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Seminararbeit

Laplace Tranformation von Borelmaßen und der Satz von Bernstein über vollständig Monotone Funktionen

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1 Introduction and notation

In the present note we will firstly introduce the notion of Laplace-Transforms of Borel measures on the Borel measurable subsets of $[0, +\infty)$. The aim of the note will be the formulation and proof of the Theorem of Bernstein on completely monotone functions stating that these functions can be written as a Laplace-Transform of a unique Borel measure. This result will strongly rely on the Theorem of Riesz-Markov and the Theorem of Banach-Alaoglu.

The following results are taken from [Kal21, Chapter 12, 14, 15, 18].

Definition 1.0.1. Let $(\Omega, \mathcal{T})$ be a locally-compact Hausdorff Space. By $C_0(\Omega, \mathbb{R})$ we denote all real valued continuous functions on Ω satisfying:

$$\forall \epsilon > 0 \; \exists K \subseteq \Omega \text{ compact} : |f(x)| \leq \epsilon \; \forall x \in \Omega \setminus K.$$

Definition 1.0.2. Let $(\Omega, \mathcal{T})$ be a locally-compact Hausdorff Space. Let μ be a complex (real) valued measure on $\mathcal{A} := \mathcal{A}(\mathcal{T})$. We call μ regular, if its variation $|\mu|$ is regular, i.e. if for every $A \in \mathcal{A}$

$$|\mu|(A) = \sup\{|\mu|(K) : K \subseteq A, \; K \text{ compact}\} = \inf\{|\mu|(O) : A \subseteq O, \; O \in \mathcal{T}\}.$$

By $M_{reg}(\Omega, \mathcal{A}, \mathbb{C}(\mathbb{R}))$ we will denote the set of all regular complex (real) measures on $\mathcal{A}$.

Lemma 1.0.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $I \subseteq \mathbb{R}$ an Interval, $s \in I$ and $f : I \times \Omega \to \mathbb{R}$ a function such that*

1. *$x \mapsto f(t, x)$ is μ-integrable for all $t \in I$,*
2. *$t \mapsto f(t, x)$ is differentiable at the point s for almost all $x \in \Omega$,*
3. *there exists a $\delta > 0$ and a μ-integrable function $g : \Omega \to \mathbb{R}$ such that for all $t \in (s - \delta, s + \delta) \cap I$ the inequality*

$$\left| \frac{f(t, .) - f(s, .)}{t - s} \right| \leq g$$

holds true μ-almost everywhere.

Then the function $F(t) := \int_\Omega f(t, .) \, d\mu$ is differentiable at the point $s \in I$ and

$$F'(s) = \int_\Omega \frac{\partial f}{\partial t}(s, .) \, d\mu.$$

Theorem 1.0.4. *(Riesz-Markov) Let* $(\Omega, \mathcal{T})$ *be a locally-compact Hausdorff Space and* $\mu \in M_{reg}(\Omega, \mathcal{A}(\mathcal{T}), \mathbb{C}(\mathbb{R}))$. *Defining* $\Phi(\mu) : C_0(\Omega, \mathbb{C}(\mathbb{R})) \to \mathbb{C}(\mathbb{R})$ *by*

$$\Phi(\mu)(f) := \int_\Omega f \, d\mu$$

we have $\Phi(\mu) \in C_0(\Omega, \mathbb{C}(\mathbb{R}))'$. *Indeed,* $\Phi := (\mu \mapsto \Phi(\mu))$ *constitutes a linear, isometric and bijective mapping from* $M_{reg}(\Omega, \mathcal{A}, \mathbb{C}(\mathbb{R}))$ *onto* $C_0(\Omega, \mathbb{C}(\mathbb{R}))'$.

We need a few results from functional analysis, which are taken from [MK23, Chapter 5].

Definition 1.0.5. We denote by $\sigma(X', X) := \sigma(X', \iota(X))$ the weak-topology on X' with respect to the subspace $\iota(X) \leq X''$ and will call it w^*-topology.

Theorem 1.0.6. *(Banach-Alaoglu) Let* $(X, ||.||)$ *be a normed space and* X' *be its dualspace. Then the closed unit ball in* X' *with respect to the operatornorm, i.e.*

$$K_1^{X'}(0) = \{f \in X' : ||f|| \leq 1\}$$

is w^**-compact.*

Definition 1.0.7. Let $(\Omega, \mathcal{T})$ be a locally-compact Hausdorff space. We say that a net $(\mu_i)_{i \in I}$ of positive measures defined on $\mathcal{A}(\mathcal{T})$ converges vaguely to a positive measure μ if and only if

$$\lim_{i \in I} \int_\Omega f \, d(\mu_i) = \int_\Omega f \, d\mu$$

for every $f \in C_0(\Omega, \mathbb{R})$.

Corollary 1.0.8. *Let* $(\Omega, \mathcal{T})$ *be a locally-compact Hausdorff space and* $(\mu_n)_{n \in \mathbb{N}}$ *be a sequence of probability measures on* $\mathcal{A}(\mathcal{T})$. *Then there exists a subnet* $(\mu_{n(i)})_{i \in I}$ *that converges vaguely to a finite positive measure on* $\mathcal{A}(\mathcal{T})$.

Proof. By $\Phi : M_{reg}(\Omega, \mathbb{R}) \longrightarrow C_0(\Omega, \mathbb{R})'$ we denote the isometric isomorphism from the Riesz-Markov Theorem 1.0.4. $||\mu_n|| = \mu_n(\Omega) = 1$ implies $\Phi(\mu_n) \in K_1^{C_0(\Omega,\mathbb{R})'}(0)$ for every $n \in \mathbb{N}$. Since $(C_0(\Omega, \mathbb{R}), ||.||_\infty)$ is a normed space, the Banach-Alaoglu Theorem 1.0.6 implies the compactness of $K_1^{C_0(\Omega,\mathbb{R})'}(0)$ with respect to the w^* topology. Hence, there exists a subnet and a functional $\phi \in K_1^{C_0(\Omega,\mathbb{R})'}(0)$ such that $\Phi(\mu_{n(i)}) \xrightarrow{i \in I} \phi$ in the w^* sense. Hence,

$$\lim_{i \in I} \int_\Omega f \, d(\mu_{n(i)}) = \lim_{i \in I} \Phi(\mu_{n(i)})(f) = \phi(f) = \int_\Omega f \, d\mu$$

for every $f \in C_0(\Omega, \mathbb{R})$ and where $\mu := \Phi^{-1}(\phi)$. The fact that μ is a positive measure follows directly from the fact that all μ_n are positive since

$$\int_\Omega f \, d\mu = \lim_{i \in I} \int_\Omega f \, d\mu_{n(i)} \geq 0$$

for every $f \in C_0(\Omega, \mathbb{R})$ that are non-negative. $\square$

2 Laplace-Transforms of measures

In the present chapter we will introduce the notion of Laplace-Transforms of positive measures. All Definition und results are based on [SSV12].

Remark 2.0.1. All measures in the forthcoming considerations will be defined on $(\mathcal{A}(\mathcal{T}) \mid_{[0,+\infty)})$, where $\mathcal{T}$ denotes the topology on $\mathbb{R}$ induced by the euclidian norm and the set $[0,+\infty)$ will only be equipped with this sigma-algebra. The set $[0,+\infty) \times [0,+\infty)$ will be equiped with $(\mathcal{A}(\mathcal{T}^2) \mid_{[0,+\infty)^2})$. By λ_1 we will denote the Lebesgue measure, restricted to $(\mathcal{A}(\mathcal{T}) \mid_{[0,+\infty)})$.

Definition 2.0.2. Let μ be a positive Borel measure on $[0,+\infty)$. We define the Laplace-Transform $\mathcal{L}(\mu) : (\rho_0, +\infty) \longrightarrow \mathbb{R}$ of μ by

$$\mathcal{L}(\mu)(x) = \int_{[0,+\infty)} e^{-xt}\, d\mu(t),$$

where $\rho_0 = \inf\{x \in \mathbb{R} : \int_{[0,+\infty)} e^{-xt}\, d\mu < +\infty\}$. Here we set $\rho_0 = +\infty$ if the corresponding set is empty.

Lemma 2.0.3. *A positive Borel measure μ is finite if and only if $\rho_0 \le 0$ and* $\lim_{x\to 0+} \mathcal{L}(\mu)(x) < +\infty$.

Proof. By monotone convergence

$$\lim_{x\to\rho+} \mathcal{L}(\mu)(x) = \int_{[0,+\infty)} e^{-\rho t}\, d\mu(t)$$

as an element of $[0,+\infty]$ for any $\rho \ge \rho_0$. In particular, μ is finite if and only if $\rho_0 \le 0$ and $\lim_{x\to 0+} \mathcal{L}(\mu)(x) < +\infty$. □

Proposition 2.0.4. *If μ is a positive Borel measure on $[0,+\infty)$ and ρ_0 is defined as in Def. 2.0.2, then $\mathcal{L}(\mu) \in C^\infty((\rho_0,+\infty),\mathbb{R})$ and for every $n \in \mathbb{N}$, $\lambda \in (\rho_0,+\infty)$ we have*

$$(-1)^n \mathcal{L}(\mu)^{(n)}(\lambda) \ge 0.$$

Proof. We define $h : (\rho_0,+\infty) \times (0,+\infty) \to \mathbb{R}$ by

$$h(\lambda, t) := e^{-\lambda t}.$$

Given $\lambda > \rho_0$ we choose $\epsilon > 0$ such that $\lambda - 2\epsilon > \rho_0$. For every $s \in (\lambda-\epsilon, \lambda+\epsilon)$ we have

$$\left|\frac{\partial^n h}{\partial s^n}(s,t)\right| = |(-t)^n e^{-st}| = t^n e^{-st} \le \frac{n!}{\epsilon^n} e^{\epsilon t} e^{-st} \le \frac{n!}{\epsilon^n} e^{\epsilon t} e^{-(\lambda-\epsilon)t} = \frac{n!}{\epsilon^n} e^{-(\lambda-2\epsilon)t}.$$

Because of

$$\int_{[0,+\infty)} \frac{n!}{\epsilon^n} e^{-(\lambda-2\epsilon)t} \, d\mu(t) = \frac{n!}{\epsilon^n} \mathcal{L}(\mu)(\lambda - 2\epsilon) < +\infty$$

we can apply Lemma 1.0.3 and obtain

$$(-1)^n \mathcal{L}(\mu)^{(n)}(\lambda) = (-1)^n \int_{[0,+\infty)} \frac{d^n}{d\lambda^n}(e^{-\lambda t}) \, d\mu(t) = \int_{[0,+\infty)} t^n e^{-\lambda t} \, d\mu(t) \geq 0.$$

□

The following lemma will be imported without proof. The proof can be found in proposition 1.2 in [SSV12, Chapter 1].

Lemma 2.0.5. *If* μ, ν *are positive Borel measures on* $[0, +\infty)$ *such that* $\mathcal{L}\mu = \mathcal{L}\nu$*, then* $\mu = \nu$*.*

3 Completely monotone functions

The following chapter is mainly based on [SSV12, Chapter 1]. Other sources will be stated explicitly.

Definition 3.0.1. We say that $f \in C^\infty((0,+\infty),\mathbb{R})$ is completely monotone if

$$(-1)^n f^{(n)} \geq 0 \text{ for all } n \in \mathbb{N}_0.$$

By $\mathcal{CM}$ we denote the set of all completely monotone functions.

Lemma 3.0.2. *Let $f \in \mathcal{CM}$ and assume that $f(+\infty) := \lim_{t\to+\infty} f(t) = 0$. Then for every $n \in \mathbb{N}$ and $\lambda > 0$ we have*

$$f(\lambda) = \int_\lambda^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}\, ds.$$

Proof. For any $a, \lambda > 0$ and $n \in \mathbb{N}$ by a version of Taylors Theorem

$$\begin{aligned} f(\lambda) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(\lambda-a)^k + \int_a^\lambda \frac{f^{(n)}(s)}{(n-1)!}(\lambda-s)^{n-1}\, ds \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(a)}{k!}(a-\lambda)^k + \int_\lambda^a \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}\, ds. \end{aligned} \tag{3.1}$$

Because of $f \in \mathcal{CM}$ for $a \geq \lambda$ all terms in (3.1) are non-negative. In particular the integral on the right hand side is bounded by $f(\lambda)$ for any $a \geq \lambda$. Hence, the limit

$$\lim_{a\to+\infty} \int_\lambda^a \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}\, ds = \int_\lambda^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}\, ds$$

exists and is smaller or equal to $f(\lambda)$. With the same argument we see that the sum in (3.1) converges to a non-negative limit for $a \longrightarrow +\infty$. Since this is true for all $n \in \mathbb{N}$, the same is true for every addend in the sum. For $k \in \mathbb{N}_0$ we set

$$\rho_k(\lambda) = \lim_{a\to+\infty} \frac{(-1)^k f^{(k)}(a)}{k!}(a-\lambda)^k.$$

Since for $x, y > 0$ we have

$$\rho_k(x) = \lim_{a\to+\infty} \frac{(-1)^k f^{(k)}(a)}{k!}(a-x)^k \quad = \lim_{a\to+\infty} \frac{(-1)^k f^{(k)}(a)}{k!}(a-y)^k \frac{(a-x)^k}{(a-y)^k} = \rho_k(y) \cdot 1,$$

ρ_k is constant. Defining $c_n = \sum_{k=0}^{n-1} \rho_k(x)$ we have

$$f(\lambda) = c_n + \int_\lambda^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}\, ds.$$

Since the integral vanishes for $\lambda \to +\infty$ and by assumption $f(+\infty) = 0$, we derive $c_n = 0$ for all $n \in \mathbb{N}$. □

The following Theorem 3.0.3 constitutes the main result of the present work. Its proof is split up into several steps and will primarily be based on Corollary 1.0.8.

Theorem 3.0.3. *Every $f \in \mathcal{CM}$ can be written as the Laplace-Transform of a unique positive Borel measure on $[0,+\infty)$.*

Proof. The desired uniqueness follows from Lemma 2.0.5.
Assume first that $\lim_{\lambda\to 0+} f(\lambda) =: f(0+) = 1$ and $f(+\infty) = 0$. Applying Lemma 3.0.2 yields

$$\begin{aligned} f(\lambda) &= \int_\lambda^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}\, ds \\ &= \int_{(0,+\infty)} \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1} \mathbb{1}_{[\lambda,+\infty)}\, d\lambda_1(s),\ \lambda \in (0,+\infty). \end{aligned}$$

By monotone convergence

$$1 = f(0+) = \lim_{k\to\infty} f\left(\frac{1}{k}\right) = \int_{(0,+\infty)} \frac{(-1)^n f^{(n)}(s)}{(n-1)!} s^{n-1}\, d\lambda_1(s),\ n \in \mathbb{N}. \tag{3.2}$$

Setting

$$f_n(s) := \frac{(-1)^n}{n!} f^{(n)}\left(\frac{n}{s}\right)\left(\frac{n}{s}\right)^{n+1},\ \ s \in (0,+\infty)$$

and substituting $s = n/t$ in (3.2) yields $\int_0^{+\infty} f_n(t)\, dt = 1$. Hence, f_n is a probability density function on $(0,+\infty)$ for every $n \in \mathbb{N}$.

In the following we employ the notation $a_+ := max(0, a)$. Also note that for $s, \lambda \in (0, +\infty)$ we have $s > \lambda$ if and only if $(1 - \lambda/s) > 0$. For fixed $\lambda > 0$ we obtain

$$
\begin{aligned}
f(\lambda) &= \int_\lambda^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}\, ds \\
&= \int_\lambda^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!} s^{n-1}(1-\frac{\lambda}{s})^{n-1}\, ds \\
&= \int_\lambda^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!} s^{n-1}((1-\frac{\lambda}{s})_+)^{n-1}\, ds + \int_0^{\lambda} \frac{(-1)^n f^{(n)}(s)}{(n-1)!}((1-\frac{\lambda}{s})_+)^{n-1}\, ds \\
&= \int_0^{+\infty} \frac{(-1)^n f^{(n)}(s)}{(n-1)!} s^{n-1}((1-\frac{\lambda}{s})_+)^{n-1}\, ds \\
&= \int_0^{+\infty} \frac{(-1)^n f^{(n)}(\frac{n}{t})}{n!}(\frac{n}{t})^{n+1}((1-\frac{\lambda t}{n})_+)^{n-1}\, ds \\
&= \int_0^{+\infty} ((1-\frac{\lambda t}{n})_+)^{n-1} f_n(t)\, dt.
\end{aligned}
$$

If we set $f_n(0) := 0$, f_n also constitutes a probability density function on $[0, +\infty)$. We can rewrite $f(\lambda)$ as

$$
f(\lambda) = \int_{[0,+\infty)} ((1-\frac{\lambda t}{n})_+)^{n-1} f_n(t)\, d\lambda_1(t)
$$

for every $n \in \mathbb{N}$. Defining $h_n(t) := (1 - \lambda t/n)_+^{n-1}$ we have

$$
\left| f(\lambda) - \int_{[0,+\infty)} e^{-\lambda t} f_n(t)\, d\lambda_1(t) \right| \leq \int_{[0,+\infty)} \left| h_n(t) - e^{-\lambda t} \right| f_n(t)\, d\lambda_1(t) \leq ||h_n - e^{-\lambda \cdot}||_\infty \quad (3.3)
$$

One can show that $(h_n)_{n\in\mathbb{N}}$ converges uniformly to $(t \mapsto e^{-\lambda t})$ on $[0, +\infty)$. Therefore, the left hand side converges to 0 for $n \to \infty$. We apply Corollary 1.0.8 with $\Omega = [0, +\infty)$ and $\mu_n = f_n \cdot \lambda_1$ and obtain a subnet $(f_{n(i)} \cdot \lambda_1)_{i\in I}$ that converges vaguely to a finite positive Borel measure μ on $[0, +\infty)$. As $(t \mapsto e^{-\lambda t}) \in C_0([0, +\infty), \mathbb{R})$

$$
\lim_{i\in I} \int_{[0,+\infty)} e^{-\lambda t} f_{n(i)}(t)\, d\lambda_1(t) = \lim_{i\in I} \int_{[0,+\infty)} e^{-\lambda t}\, d(f_{n(i)} \cdot \lambda_1)(t) = \int_{[0,+\infty)} e^{-\lambda t}\, d\mu(t).
$$

Taking (3.3) into account we obtain

$$
\int_{[0,+\infty)} e^{-\lambda t}\, d\mu(t) = \lim_{i\in I} \int_{[0,+\infty)} e^{-\lambda t} f_{n(i)}(t)\, d\lambda_1(t) = \lim_{n\to\infty} \int_{[0,+\infty)} e^{-\lambda t} f_n(t)\, d\lambda_1(t) = f(\lambda)
$$

for every $\lambda \in (0, +\infty)$.

In the case $0 < f(0+) < +\infty$ and $f(+\infty) = 0$ we define $g : (0, +\infty) \to \mathbb{R}$ by $g(\lambda) = f(\lambda)/f(0+)$. Because of $g(0+) = 1$, by the first part of the proof exists a positive Borel measure μ' satisfying

$$
g(\lambda) = \int_{[0,+\infty)} e^{-\lambda t}\, \mu'(t)
$$

for all $\lambda > 0$. Defining $\mu := f(0+) \cdot \mu$ we obtain $\mathcal{L}(\mu)(\lambda) = f(\lambda),\ \lambda > 0$. If $f(0+) = 0$, then f must be identically zero, due to the fact that it is monotonically decreasing and non-negative. Hence, $\mathcal{L}(\mu) = f$ for $\mu = 0$.
For a function $f \in \mathcal{CM}$, which only satisfies $f(+\infty) = 0$, and $a > 0$ we set $f_a(\lambda) := f(a+\lambda)$. Clearly, $f_a(0+) = f(a) < +\infty$ and $f(+\infty) = 0$. From the previous considerations for any $a > 0$ we obtain a finite positive Borel measure μ_a satisfying $\mathcal{L}(\mu_a) = f_a$. For $b, a > 0$ and $\lambda > 0$ we have

$$\mathcal{L}(\mu_b)(\lambda) = f(a + (\lambda + b - a)) = \int_{[0,+\infty)} e^{-(\lambda+b-a)t}\, d\mu_a = \int_{[0,+\infty)} e^{-\lambda t} e^{t(a-b)}\, d\mu_a$$

From Lemma 2.0.5 we derive $\mu_b = e^{\cdot(a-b)} \cdot \mu_a$ and in turn $e^{a\cdot} \cdot \mu_a = e^{b\cdot} \cdot \mu_b$. Consequently, the definition $\mu := e^{a\cdot} \cdot \mu_a$ of a positive Borel measure does not depend on $a > 0$. For $\lambda \in (0, +\infty)$ we conclude

$$f(\lambda) = f_{\lambda/2}(\lambda/2) = \int_{[0,+\infty)} e^{-\frac{\lambda}{2}t}\, d\mu_{\lambda/2}(t) = \int_{[0,+\infty)} e^{-\lambda t} e^{\frac{\lambda}{2}t}\, d\mu_{\lambda/2}(t) = \int_{[0,+\infty)} e^{-\lambda t}\, d\mu(t).$$

Finally, in the case $f(+\infty) = c > 0$ we define $g : (0, +\infty) \to \mathbb{R}$ by $g(\lambda) := f(\lambda) - c$. The function g belongs to $\mathcal{CM}$ and satisfies $g(+\infty) = 0$. Hence, $g = \mathcal{L}(\mu_g)$ for a positive Borel measure μ_g. From

$$f(\lambda) = g(\lambda) + c = \int_{[0,+\infty)} e^{-\lambda t}\, d\mu_g(t) + c = \int_{[0,+\infty)} e^{-\lambda t}\, d\mu_g(t) + \int_{[0,+\infty)} e^{-\lambda t}\, d(c \cdot \delta_0)(t)$$

we conclude $\mathcal{L}(\mu) = f$, where $\mu =:= \mu_g + c \cdot \delta_0$. □

Remark 3.0.4. From the previous theorem and Proposition 2.0.4 we nearly obtain an equivalence of Laplace-Transforms of positive Borel measures and the notion of completely monotone functions introduced in Definition 2.0.2. Note that we demanded a $\mathcal{CM}$ function to be defined on all positive real numbers. If one starts with a positive Borel measure, its Laplace-Transform will be of class $C^\infty((\rho_0, +\infty), \mathbb{R})$, but might not be defined for all positive numbers and, in turn, might not be a completely monotone function.

Example 3.0.5. The following examples are taken from [Mer14].

1. Clearly, all constant non-negative functions $f = c$ belong to $\mathcal{CM}$. From

   $$f(\lambda) = \int_{[0,+\infty)} e^{-\lambda t}\, d(c \cdot \delta_0)(t) = e^0 c = c$$

   we conclude $f = \mathcal{L}(c \cdot \delta_0)$.

2. The function $f : (0, +\infty) \to \mathbb{R}$ defined by

   $$f(x) = \frac{1}{x},\ x > 0$$

   belongs to $f \in \mathcal{CM}$. By

   $$f(x) = \int_0^{+\infty} e^{-xt}\, dt = \int_{[0,+\infty)} e^{-xt}\, d\lambda_1(t),\ x > 0,$$

   we see that $f = \mathcal{L}(\lambda_1|_{[0+\infty)})$.

Corollary 3.0.6. *If $f \in \mathcal{CM}$ is a non-constant function, then $f^{(n)}(\lambda) \neq 0$ for all $n \in \mathbb{N}$ and all $\lambda > 0$.*

Proof. By Theorem 3.0.3 we have $f = \mathcal{L}(\mu)$ for some positive Borel measure μ. By assumption and Lemma 2.0.5 μ can not be of the form $c \cdot \delta_0$. Applying Proposition 2.0.4 gives

$$(-1)^n f^{(n)}(\lambda) = (-1)^n \int_{[0,+\infty)} \frac{d^n}{d\lambda^n} e^{-\lambda t} \, d\mu(t) = \int_{[0,+\infty)} t^n e^{-\lambda t} \, d\mu(t) > 0$$

for all $n \in \mathbb{N}$ and all $\lambda > 0$. □

The following result is taken from [Mer14, Chapter 1].

Definition 3.0.7. Let μ, ν be positive measures on $[0, +\infty)$. We define the *convolution* of μ and ν as

$$(\mu \star \nu)(B) := \int_{[0,+\infty)} \int_{[0,+\infty)} \mathbb{1}_B(x + y) \, d\mu(x) d\nu(y)$$

for all Borel measurable $B \subseteq [0, +\infty)$.

Remark 3.0.8. Note that we can write the convolution of two positive measures μ and ν in the form

$$(\mu \star \nu)(B) = \int_{[0,+\infty)^2} \mathbb{1}_B(x + y) \, d(\mu \times \nu)(x, t) = \int_{[0,+\infty)^2} \mathbb{1}_B(T(x, t)) \, d(\mu \times \nu)(x, t),$$

where $T : [0, +\infty)^2 \longrightarrow [0, +\infty)$ is defined by $(x, t) \mapsto x + t$. Recall that T is continuous and therefore measurable. Therefore,

$$\mu \star \nu = (\mu \times \nu) \circ T^{-1}.$$

Corollary 3.0.9. *The set $\mathcal{CM}$ is closed under pointwise multiplication, i.e. $fg \in \mathcal{CM}$ for $f, g \in \mathcal{CM}$.*

Proof. By Theorem 3.0.3, there exist positive Borel measures μ and ν such that $f = \mathcal{L}(\mu)$ and $g = \mathcal{L}(\nu)$. For $\lambda > 0$ we derive

$$\begin{aligned}
(fg)(\lambda) = (\mathcal{L}(\mu)\mathcal{L}(\nu))(\lambda) &= \int_{[0,+\infty)} \int_{[0,+\infty)} e^{-\lambda(t_1+t_2)} \, d\mu(t_1) d\nu(t_2) \\
&= \int_{[0,+\infty)} e^{-\lambda t} \, d((\mu \times \nu) \circ T^{-1})(t) \\
&= \int_{[0,+\infty)} e^{-\lambda t} \, d(\mu \star \nu)(t)
\end{aligned}$$

as elements of $[0, +\infty]$. As $f(\lambda)g(\lambda) < +\infty$ the right hand side is finite for all $\lambda > 0$. Hence, $\inf\{\lambda \in \mathbb{R} : \int_{[0,+\infty)} e^{-\lambda t} \, d(\mu \star \nu)(t) < +\infty)\}$ is smaller or equal to zero and by Proposition 2.0.4 we have $fg \in \mathcal{CM}$. □

Corollary 3.0.10. *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of completely monotone functions satisfying $\sup_{n\in\mathbb{N}} f_n(0+) < +\infty$. Then there exists a subnet $\left(f_{n(k)}\right)_{k\in I}$ and a completely monotone function f, such that $\lim_{k\in I} f_{n(k)}(\lambda) = f(\lambda)$ for all $\lambda > 0$.*

Proof. Applying Theorem 3.0.3 for every $n \in \mathbb{N}$ gives a sequence of representing positive measures $(\mu_n)_{n\in\mathbb{N}}$, for which $f_n = \mathcal{L}(\mu_n)$ and

$$\mu_n([0,+\infty)) = f_n(0+) \le c := \sup_{n\in\mathbb{N}} f_n(0+) < +\infty.$$

Employing the same arguments as in Corollary 1.0.8 there exists a subnet $(\mu_{n(k)})_{k\in I}$ that converges vaguely to a finite positive Borel measure μ. In consequence

$$\lim_{k\in I} f_{n(k)}(\lambda) = \lim_{k\in I} \int_{[0,+\infty)} e^{-\lambda t}\, d\mu_{n(k)}(t) = \int_{[0,+\infty)} e^{-\lambda t}\, d\mu(t) = \mathcal{L}(\mu)(\lambda) =: f(\lambda)$$

for every $\lambda > 0$. By Proposition 2.0.4, we conclude $f \in \mathcal{CM}$. □

The following result will be presented without proof. Its proof can be found in [SSV12, Chapter 1].

Corollary 3.0.11. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of completely monotone functions such that the limit $\lim_{n\to\infty} f_n(\lambda) = f(\lambda)$ exists for all $\lambda \in (0,+\infty)$. Then $f \in \mathcal{CM}$ and $\lim_{n\to\infty} f_n^{(k)}(\lambda) = f^{(k)}(\lambda)$ for all $k \in \mathbb{N}_0$ locally uniformly in $\lambda \in (0,+\infty)$.*

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