{j_s} >),$$

then this mapping is homogeneous and inherits the bilinearity and symmetry from $< \cdot, \cdot >$ defined on V. With $\det(< b^*_{i_s}, b^*_{i_s} >) = 1$ we have that $< \cdot, \cdot >_{\Lambda_k}$ is positive definite. So it is a inner product and B_{Λ_k} is an ONB for $\Lambda_k(V)$.

Since any other ONB C is related to B via an orthogonal transformation T with $\det(T) = 1$, C_{Λ_k} is also an ONB for $\Lambda_k(V)$. To obtain uniqueness, we can just suppose the existence of another inner product fulfilling the required properties and see, that it must take the same values on all basis elements, so it must be exactly equal to the the above defined one. $\square$

We will now define an operator that allows us to extend this inner product to the exterior algebra $\Lambda_\bullet(V)$. In particular we will do that by constructing an isomorphism $* : \Lambda_k(V) \longrightarrow \Lambda_{n-k}(V)$ that is known as the Hodge-$*$-Operator. The following is the main theorem of this subsection.

Theorem 3.20 (The Hodge-$*$-Operator)**.** *Let $(V, < \cdot, \cdot >, [B])$ be an oriented inner product space and dV the respective volume element. Also let $0 \leq k \leq n$. Then there exists a unique isomorphism $* : \Lambda_k(V) \longrightarrow \Lambda_{n-k}(V)$ satisfying*

$$\forall \omega, \tau \in \Lambda_k(V) : \quad < \omega \wedge *\tau, dV >_{\Lambda_n} = < \omega, \tau >_{\Lambda_k}$$

Proof. Step 1: At first we would like to define a few mappings, isomorphisms to be precise, that will enable us to construct the desired operator. Naturally we can define the canonical isomorphism

$$\begin{aligned} \kappa_k : \Lambda_k(V) &\longrightarrow (\Lambda_k(V))^* \\ \omega &\mapsto < \omega, \cdot >_{\Lambda_k} . \end{aligned}$$

Further we define the map $\psi_k : \Lambda_{n-k}(V) :\longrightarrow (\Lambda_k(V))^*$ by

$$\tau \mapsto < \cdot \wedge \tau, dV >_{\Lambda_n}$$

that is based on the bilinear pairing $\beta_k : \Lambda_{n-k}(V) \times \Lambda_k :\longrightarrow \mathbb{R}$ given by

$$(\tau, \omega) \mapsto < \omega \wedge \tau, dV >_{\Lambda_n} .$$

We want ψ_k to be an isomorphism. Since $\Lambda_k(V)$ and Λ_{n-k} have equal dimension it suffices to show, that ψ_k is linear and injective. The linearity is inherited by the bilinearity of $< \cdot, \cdot >_{\Lambda_n}$. Injectivity would mean that

$$(0 = \psi_k(\tau) = < \omega \wedge \tau, dV >_{\Lambda_n}, \forall \omega \in \Lambda_k(V)) \Rightarrow \tau = 0.$$

So if we find β_k to be a regular bilinear form, it follows that $\ker(\psi_k) = 0$. The bilinearity of β_k is obvious, but to see its regularity we can check it on a basis. Suppose that β_k is not regular for τ. Since the inner product in $\Lambda_n(V)$ is regular, that would imply that

$$\forall \omega \in \Lambda_k(V) : \quad \omega \wedge \tau \equiv 0$$

Without loss of generality, we can assume that $\tau = (b^*_{i_1} \wedge \cdots \wedge b^*_{i_{n-k}})$ with an increasing index set. If we now take the complementary index set $\{j_1, \ldots, j_k\}$

such that $\{i_1, \dots, i_{n-k}\} \cup \{j_1, \dots, j_k\} = \{1, \dots, n\}$ and define $\omega := (b^*_{j_1} \wedge \dots \wedge b^*_{j_k})$, the exterior product gives

$$\omega \wedge \tau = b^*_{j_1} \wedge \dots \wedge \omega^*_{j_k} \wedge \tau^*_{i_1} \wedge \dots \wedge \tau^*_{i_{n-k}} = \pm b^*_1 \wedge \dots \wedge b^*_n \neq 0$$

which is a contradiction. So β_k is regular and ψ_k is an isomorphism.
Step 2: construction of the operator. We define the Hodge-$*$-operator by

$$\begin{aligned} * : \Lambda_k(V) &\longrightarrow \Lambda_{n-k}(V) \\ * :=& \psi_k^{-1} \circ \kappa_k. \end{aligned}$$

As composition of such, $*$ is also an isomorphism and a computation yields $\forall \omega, \tau \in \Lambda_k(V)$:

$$< \omega \wedge * \tau, dV >_{\lambda_n} = \psi_k(*\tau)(\omega) = \psi_k(\psi_k^{-1}(\kappa_k(\tau)))(\omega) = \kappa_k(\tau)(\omega) =< \tau, \omega >_{\Lambda_k}$$

$\square$

4 Extension to the Cotangent Bundle

We are already familiar with partitions of unity for sets in the Euclidean topology on $\mathbb{R}$ and will now construct a partition of unity for a manifold, *subordinate* to a given atlas.

4.1 Partitions of Unity for Manifolds

In the following Section 4.2 we employ partitions of unity to define an integral over differential forms on a manifold. Based on our theoretical grounds, partitions of unity are known only for subsets of vector spaces.

Definition 4.1. *Let $(X, \mathfrak{T})$ be a topological space. A family of subsets $(O_i)_{i \in I} \in \mathfrak{T}$ is called locally finite, if for every point $x \in X$ there exists a neighbourhood of x, that intersects at most finitely many subsets of $(O_i)_{i \in I}$.*

Definition 4.2. *A partition of unity subordinate to an open cover $\bigcup_{\alpha \in A} U_\alpha \supset M$ is a collection of smooth functions $(f_i)_{i \in I} : f_i : M \longrightarrow \mathbb{R}$, such that*

1. $\forall i \in I \; \exists \alpha \in A : \operatorname{supp}(f_i) \subset U_\alpha$
2. $0 \leq f_i \leq 1$ *on M*
3. *for all $x \in M$ there is an open neighbourhood V_x, such that $supp(f_i) \cap V_x \neq \emptyset$ for only finitely many $i \in I$*
4. $\sum_{i \in I} f_i = 1$.

Using some basic facts from general topology we show that partitions of unity exist.

Lemma 4.3. *Compact Hausdorff spaces are normal.*

Lemma 4.4 (Urysohn)**.** *Let X be normal and A, B closed subspaces of X such that $A \cap B = \emptyset$. Then there exists a continuous function $f : X \longrightarrow \mathbb{R}$ that satisfies $f \restriction_A = 1$ and $f \restriction_B = 0$.*

For a proof we refer to [2]

Corollary 4.5. *Let $U \subset \mathbb{R}^n$ be open and K compact with $K \subset U$. Then there exists a smooth function $f : \mathbb{R}^n \longrightarrow [0, 1]$ such that $f \restriction_K = 1$ and* $\operatorname{supp}(f) \subset U$.

Lemma 4.6. *Let $C_0, C_1 \subset M$ be closed sets, such that $C_0 \cap C_1 = \emptyset$. Then there exists a smooth function $f : M \longrightarrow \mathbb{R}$, that satisfies $f(C_0) = 0$ and $f(C_1) = 1$.*

Proof. We claim that for any open set $O_M \subset M$ there is a smooth function $f : M \longrightarrow [0, \infty)$, such that $f^{-1}(0) = M - O$.
To prove that, note that for any open cube

$$I = (a_1, b_1) \times \cdots \times (a_n, b_n) \tag{14}$$

there is a function $\tilde{f} \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\tilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ that satisfies $\forall x \in I : \tilde{f} > 0$ and $\forall x \notin I : \tilde{f} = 0$. For a proof we refer to [4].
Since $O_M \subset M$ is open in the manifold topology if and only if it is open in the restricted Euclidean topology $\mathcal{E}$, we can write it as an intersection $O_M = M \cap O_{eu}$ with some open $O_{eu} \in \mathcal{E}$. As the cubes of the form (14) constitute a basis for the euclidean topology on $\mathbb{R}^n$ we can write any O_{eu} as union of such, that is $O_{eu} = \bigcup_{j \in J} I_j$. Next we want $\{I_j | j \in \tilde{J}\}$ to be a set of cubes such that for any $p \in O_M$ there exists some neighbourhood $U \in \mathfrak{U}(p)$ (where $\mathfrak{U}(p)$ denotes the neighbourhood filter in the point p) that only intersects with finitely many $\{I_{j_{1,p}}, \ldots, I_{j_{m,p}}\} \subset \{I_j | j \in \tilde{J}\}$. Since M is compact we can find such a locally finite collection. On each of these cubes we use a bump function as was stated in the beginning of the proof, that satisfies

$$\tilde{f}_j : I_j \longrightarrow (0, \infty), \quad \tilde{f}_j(\mathbb{R}^n - I_j) = 0, \quad \forall j \in \tilde{J}.$$

Now we can construct a function that is positive on O_M and vanishes on its complement by

$$f_{O_M} := \sum_{\tilde{J}} \tilde{f}_j \restriction_M, \quad f_{O_M}(O_M) \subset (0, \infty), f_{O_M}(M - O_M) = 0.$$

This function is well defined, as only finitely many terms in the sum are non-zero. Finally we define two functions f_{M-C_0} and f_{M-C_1} in the above mentioned manner and write the desired Urysohn function as

$$f(x) := \frac{f_{M-C_0}(x)}{f_{M-C_0}(x) + f_{M-C_1}(x)}$$

and it clearly fulfils our requirements. □

We can now go on to construct the partitions of unity for a smooth manifold.

Theorem 4.7 (partitions of unity)**.** *For any locally finite cover of M there exists a partition of unity subordinate to that cover, i.e., if* $\bigcup_{\alpha \in A} U_\alpha \supset M$ *is a cover of M then we can construct functions* $(\phi_\alpha)_{\alpha \in A} : M \longrightarrow \mathbb{R}$, *such that* $\phi_\alpha^{-1}(0) = M - U_\alpha$ *and* $\sum_{\alpha \in A} \phi_\alpha = 1$.

Proof. The previous lemma ascertains the existence of functions $\lambda_\alpha : M \longrightarrow [0, 1]$ with $\lambda_\alpha^{-1}(0) = M - U_\alpha$. As the cover is locally finite, the sum $\sum_{\alpha \in A} \lambda_\alpha$ is everywhere well defined on M. We can therefore define the required

$$\phi_\alpha := \frac{\lambda_\alpha}{\sum_{\alpha \in A} \lambda_\alpha}$$

$\square$

4.2 Integration of Differential Forms on a Manifold

We recall again that for the space $\Lambda_k(T_pM)$ of covariant skew-symmetric tensors of degree k acting on the tangent space T_pM, the basis consists of exterior products of the basis elements from the cotangent space T_p^*M.

To define an integral over differential forms of degree n on a n-dimensional manifold, let us first recall an important theorem (which can be found in ([3]). In the following sections $J\phi$ will denote the Jacobian of a function ϕ.

Theorem 4.8 (Change of Variable). *Let $U \subset \mathbb{R}^n$ be an open set and $\phi : U \longrightarrow \phi(U)$ a $C^{(k)}$-diffeomorphism for $k \geq 1$. Then for any $f \in L^1(\phi(U))$ we have $(f \circ \phi)|\det J\phi| \in L^1(U)$ and*

$$\int_U (f \circ \phi)(x)|\det J\phi| d\lambda^n(x) = \int_{\phi(U)} f(y) d\lambda^n(y).$$

Here $d\lambda^n$ is the n-dimensional Lebesgue-measure.

Definition 4.9. *We define a differential form of degree k to be a smooth section of the bundle of alternating k-degree tensors acting on M. These sections are denoted by $\Gamma^\infty(\Lambda_k(TM))$. Equivalently, $\omega \in \Gamma^\infty(\Lambda_k(TM))$ assigns to each $p \in M$ a skew-symmetric tensor $\omega_p \in \Lambda_k(T_pM)$ in such a way that in any chart of M the coefficients $\omega_{i_1}, \ldots, \omega_{i_k}$ are C^∞-functions.*

By performing the exterior product on the vector space of the differential k-forms on the tangent bundle $\Gamma^\infty(\Lambda_k(TM))$ we get another vector space, that we want do endow with an inner product.

Definition 4.10. *The associative, graded algebra defined by*

$$\Gamma^\infty(\Lambda_\bullet(TM)) := \bigoplus_{k=1}^{\infty} \Gamma^\infty(\Lambda_k(TM))$$

is called the exterior algebra, or Grassmann algebra of the manifold.

We will now introduce a map, called the *exterior differential*, that will help us in introducing the integral for differential forms. Although this map has a lot of interesting properties, here we only prove what will be used in the later sections.

Definition 4.11 (exterior differential)**.** *Let*

$$\omega = \sum_{\substack{I \text{ increasing index set} \\ |I|=k}} \alpha_I dx_{i_1} \wedge \cdots \wedge \omega_{i_k}$$

be a differential k-form. Then we define a map

$$d : \begin{cases} \Gamma^\infty(\Lambda_\bullet(TM)) \longrightarrow \Gamma^\infty(\Lambda_\bullet(TM)) \\ \omega \mapsto \sum_{l=1}^n \sum_I \frac{\partial \alpha_I}{\partial x_l} dx_l \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \end{cases}$$

which is called exterior differential.

Note that for the subspace of alternating k-tensors in the Grassmann algebra

$$d : \Gamma^\infty(\Lambda_k(TM)) \longrightarrow \Gamma^\infty(\Lambda_{k+1}(TM)).$$

Definition 4.12 (pull-back)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map and $\omega \in \Lambda_p(\mathbb{R}^m)$ be a skew-symmetric tensor of degree $0 \leq p \leq n$. We define the pull-back $\phi^*\omega \in \Lambda_p(\mathbb{R}^n)$ of ω by ϕ as*

$$(\phi^*\omega)(v_1, \ldots, v_n) := \omega(\phi(v_1), \ldots, \phi(v_n)),$$

with $v_1, \ldots, v_n \in \mathbb{R}^n$.

Definition 4.13 (pull-back for differential forms)**.** *Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ be two open subsets and $\phi : U \longrightarrow V, \phi \in C^\infty(U, V)$ be smooth. For $\omega \in \Gamma^\infty(\Lambda_k(TM))$ we define the pull-back by ϕ to be*

$$\phi^*\omega := (d\phi)^*\omega_{\phi(x)}.$$

Remark 4.14. *By application of Definition 4.12 we can see that this operation is well defined.*

Definition 4.15. *Let $\omega = f dx_1 \wedge \cdots \wedge dx_n$ be a differential form on $U \subset \mathbb{R}^n$ and $f \in L^1(U)$. Then we define the integral of ω on U as*

$$\int_U \omega = \int_U f d\lambda^n.$$

We write $\omega \in \Lambda_{n,L1}(U)$.

Lemma 4.16. *Let $U \subset \mathbb{R}^n$ be open and $\phi : U \longrightarrow \phi(U)$ a $C^{(k)}$-diffeomorphism. Also assume that the Jacobian is everywhere positive, that is $J\phi > 0$ $\forall x \in U$. Then for $\omega \in \Lambda_{n,L1}(U)$ the following formula holds:*

$$\int_{\phi(U)} \omega = \int_U \phi^*\omega. \tag{15}$$

Proof. STEP 1: First observe that for any $v_1, \ldots, v_n \in \mathbb{R}^n, v_i = \sum_{j=1}^n v_i^j e_j$ the exterior product of the n-tensor can be written as

$$e_1^* \wedge \cdots \wedge e_n^*(v_1, \ldots, v_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) v_1^{j_{\sigma(1)}} \cdots v_n^{j_{\sigma(n)}} = \det(v_1^j \cdots v_n^j).$$

Next we have that for a linear map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the computation of the pull-back yields

$$\begin{aligned}(\psi^*(e_1^* \wedge \cdots \wedge e_n^*))(e_1, \ldots, e_n) &= e_1^* \wedge \cdots \wedge e_n^*(\psi(e_1), \ldots, \psi(e_n)) \\ &= \det(\psi^j(e_1) \cdots \psi^j(e_n))\end{aligned}$$

and thus

$$\psi^*(e_1^* \wedge \cdots \wedge e_n^*) = \det(\psi) e_1^* \wedge \cdots \wedge e_n^*.$$

So if we instead of a linear map take a maximal differential form of degree n, and observe by the definition of pull-backs for differential forms (4.13) that in every point it is defined to be a linear pull-back, we also know that the pull-back for n-differential forms (on a n-dimensional manifold) is just the multiplication by the determinant of the Jacobian. In taking the Jacobian we implicitly made use of the fact that for functions (or equivalently forms in $\Lambda_0(T_pM)$) the exterior differential coincides with the usual differentiation. So we obtain

$$\phi^*\omega = f \circ \phi \cdot \det(J\phi) dx_1 \wedge \cdots \wedge dx_n$$

STEP 2: By the definition of the integral for differential forms we obtain

$$\int_U \phi^*\omega = \int_U f \circ \phi \cdot \det(J\phi) d\lambda^n,$$

which we can further rewrite using the fact that $\det(J\phi) > 0$ and applying the change of variable theorem as

$$\int_U f \circ \phi \cdot \det(J\phi) d\lambda^n = \int_{\phi(U)} f d\lambda^n = \int_{\phi(U)} \omega.$$

□

Now we collected the tools that enable us to construct the desired Hilbert space.

Definition 4.17. *Let $\omega = f dx_1 \wedge \cdots \wedge dx_n$ be a differential form on the manifold. Also let $(U_i, \phi_i)_{i \in I}$ be a positively oriented atlas and $(U_i, \alpha_i)_{i \in I}$ be a partition of unity subordinate to it. Assume further that $\alpha_i f \circ \phi^{-1} \in L^1(\phi(U)), \forall i \in I$. Then we define the integral of ω over the manifold as*

$$\int_M \omega = \sum_I \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\alpha_i \omega).$$

We write $\omega \in \Lambda_{n,L^1}(M)$.

Theorem 4.18. *The integral defined in Definition 4.17 is well-defined and finite. It does neither depend on the choice of the partition of unity, nor on the choice of a positively oriented atlas.*

Proof. The condition $\alpha_i f \circ \phi^{-1} \in L^1(\phi(U)), \forall i \in I$ is clearly fulfilled, since in our case f is a C^∞-function on a compact manifold. Note further that by our assumption in the Definition 4.17 we immediately obtain $(\phi_i^{-1})^*(\alpha_i \omega) \in \Lambda_{n,L^1}(\phi_i(U)), \forall i \in I$.
SETP 1: The integral

$$\int_{\phi_i(U_i)} (\phi_i^{-1})^*(\alpha_i \omega)$$

can only be non-zero for a finite number of indices. To see that, recall that, without loss of generality, we can assume $(U_i)_{i\in I}$ to be a locally finite cover of M. Moreover, $K := \mathrm{supp}(\omega) \subset M$ is compact by assumption. For every $p \in M$ there exists an open neighbourhood U_p of p such that $(U_p \cap U_i) \neq \emptyset$ for only finitely many $i \in I$. Therefore we can use the compactness of $K \subset \bigcup_{p\in K} U_p$ and extract a finite number of sets that cover $K \subset U_{p_1} \cup \cdots \cup U_{p_m}$. Then also the neighbourhoods $U_{p_j}, j = 1, \ldots, m$ only intersects finitely many sets in $(U_i)_{i\in I}$. Thus the first claim is proved, since the partition of unity is subordinate to the cover, i.e. satisfies $\mathrm{supp}(\alpha_i) \subset U_i, \forall i \in I$.
STEP 2: For the second claim let $(V_j, \eta_j)_j \in J$ be another positively oriented atlas of M and $(V_j, \beta_j)_{j\in J}$ be the partition of unity subordinated to it. We recall that when both are positively oriented, they must satisfy that the Jacobian of the transition functions have positive determinant $J(\phi_i \circ \eta_j^{-1}) > 0, \forall i, j\ \forall x \in \eta(V_j) \cap \phi(U_i)$. Since $\sum_J \eta_j = 1, \forall x \in M$, we can write

$$\int_{\phi_i(U)} (\phi_i^{-1})^*(\alpha_i \omega) = \sum_J \int_{\phi_i(U_i \cap V_j)} (\phi_i^{-1})^*(\eta_j \alpha_i \omega).$$

Now we apply Lemma 4.16 to the differential n-form $(\phi_i^{-1})^*(\eta_j \alpha_i \omega)$ on $\mathbb{R}^n$ and the diffeomorphism

$$\phi_i \circ \eta_j^{-1} : \eta_j(U_i \cap V_j) \longrightarrow \phi_i(U_i \cap V_j)$$

to obtain

$$\begin{aligned} \int_{\phi_i(U_i \cap V_j)} (\phi_i^{-1})^*(\eta_j \alpha_i \omega) &= \int_{\eta_j \circ \phi^{-1}(\phi(U_i \cap V_j))} (\phi_i \circ \eta_j^{-1})((\phi_i^{-1})^*(\alpha_i \beta_j \omega)) \\ &= \int_{\eta_j(U_i \cap V_j)} (\eta_j^{-1})^*(\alpha_i \beta_j \omega)). \end{aligned}$$

By taking the sum over all $i \in I$ it follows that

$$\begin{aligned} \sum_I \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\alpha_i \omega) &= \sum_I \sum_J \int_{\phi_i(U_i \cap V_j)} (\eta_j^{-1})^*(\alpha_i \beta_j \omega) \\ &= \sum_I \int_{\eta_j(V_j)} (\eta_j^{-1})^*(\beta_j \omega) = \int_M \omega. \end{aligned}$$

□

4.3 Completion of the Inner Product Space

Remark 4.19. *In the previous section we introduced the Hodge-$*$-operator as isomorphism between the skew-symmetric forms on a vector space, in particular*

$$* : \Lambda_k(T_pM) \longrightarrow \Lambda_{n-k}(T_pM).$$

In a natural way, by pointwise definition, we can extend $*$ to a mapping

$$* : \Lambda_k(TM) \longrightarrow \Lambda_{n-k}(TM)$$

between the cotangent bundles. Moreover, since we are treating the case of a compact smooth manifold, remember Remark 2.8, we can even define it on the smooth sections of the cotangent Grassmann bundle by

$$*\omega := f * (v_{i_1} \wedge \cdots \wedge v_{i_k})$$

for any $\omega \in \Gamma^\infty(\Lambda_\bullet(TM)),\ \omega = f v_{i_1} \wedge \cdots \wedge v_{i_k}$

Theorem 4.20. *Let again $\Gamma^\infty(\Lambda_\bullet(TM))$ denote the smooth sections on the Grassmann algebra of the cotangent bundle and let*

$$*_k : \Lambda_k(TM) \longrightarrow \Lambda_{n-k}(TM)$$

denote the Hodge-$*$-operator as introduced in the previous section. Then the mapping

$$< \cdot, \cdot >_{L^2(M)} : \Gamma^\infty(\Lambda_\bullet(TM)) \times \Gamma^\infty(\Lambda_\bullet(TM)) \longrightarrow \mathbb{R}$$

$$(\omega, \tau) \mapsto \begin{cases} \int_M \omega \wedge *\tau & \text{if } \omega \wedge *\tau \text{ has degree } n \\ 0 & \text{otherwise} \end{cases}$$

is an inner product on $\Gamma^\infty(\Lambda_\bullet(TM))$.

Proof. To see that the written map is symmetric, bilinear and non-degenerate we simply have to observe that for each $k \in \{1, \ldots, , n\}$ and $\omega, \tau \in \Lambda_k(T^*M)$ the expression

$$\omega \wedge *\tau =< \omega, \tau >_{\Lambda_k}$$

is nothing else but the inner product on the k-degree skew-symmetric tensors. To extend the definition to the space $\Gamma^\infty(\Lambda_\bullet(TM))$ we have to take note of the fact that we can write each n-degree tensor $\omega \wedge *\tau$ as $\omega \wedge *\tau = f \cdot g dx_1 \wedge \cdots \wedge dx_n$. We use compactness of the manifold to get the bound

$$\int_M \omega \wedge *\tau = \int_M |f| \cdot |g| d\lambda^n \leq \int_M \sup_{x \in M} |f| \cdot \sup_{y \in M} |g| d\lambda^n = \sup_{x \in M} |f| \cdot \sup_{y \in M} |g| \cdot \mu(M),$$

where $\mu(M) < \infty$ is the surface measure of the manifold. Thus it is well defined for all differential forms and we have an inner product on $\Gamma^\infty(\Lambda_\bullet(TM))$. □

Starting from a vector space endowed with an inner product, we can always define a norm on it via $\|x\|_V := \sqrt{< x, x >_V}$ and thus we obtain a normed space, in our case $(\Gamma^\infty(\Lambda_\bullet(TM)), \|\cdot\|_{\Gamma^\infty(\Lambda_\bullet(TM))})$. To finally perform the last step in the construction of our Hilbert space, we have to recall that for every normed space $(X, \|\cdot\|)$ there exists a completion $((\hat{X}, \|\cdot\|), \iota)$ (see [5]). Therefore we can define a Hilbert space as completion of the space of differentiable forms on the manifold. This space is commonly denoted by $L^2(\Gamma^\infty(\Lambda_\bullet(TM)))$.

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