SEMINARARBEIT

Absolutely Summing Operators

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1 Introduction and Notation

This paper aims to provide a brief insight into absolutely summing operators in Banach spaces. Chapter 2 contains a concise discussion of Banach-valued sequence spaces. The notation introduced in this chapter, along with the proven results will allow us to introduce the theory of absolutely summing operators in an elegant way. In Chapter 3 we proceed with the study of absolutely summing operators. In Section 3.1 we will show that a linear operator is absolutely summing if and only if it takes unconditionally summable sequences to absolutely summable sequences. We will use this equivalence in Section 3.2 to show that in finite-dimensional Banach spaces the unconditional convergence of a series is equivalent to its absolute convergence. Lastly, Section 3.3 provides typical examples of absolutely summing operators.

This paper is predominantly based on [DJT95, Chapter 2]. Other sources are explicitly cited in the text.

For the sake of simplicity, we will solely consider Banach spaces over $\mathbb{C}$ throughout this paper. However, with slight adjustments to the presented proofs, one can quickly obtain the same results for Banach spaces over $\mathbb{R}$.

For a given a normed vector space X, we will denote the closed unit ball in its topological dual X' by $K_1^{X'}(0)$, i.e. $K_1^{X'}(0) = \{f \in X' : \|f\| \leq 1\}$.

Given normed vector spaces X and Y we denote by $L(X,Y)$ the space of linear operators from X to Y and by $L_b(X,Y)$ the space of bounded linear operators from X to Y.

We will need the following corollary of the Hahn-Banach theorem implying that the dual space of a normed vector space X is comprehensive enough to encode properties of the elements of X, see [BKW20, 5.2.4].

Proposition 1.1. *Let X be a normed vector space. Then*

$$\|x\| = \sup\{|f(x)| : f \in K_1^{X'}(0)\} \tag{1.1}$$

for every x in X.

Furthermore, we will make use of the closed graph theorem, see [BKW20, 4.4.2].

Theorem 1.2 (Closed Graph Theorem). *Let X and Y be Banach spaces and $T \in L(X,Y)$. If the graph of T is closed in $X \times Y$, then T is bounded.*

2 Banach-Valued Sequence Spaces

For simplicity of notation we set $l^1 := \{(z_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} |z_n| < +\infty\}$. Renownedly, l^1 forms a Banach space if provided with $\|(z_n)_{n\in\mathbb{N}}\|_{l^1} := \sum_{n=1}^{\infty} |z_n|$.

Definition 2.1. Let X be a Banach space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X

- is called strongly summable (or absolutely summable), if $(\|x_n\|)_{n\in\mathbb{N}} \in l^1$.
- is called weakly summable, if $(f(x_n))_{n\in\mathbb{N}} \in l^1$ for all $f \in X'$.

Let $l^1_s(X)$ denote the set of all strongly summable sequences in X and $l^1_w(X)$ the set of all weakly summable sequences in X.

Example 2.2. Sequences $(x_n)_{n\in\mathbb{N}}$ in Banach spaces that are eventually 0, i.e. there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N$, are both strongly and weakly summable. By abuse of notation, we will denote such sequences by $(x_n)_{n=1}^{N}$.

Remark 2.3. The sets $l^1_s(X)$ and $l^1_w(X)$ are vector spaces under pointwise addition and scalar multiplication. Moreover, it is easy to verify that

$$\|(x_n)_{n\in\mathbb{N}}\|_s := \sum_{n=1}^{\infty} \|x_n\|$$

constitutes a norm on $l^1_s(X)$. We call $\|.\|_s$ the strongly summable norm.

Proposition 2.4. *$l^1_s(X)$ is a Banach space with norm $\|.\|_s$.*

Proof. The proof is almost identical to the proof that l^1 is a Banach space, one merely has to exchange the absolute value bars with the norm in X. For the sake of completeness, we provide a rigorous proof.
Let $((x_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ be a Cauchy sequence in $l^1_s(X)$. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$\|(x_{n,k})_{k\in\mathbb{N}} - (x_{m,k})_{k\in\mathbb{N}}\|_s < \epsilon \text{ for all } m > n \geq N.$$

By definition of $\|.\|_s$ we obtain

$$\|x_{n,k} - x_{m,k}\| \leq \sum_{j=1}^{\infty} \|x_{n,j} - x_{m,j}\| = \|(x_{n,j})_{j\in\mathbb{N}} - (x_{m,j})_{j\in\mathbb{N}}\|_s < \epsilon \tag{2.1}$$

for all $m > n \geq N$ and $k \in \mathbb{N}$. Hence, $(x_{n,k})_{n\in\mathbb{N}}$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. Since X is a Banach space, the limit $\lim_{n\to\infty} x_{n,k}$ exists for all $k \in \mathbb{N}$. Let us denote these

limits by s_k.
The sequence $(s_k)_{k\in\mathbb{N}}$ is strongly summable. To see this, recall that Cauchy sequences are bounded. Hence, there exists $C > 0$ such that $\|(x_{n,k})_{k\in\mathbb{N}}\|_s \le C$ for all $n \in \mathbb{N}$. For fixed $m \in \mathbb{N}$ and any $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{m} \|x_{n,k}\| \le \|(x_{n,k})_{k\in\mathbb{N}}\|_s \le C.$$

As n tends to infinity we obtain $\sum_{k=1}^{m} \|s_k\| \le C$. m being arbitrary yields $\sum_{k=1}^{\infty} \|s_k\| = \|(s_k)_{k\in\mathbb{N}}\|_s \le C$ and $(s_k)_{k\in\mathbb{N}} \in l_s^1(X)$.
From (2.1) we conclude for $J \in \mathbb{N}$

$$\sum_{k=1}^{J} \|x_{n,k} - x_{m,k}\| < \epsilon$$

as long as $m > n \ge N$. For $m \to \infty$ we obtain

$$\sum_{k=1}^{J} \|x_{n,k} - s_k\| \le \epsilon. \tag{2.2}$$

J being arbitrary in (2.2) yields $\|(x_{n,k})_{k\in\mathbb{N}} - (s_k)_{k\in\mathbb{N}}\|_s \le \epsilon$ for all $n \ge N$. Thus, $((x_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ converges to $(s_k)_{k\in\mathbb{N}}$ in $l_s^1(X)$. □

Definition 2.5. Let X be a Banach space. The weakly summable norm of a sequence $(x_n)_{n\in\mathbb{N}} \in l_w^1(X)$ is defined by

$$\|(x_n)_{n\in\mathbb{N}}\|_w := \sup\{\sum_{n=1}^{\infty} |f(x_n)| : f \in K_1^{X'}(0)\}.$$

Lemma 2.6. *Given a Banach space X, the weakly summable norm is a well-defined norm on $l_w^1(X)$.*

Proof. Any weakly summable $(x_n)_{n\in\mathbb{N}}$ gives rise to a linear operator $R : X' \to l^1$, $Rf = (f(x_n))_{n\in\mathbb{N}}$. We use the closed graph theorem to show that R is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence that converges to f in X' such that $(Rf_k)_{k\in\mathbb{N}}$ converges to $(s_n)_{n\in\mathbb{N}}$ in l^1. For fixed $n \in \mathbb{N}$ we have

$$\begin{aligned}|s_n - f(x_n)| &\le |s_n - f_k(x_n)| + |f_k(x_n) - f(x_n)| \\ &\le \|(s_m)_{m\in\mathbb{N}} - Rf_k\|_{l^1} + \|f - f_k\|_{X'} \cdot \|x_n\| \xrightarrow{k\to\infty} 0.\end{aligned}$$

Hence, $s_n = f(x_n)$ for all $n \in \mathbb{N}$. Therefore, R has a closed graph and $\|(x_n)_{n\in\mathbb{N}}\|_w = \|R\| < +\infty$.
It is routine to show that the weakly summable norm satisfies all norm axioms. □

Remark 2.7. Given a Banach space X, $l_s^1(X)$ is a linear subspace of $l_w^1(X)$ whereby the inclusion map $\iota : l_s^1(X) \to l_w^1(X)$ satisfies $\|\iota\| \leq 1$. If $X \neq \{0\}$, then $\|\iota\| = 1$.
In fact, given $(x_n)_{n\in\mathbb{N}}$ in $l_s^1(X)$ we have

$$\left\|(f(x_n))_{n\in\mathbb{N}}\right\|_{l^1} \leq \|f\| \cdot \|(x_n)_{n\in\mathbb{N}}\|_s \leq \|(x_n)_{n\in\mathbb{N}}\|_s.$$

for all f in $K_1^{X'}(0)$. Hence, $(x_n)_{n\in\mathbb{N}}$ is weakly summable and $\|\iota\| \leq 1$. Assuming that $X \neq \{0\}$, let u be a vector in X of length 1 and $(s_n)_{n\in\mathbb{N}}$ the sequence with $s_1 = u$ and $s_n = 0$ for $n > 1$. As a consequence of Proposition 1.1, $\|(s_n)_{n\in\mathbb{N}}\|_w = \|(s_n)_{n\in\mathbb{N}}\|_s$ and $\|\iota\| = 1$.

Proposition 2.8. *Let X be a Banach space. $l_w^1(X)$ is a Banach space when provided with $\|.\|_w$.*

Proof. Let $((x_{n,k})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ be a Cauchy sequence in $l_w^1(X)$. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|(x_{n,k})_{k\in\mathbb{N}} - (x_{m,k})_{k\in\mathbb{N}}\|_w = \sup_{f\in K_1^{X'}(0)} \sum_{k=1}^{\infty} |f(x_{n,k}) - f(x_{m,k})| < \epsilon \tag{2.3}$$

for all $m, n \geq N$. Due to (1.1) we have

$$\|x_{n,k} - x_{m,k}\| = \sup_{f\in K_1^{X'}(0)} |f(x_{n,k} - x_{m,k})| < \epsilon$$

for all $m, n \geq N$ and all $k \in \mathbb{N}$. Hence, $(x_{n,k})_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is a Banach space, we have $x_k := \lim_{n\to\infty} x_{n,k} \in X$.
In order to see the weak summability of $(x_k)_{k\in\mathbb{N}}$, fix $L \in \mathbb{N}$ and $f \in K_1^{X'}(0)$. Since Cauchy sequences are bounded, we have $\|(x_{n,k})_{k\in\mathbb{N}}\|_w \leq C$ for all $n \in \mathbb{N}$ and some $C \geq 0$. By definition of the weakly summable norm

$$\sum_{k=1}^{L} |f(x_{n,k})| \leq \|(x_{n,k})_{k\in\mathbb{N}}\|_w \leq C.$$

Letting n tend to infinity yields $\sum_{k=1}^{L} |f(x_k)| \leq C$ for all $L \in \mathbb{N}$ and $f \in K_1^{X'}(0)$ which implies $\|(x_k)_{k\in\mathbb{N}}\|_w \leq C$.
In order to show convergence, again fix $L \in \mathbb{N}$ and let $m, n \geq N$ be as in (2.3). For any $f \in K_1^{X'}(0)$ we have $\sum_{k=1}^{L} |f(x_{n,k}) - f(x_{m,k})| < \epsilon$. Letting m tend to infinity yields $\sum_{k=1}^{L} |f(x_{n,k}) - f(x_k)| \leq \epsilon$. Since L and f were arbitrary, we obtain

$$\|(x_{n,k})_{k\in\mathbb{N}} - (x_k)_{k\in\mathbb{N}}\|_w = \sup_{f\in K_1^{X'}(0)} \sum_{k=1}^{\infty} |f(x_{n,k}) - f(x_k)| \leq \epsilon$$

for all $n \geq N$. □

3 Absolutely Summing Operators

3.1 Basic Concepts

Definition 3.1. Let X and Y be Banach spaces. A linear operator $T : X \to Y$ is called absolutely summing if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$

$$\sum_{j=1}^{n} \|Tx_j\| \leq C \cdot \sup\{\sum_{j=1}^{n} |f(x_j)| : f \in K_1^{X'}(0)\}. \tag{3.1}$$

We shall write $\pi(T)$ for the smallest such C and $\Pi(X, Y)$ for the set of all absolutely summing operators from X into Y.

Remark 3.2. The supremum in (3.1) exists for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$, since it is bounded by $\sum_{j=1}^{n} \|x_j\|$.

Remark 3.3. Using the same notation as in Section 2.2 we can express (3.1) by

$$\left\|(Tx_j)_{j=1}^n\right\|_s \leq C\left\|(x_j)_{j=1}^n\right\|_w$$

where $\|.\|_s$ is the strongly summable norm on $l_s^1(Y)$ and $\|.\|_w$ is the weakly summable norm on $l_w^1(X)$.

Lemma 3.4. $\Pi(X, Y)$ *is a linear subspace of* $L_b(X, Y)$ *and* $\pi(.)$ *defines a norm on* $\Pi(X, Y)$ *satisfying* $\|.\|_{L_b(X,Y)} \leq \pi(.)$

Proof. By (1.1) $\|x\| = \sup\{|f(x)| : f \in K_1^{X'}(0)\}$. As an immediate consequence from (3.1) with $n = 1$ we obtain $\|.\|_{L_b(X,Y)} \leq \pi(.)$.
Suppose $T, S \in \Pi(X, Y)$ and $\lambda \in \mathbb{C}$. For $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$ we obtain

$$\begin{aligned}\sum_{j=1}^{n} \|(T + \lambda S)x_j\| &\leq \sum_{j=1}^{n} \|Tx_j\| + |\lambda| \cdot \sum_{j=1}^{n} \|Sx_j\| \\ &\leq (\pi(T) + |\lambda| \cdot \pi(S)) \cdot \sup\{\sum_{j=1}^{n} |f(x_j)| : f \in K_1^{X'}(0)\}.\end{aligned}$$

This shows that $\pi(T) + |\lambda| \cdot \pi(S)$ is an appropriate constant such that (3.1) holds true for $T + \lambda S$ and consequently

$$\pi(T + \lambda S) \leq \pi(T) + |\lambda|\pi(S). \tag{3.2}$$

Hence, $\Pi(X, Y)$ is a linear subspace of $L_b(X, Y)$ and $\pi(.)$ satisfies the triangle inequality. Clearly, $\pi(0) = 0$. The absolute homogeneity of $\pi(.)$ follows from (3.2). Together with $\|.\|_{L_b(X,Y)} \leq \pi(.)$ we see that $\pi(.)$ is indeed a norm. $\square$

Lemma 3.5. *Let X and Y be Banach spaces and $T \in L_b(X, Y)$. Then the induced mapping $\hat{T}$ given by $\hat{T}((x_n)_{n\in\mathbb{N}}) = (Tx_n)_{n\in\mathbb{N}}$ constitutes a bounded linear operator from $l^1_w(X)$ to $l^1_w(Y)$ and from $l^1_s(X)$ to $l^1_s(Y)$ with $\|\hat{T}\|_{L_b(l^1_w(X),l^1_w(Y))} \leq \|T\|$ and $\|\hat{T}\|_{L_b(l^1_s(X),l^1_s(Y))} \leq \|T\|$.*

Proof. The linearity of $\hat{T}$ is clear. Given any $(x_n)_{n\in\mathbb{N}} \in l^1_s(X)$ we have

$$\|(Tx_n)_{n\in\mathbb{N}}\|_s = \sum_{n=1}^{\infty} \|Tx_n\| \leq \|T\| \cdot \sum_{n=1}^{\infty} \|x_n\| = \|T\| \cdot \|(x_n)_{n\in\mathbb{N}}\|_s$$

showing that $\|\hat{T}\|_{L_b(l^1_s(X),l^1_s(Y))} \leq \|T\|$.
Since for any $g \in K^{Y'}_1(0)$ the functional $g \circ T$ is an element of $K^{X'}_{\|T\|}(0)$, we derive for $(x_n)_{n\in\mathbb{N}} \in l^1_w(X)$

$$\begin{aligned}
\|(Tx_n)_{n\in\mathbb{N}}\|_w &= \sup\{\sum_{n=1}^{\infty} |g(Tx_n)| : g \in K^{Y'}_1(0)\} \\
&\leq \sup\{\sum_{n=1}^{\infty} |f(x_n)| : f \in K^{X'}_{\|T\|}(0)\} \\
&= \|T\| \cdot \sup\{\sum_{n=1}^{\infty} |f(x_n)| : f \in K^{X'}_1(0)\} = \|T\| \cdot \|(x_n)_{n\in\mathbb{N}}\|_w.
\end{aligned}$$

Thus, $\|\hat{T}\|_{L_b(l^1_w(X),l^1_w(Y))} \leq \|T\|$. $\square$

Theorem 3.6. *Let X and Y be Banach spaces. An operator $T \in L_b(X, Y)$ is absolutely summing if and only if $\hat{T}(l^1_w(X)) \subseteq l^1_s(Y)$ where $\hat{T}$ is defined as in Lemma 3.5. In this case, $\|\hat{T}\|_{L_b(l^1_w(X),l^1_s(Y))} = \pi(T)$.*

Proof. Let us first assume that $T \in \Pi(X, Y)$. Given $(x_n)_{n\in\mathbb{N}} \in l^1_w(X)$ we obtain

$$\begin{aligned}
\left\|\hat{T}((x_n)_{n\in\mathbb{N}})\right\|_s &= \sum_{n=1}^{\infty} \|Tx_n\| = \sup_{m\in\mathbb{N}} \sum_{n=1}^{m} \|Tx_n\| \\
&\leq \pi(T) \cdot \sup_{m\in\mathbb{N}} \sup_{f\in K^{X'}_1(0)} \sum_{n=1}^{m} |f(x_n)| = \pi(T) \cdot \sup_{f\in K^{X'}_1(0)} \sup_{m\in\mathbb{N}} \sum_{n=1}^{m} |f(x_n)| \\
&= \pi(T) \cdot \sup_{f\in K^{X'}_1(0)} \sum_{n=1}^{\infty} |f(x_n)| = \pi(T) \cdot \|(x_n)_{n\in\mathbb{N}}\|_w < +\infty.
\end{aligned}$$

Therefore, $\hat{T}(l^1_w(X)) \subseteq l^1_s(Y)$ and $\|\hat{T}\| \leq \pi(T)$.
Conversely, assume that $\hat{T}(l^1_w(X)) \subseteq l^1_s(Y)$. At first we show that $\hat{T} : l^1_w(X) \to l^1_s(Y)$ is

bounded. To see this, let $((x_{j,n})_{n\in\mathbb{N}})_{j\in\mathbb{N}}$ be a sequence that converges to $(x_n)_{n\in\mathbb{N}}$ in $l_w^1(X)$ such that $((Tx_{j,n})_{n\in\mathbb{N}})_{j\in\mathbb{N}}$ converges to $(s_n)_{n\in\mathbb{N}}$ in $l_s^1(Y)$. For fixed $n_0 \in \mathbb{N}$ we obtain from (1.1)

$$\begin{aligned}\|Tx_{n_0} - s_{n_0}\| &\leq \|Tx_{n_0} - Tx_{j,n_0}\| + \|Tx_{j,n_0} - s_{n_0}\| \\ &\leq \|T\| \cdot \|x_{n_0} - x_{j,n_0}\| + \|(Tx_{j,n})_{n\in\mathbb{N}} - (s_n)_{n\in\mathbb{N}}\|_s \\ &= \|T\| \cdot \sup\{|f(x_{n_0} - x_{j,n_0})| : f \in K_1^{X'}(0)\} + \|(Tx_{j,n})_{n\in\mathbb{N}} - (s_n)_{n\in\mathbb{N}}\|_s \\ &\leq \|T\| \cdot \|(x_n)_{n\in\mathbb{N}} - (x_{j,n})_{n\in\mathbb{N}}\|_w + \|(Tx_{j,n})_{n\in\mathbb{N}} - (s_n)_{n\in\mathbb{N}}\|_s.\end{aligned}$$

By assumption, the right side converges to 0 if j tends to infinity. $Tx_{n_0} = s_{n_0}$ for any $n_0 \in \mathbb{N}$ shows that $\hat{T}$ has a closed graph. By the closed graph theorem, $\hat{T}$ is bounded. Finally, for vectors $u_1, \ldots, u_n \in X$ seen as a sequence $(u_j)_{j=1}^n \in l_w^1(X)$ as in Example 2.2

$$\|(Tu_j)_{j=1}^n\|_s = \|\hat{T}(u_j)_{j=1}^n\|_s \leq \|\hat{T}\| \cdot \|(u_j)_{j=1}^n\|.$$

Therefore, T is absolutely summing with $\pi(T) \leq \|\hat{T}\|$. □

Corollary 3.7 (Ideal Property of Absolutely Summing Operators)**.** *Let W, X, Y, Z be Banach spaces and $R \in L_b(W, X)$, $T \in L_b(X, Y)$ and $S \in L_b(Y, Z)$. If T is absolutely summing, then STR is absolutely summing satisfying $\pi(STR) \leq \|S\|\,\pi(T)\|R\|$.*

Proof. By Lemma 3.5 and Theorem 3.6 the operators R, T, S induce operators $\hat{R} : l_w^1(W) \to l_w^1(X), \hat{T} : l_w^1(X) \to l_s^1(Y)$ and $\hat{S} : l_s^1(Y) \to l_s^1(Z)$ with $\|\hat{R}\| \leq \|R\|, \|\hat{T}\| = \pi(T)$ and $\|\hat{S}\| \leq \|S\|$. Consequently, $\widehat{STR} = \hat{S}\hat{T}\hat{R}$ maps $l_w^1(W)$ to $l_s^1(Z)$. According to Theorem 3.6, STR is absolutely summing and $\pi(STR) = \|\widehat{STR}\|$ where

$$\pi(STR) = \|\widehat{STR}\| \leq \|STR\| \leq \|S\|\|T\|\|R\| = \|S\|\pi(T)\|R\|.$$

□

Corollary 3.8. *Let X and Y be Banach spaces, then $\Pi(X, Y)$ provided with $\pi(.)$ is a Banach space.*

Proof. We merely need to show completeness, since $\Pi(X, Y)$ is a normed space as seen in Lemma 3.4. By this Lemma we also have $\|.\|_{L_b(X,Y)} \leq \pi(.)$ which implies that a Cauchy sequence $(T_n)_{n\in\mathbb{N}}$ in $\Pi(X, Y)$ with respect to $\pi(.)$ is also a Cauchy sequence in $L_b(X, Y)$ with respect to $\|.\|$. Since $L_b(X, Y)$ is complete, $(T_n)_{n\in\mathbb{N}}$ converges to some $T \in L_b(X, Y)$ in the operator norm. By Lemma 3.5 and Theorem 3.6 we have $\hat{T} \in L_b(l_w^1(X), l_w^1(Y))$ and $\hat{T}_n \in L_b(l_w^1(X), l_s^1(Y))$ for all $n \in \mathbb{N}$. $\|\hat{T}_n - \hat{T}_m\| = \|\widehat{T_n - T_m}\| = \pi(T_n - T_m)$ for all $n, m \in \mathbb{N}$ shows that $(\hat{T}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L_b(l_w^1(X), l_s^1(Y))$ and thus converges to some $S \in L_b(l_w^1(X), l_s^1(Y))$. Recall that by Remark 2.7 the inclusion map $\iota : l_s^1(X) \to l_w^1(X)$ is bounded with $\|\iota\| \leq 1$. For any $n \in \mathbb{N}$ we obtain

$$\begin{aligned}\|\hat{T} - \iota S\|_{L_b(l_w^1(X),l_w^1(Y))} &\leq \|\hat{T} - \iota\hat{T}_n\|_{L_b(l_w^1(X),l_w^1(Y))} + \|\iota(\hat{T}_n - S)\|_{L_b(l_w^1(X),l_w^1(Y))} \\ &\leq \|\hat{T} - \iota\hat{T}_n\|_{L_b(l_w^1(X),l_w^1(Y))} + \|\iota\|\,\|\hat{T}_n - S\|_{L_b(l_w^1(X),l_s^1(Y))}\end{aligned}$$

$$\leq \|\widehat{T - T_n}\|_{L_b(l^1_w(X),l^1_w(Y))} + \|\hat{T}_n - S\|_{L_b(l^1_w(X),l^1_s(Y))}$$
$$\leq \|T - T_n\|_{L_b(X,Y)} + \|\hat{T}_n - S\|_{L_b(l^1_w(X),l^1_s(Y))}$$

which yields $\hat{T} = S$ when n tends to infinity. Consequently, $\hat{T}$ maps $l^1_w(X)$ to $l^1_s(Y)$ and T is absolutely summing by Theorem 3.6. Finally, with T_n and T also $T_n - T$ is absolutely summing with $\pi(T_n - T) = \|\widehat{T_n - T}\|$. Henceforth,

$$\pi(T_n - T) = \|\hat{T}_n - \hat{T}\|_{L_b(l^1_w(X),l^1_s(Y))} = \|\hat{T}_n - S\|_{L_b(l^1_w(X),l^1_s(Y))} \xrightarrow{n\to\infty} 0.$$

□

Remark 3.9. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed vector space X. Let $\mathcal{E}(\mathbb{N})$ denote the set of all finite subsets of $\mathbb{N}$. If the net $(\sum_{k\in A} x_k)_{A\in\mathcal{E}(\mathbb{N})}$ converges in X where $\mathcal{E}(\mathbb{N})$ is directed by the set inclusion, we denote its limit by $\sum_{n\in\mathbb{N}} x_n$ and say that $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally in X.

Lemma 3.10. *Let X be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. The following assertions are equivalent:*

(i) $\sum_{n\in\mathbb{N}} x_n$ *converges unconditionally.*

(ii) $\forall\epsilon > 0 \exists N \in \mathbb{N} : M \in \mathcal{E}(\{k \in \mathbb{N} : k \geq N\}) \implies \|\sum_{n\in M} x_n\| < \epsilon.$

(iii) $\forall\epsilon > 0 \exists N \in \mathbb{N} : \sup\{\sum_{n=N}^{\infty} |f(x_n)| : f \in K_1^{X'}(0)\} < \epsilon.$

Proof. (i)⇒(ii): Since $(\sum_{k\in A} x_k)_{A\in\mathcal{E}(\mathbb{N})}$ converges, it is a Cauchy net. Given $\epsilon > 0$, we therefore find $A_0 \in \mathcal{E}(\mathbb{N})$ such that

$$\left\|\sum_{k\in A} x_k - \sum_{k\in B} x_k\right\| = \left\|\sum_{k\in A\setminus B} x_k - \sum_{k\in B\setminus A} x_k\right\| < \epsilon \tag{3.3}$$

for all finite $A, B \supseteq A_0$. Set $N := \max(A_0) + 1$ and let $M \in \mathcal{E}(\{k \in \mathbb{N} : k \geq N\})$. Using (3.3) with $B := \{1, \ldots, \max(A_0)\}$ and $A := M \cup B$ we obtain $\|\sum_{k\in M} x_k\| < \epsilon$ since $A, B \supseteq A_0$, $A \setminus B = M$ and $B \setminus A = \emptyset$.
(ii)⇒(i): Since X is a Banach space, it is sufficient to show that $(\sum_{k\in A} x_k)_{A\in\mathcal{E}(\mathbb{N})}$ is a Cauchy net. By (ii) for $\epsilon > 0$ we find $N \in \mathbb{N}$ such that $\|\sum_{n\in M} x_n\| < \epsilon/2$ for all $M \in \mathcal{E}(\{k \in \mathbb{N} : k \geq N\})$. Set $A_0 := \{1, \ldots, N\}$. For any finite $A, B \supseteq A_0$ we conclude

$$\left\|\sum_{k\in A} x_k - \sum_{k\in B} x_k\right\| = \left\|\sum_{k\in A\setminus B} x_k - \sum_{k\in B\setminus A} x_k\right\| \leq \left\|\sum_{k\in A\setminus B} x_k\right\| + \left\|\sum_{k\in B\setminus A} x_k\right\| < \epsilon$$

since $A \setminus B$ and $B \setminus A$ are elements of $\mathcal{E}(\{k \in \mathbb{N} : k \geq N\})$. Thus, $(\sum_{k\in A} x_k)_{A\in\mathcal{E}(\mathbb{N})}$ is a Cauchy net.

(ii)$\Rightarrow$(iii): Fix $\epsilon > 0$ and take $N \in \mathbb{N}$ such that (ii) holds true for $\epsilon/5$ instead of ϵ. For $f \in K_1^{X'}(0)$ and $L \geq N$ we define subsets of $\mathcal{E}(\{k \in \mathbb{N} : k \geq N\})$ by

$$\mathcal{M}_{\mathrm{Re}}^{+} := \{N \leq k \leq L : \operatorname{Re} f(x_k) > 0\},\ \mathcal{M}_{\mathrm{Re}}^{-} := \{N \leq k \leq L : \operatorname{Re} f(x_k) < 0\}$$
$$\mathcal{M}_{\mathrm{Im}}^{+} := \{N \leq k \leq L : \operatorname{Im} f(x_k) > 0\},\ \mathcal{M}_{\mathrm{Im}}^{-} := \{N \leq k \leq L : \operatorname{Im} f(x_k) < 0\}.$$

Notice that $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathbb{R}$-linear functionals satisfying $\|\operatorname{Re} f\|, \|\operatorname{Im} f\| \leq \|f\|$. Hence, by using (ii) we obtain

$$\begin{aligned}
&\sum_{k=N}^{L} |f(x_k)| \leq \sum_{k=N}^{L} |\operatorname{Re} f(x_k)| + \sum_{k=N}^{L} |\operatorname{Im} f(x_k)| \\
&= \sum_{k\in\mathcal{M}_{\mathrm{Re}}^{+}} \operatorname{Re} f(x_k) - \sum_{k\in\mathcal{M}_{\mathrm{Re}}^{-}} \operatorname{Re} f(x_k) + \sum_{k\in\mathcal{M}_{\mathrm{Im}}^{+}} \operatorname{Im} f(x_k) - \sum_{k\in\mathcal{M}_{\mathrm{Im}}^{-}} \operatorname{Im} f(x_k) \\
&= |\operatorname{Re} f(\sum_{k\in\mathcal{M}_{\mathrm{Re}}^{+}} x_k)| + |\operatorname{Re} f(\sum_{k\in\mathcal{M}_{\mathrm{Re}}^{-}} -x_k)| + |\operatorname{Im} f(\sum_{k\in\mathcal{M}_{\mathrm{Im}}^{+}} x_k)| + |\operatorname{Im} f(\sum_{k\in\mathcal{M}_{\mathrm{Im}}^{-}} -x_k)| \\
&\leq \|f\| \left(\|\sum_{k\in\mathcal{M}_{\mathrm{Re}}^{+}} x_k\| + \|\sum_{k\in\mathcal{M}_{\mathrm{Re}}^{-}} x_k\| + \|\sum_{k\in\mathcal{M}_{\mathrm{Im}}^{+}} x_k\| + \|\sum_{k\in\mathcal{M}_{\mathrm{Im}}^{-}} x_k\| \right) < \|f\| \frac{4\epsilon}{5} \leq \frac{4\epsilon}{5}.
\end{aligned}$$

Thus, $\sum_{k=N}^{\infty} |f(x_n)| \leq 4\epsilon/5$ for all f in $K_1^{X'}(0)$ which implies (iii).
(iii)$\Rightarrow$(ii): Given $\epsilon > 0$ take $N \in \mathbb{N}$ such that (iii) holds true. For any $M \in \mathcal{E}(\{k \in \mathbb{N} : k \geq N\})$ we obtain

$$\left\| \sum_{n\in M} x_n \right\| \overset{(1.1)}{=} \sup_{f\in K_1^{X'}(0)} |f(\sum_{n\in M} x_n)| \leq \sup_{f\in K_1^{X'}(0)} \sum_{n\in M} |f(x_n)| \leq \sup_{f\in K_1^{X'}(0)} \sum_{n=N}^{\infty} |f(x_n)| < \epsilon.$$

$\square$

Definition 3.11. We say that an operator $T : X \to Y$ takes unconditionally summable sequences to absolutely summable sequences, if $\sum_{n\in\mathbb{N}} \|Tx_n\| < +\infty$ whenever $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally in X.

Theorem 3.12. *Let X and Y be Banach spaces. A linear $T : X \to Y$ is absolutely summing if and only if it takes unconditionally summable sequences to absolutely summable sequences.*

Proof. Let us first assume that T is absolutely summing. Given an unconditionally convergent $\sum_{n\in\mathbb{N}} x_n$ in X, according to Lemma 3.10 we find $N \in \mathbb{N}$ such that $\sup\{\sum_{n=N}^{\infty} |f(x_n)| : f \in K_1^{X'}(0)\} < 1$. Due to

$$\|(x_n)_{n\in\mathbb{N}}\|_w = \sup_{f\in K_1^{X'}(0)} \sum_{n=1}^{\infty} |f(x_n)| \leq \sum_{n=1}^{N-1} \|x_n\| + 1 < +\infty,$$

$(x_n)_{n\in\mathbb{N}}$ is weakly summable. As T is bounded by Lemma 3.4, we can make use of Theorem 3.6 and conclude $\hat{T}(l_w^1(X)) \subseteq l_s^1(Y)$. In particular, $\sum_{n\in\mathbb{N}} \|Tx_n\| < +\infty$.
If, on the contrary, T is not absolutely summing, then for all $k \in \mathbb{N}$ there exist $x_1^k, \ldots, x_{n(k)}^k$ in X such that

$$\sum_{j=1}^{n(k)} \|Tx_j^k\| > 2^k \cdot \sup\{\sum_{j=1}^{n(k)} |f(x_j^k)| : f \in K_1^{X'}(0)\}.$$

Since we can multiply $x_1^k, \ldots, x_{n(k)}^k$ by an appropriate constant, we can assume that $\sum_{j=1}^{n(k)} \|Tx_j^k\| = 1$ for each k and therefore have

$$\sup\{\sum_{j=1}^{n(k)} |f(x_j^k)| : f \in K_1^{X'}(0)\} < 2^{-k}. \tag{3.4}$$

We define a sequence $(u_n)_{n\in\mathbb{N}}$ by $u(\sum_{r=1}^{s} n(r) + j) := x_j^{s+1}$ where $s \in \mathbb{N}_0$ and $j = 1, \ldots, n(s+1)$. This sequence is of the form $(x_1^1, \ldots, x_{n(1)}^1, x_1^2, \ldots, x_{n(2)}^2, \ldots)$ and will turn out to be unconditionally convergent whereas $\sum_{n\in\mathbb{N}} \|Tx_n\| = +\infty$. To see this, fix $\epsilon > 0$ and choose $K \in \mathbb{N}$ such that $2^{-K+1} < \epsilon$. With $N := \sum_{r=1}^{K-1} n(r) + 1$ and using (3.4) we get

$$\sum_{n=N}^{\infty} |f(u_n)| = \sum_{k=K}^{\infty} \sum_{j=1}^{n(k)} |f(x_j^k)| < \sum_{k=K}^{\infty} 2^{-k} = 2^{-K+1} < \epsilon$$

for all f in $K_1^{X'}(0)$. Therefore, $(u_n)_{n\in\mathbb{N}}$ satisfies property (iii) in Lemma 3.10 and is thus unconditionally summable. $\sum_{j=1}^{n(k)} \|Tx_j^k\| = 1$ implies that $(Tu_n)_{n\in\mathbb{N}}$ is not strongly summable. □

3.2 Finite Dimension

Remark 3.13. If $T \in L_b(X, Y)$ has finite rank n, then $T = \sum_{j=1}^{n} f_j y_j$ for some $f_1, \ldots f_n \in X'$ and $y_1, \ldots, y_n \in T(X)$. To see this, let $\{y_1, \ldots, y_n\}$ be a basis of $T(X)$. Since $T(X)$ is finite-dimensional, $y_j^* : T(X) \to \mathbb{C}, \sum_{k=1}^{n} \alpha_k y_k \mapsto \alpha_j$, is continuous and linear for every $j = 1, \ldots, n$. Hence, $Tx = \sum_{j=1}^{n} y_j^*(Tx) y_j = \sum_{j=1}^{n} f_j(x) y_j$ if we set $f_j := y_j^* \circ T$.

Proposition 3.14. *Let X and Y be Banach spaces and $T \in L_b(X, Y)$ with finite rank. Then T is absolutely summing.*

Proof. We first assume that $T \in L_b(X, Y)$ has rank one. As seen in the above remark, T can be represented in the form $Tx = f(x) \cdot y_0$ for some $f \in X' \setminus \{0\}$ and $y_0 \in T(X) \setminus \{0\}$. For any $x_1, \ldots, x_n \in X$ we obtain

$$\sum_{j=1}^{n} \|Tx_j\| = \sum_{j=1}^{n} |f(x_j)| \|y_0\| = \|f\| \|y_0\| \sum_{j=1}^{n} \frac{|f(x_j)|}{\|f\|}$$

$$\leq \|f\| \|y_0\| \cdot \sup\{\sum_{j=1}^{n} |g(x_j)| : g \in K_1^{X'}(0)\}$$

which shows that T is absolutely summing with $\pi(T) \leq \|f\| \, \|y_0\|$. According to Remark 3.13, a bounded operator with finite rank can be written as the sum of bounded rank one operators. Therefore, it is an element of the vector space $\Pi(X, Y)$ and is thus absolutely summing. □

Corollary 3.15. *A finite-dimensional Banach space X satisfies $l_w^1(X) = l_s^1(X)$.*

Proof. Clearly $l_s^1(X) \subseteq l_w^1(X)$. By Proposition 3.14 the identity I on X is absolutely summing. Hence, $l_w^1(X) = \hat{I}(l_w^1(X)) \subseteq l_s^1(X)$ according to Theorem 3.6. □

Corollary 3.16. *Let X be a finite-dimensional Banach space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally if and only if $\sum_{n\in\mathbb{N}} \|x_n\| < +\infty$.*

Proof. Let us assume that $\sum_{n\in\mathbb{N}} \|x_n\| < +\infty$. Let $\epsilon > 0$ and choose $A_0 \in \mathcal{E}(\mathbb{N})$ such that $|\sum_{n\in A\setminus B} \|x_n\| - \sum_{n\in B\setminus A} \|x_n\|| < \epsilon$ for all $A, B \in \mathcal{E}(\mathbb{N})$ with $A, B \supseteq A_0$. Then

$$\left\| \sum_{n\in A\setminus B} x_n - \sum_{n\in B\setminus A} x_n \right\| \leq \sum_{n\in A\triangle B} \|x_n\| = \sum_{n\in A\cup B} \|x_n\| - \sum_{n\in B\cap A} \|x_n\| < \epsilon$$

shows that $(\sum_{n\in A} x_n)_{A\in\mathcal{E}(\mathbb{N})}$ is a Cauchy net in X and thus converges.
Conversely, assume that $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally. By Proposition 3.14 the identity operator I on X is absolutely summing. As seen in Theorem 3.12, I takes unconditionally summable sequences to absolutely summable sequences which means $\sum_{n\in\mathbb{N}} \|x_n\| < +\infty$. □

Remark 3.17. Let H be a separable, infinite-dimensional Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. By Parseval's identity, $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges unconditionally in H. Since $\sum_{n\in\mathbb{N}} \left\|\frac{1}{n} e_n\right\| = \sum_{n\in\mathbb{N}} \frac{1}{n} = +\infty$, the identity operator on H is not absolutely summing, see Theorem 3.12. In particular, according to Theorem 3.6 we have $l_s^1(H) \subsetneq l_w^1(H)$.

3.3 Examples

The present section is again mainly based on [DJT95, Chapter 2] with a few ideas taken from [Sch22].

Lemma 3.18. *For complex numbers $z_1, \ldots, z_n \in \mathbb{C}$ we have*

$$\sum_{j=1}^{n} |z_j| = \sup\{|\sum_{j=1}^{n} \lambda_j z_j| : \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0)\}. \tag{3.5}$$

Proof. Clearly,

$$\sup\{|\sum_{j=1}^{n}\lambda_j z_j| : \lambda_1,\dots,\lambda_n \in K_1^{\mathbb{C}}(0)\} \le \sup\{\sum_{j=1}^{n}|\lambda_j||z_j| : \lambda_1,\dots,\lambda_n \in K_1^{\mathbb{C}}(0)\} \le \sum_{j=1}^{n}|z_j|.$$

On the other hand, with $\zeta_j := |z_j|z_j^{-1}$ if $z_j \neq 0$ and $\zeta_j := 0$ otherwise we obtain

$$\sum_{j=1}^{n}|z_j| = |\sum_{j=1}^{n}\zeta_j z_j| \le \sup\{|\sum_{j=1}^{n}\lambda_j z_j| : \lambda_1,\dots,\lambda_n \in K_1^{\mathbb{C}}(0)\}.$$

□

Definition 3.19. Let X be a Banach space. $N \subseteq X'$ is called a norming subset of X', if

$$\|x\| = \sup\{|f(x)| : f \in N\} \tag{3.6}$$

holds true for all $x \in X$.

Lemma 3.20. *Let X be a Banach space and $N \subseteq X'$ a norming subset. Then the weakly summable norm of $(x_n)_{n\in\mathbb{N}} \in l_w^1(X)$ can be expressed in the form*

$$\|(x_n)_{n\in\mathbb{N}}\|_w = \sup\{\sum_{n=1}^{\infty}|f(x_n)| : f \in N\}.$$

Proof. For $(x_n)_{n\in\mathbb{N}} \in l_w^1(X)$ the desired equality is a consequence of the following interchange of suprema argument:

$$\begin{aligned}
\|(x_n)_{n\in\mathbb{N}}\|_w &= \sup\{\sum_{n=1}^{\infty}|f(x_n)| : f \in K_1^{X'}(0)\} = \sup\{\sum_{n=1}^{L}|f(x_n)| : f \in K_1^{X'}(0), L \in \mathbb{N}\}\\
&\overset{(3.5)}{=} \sup\{|\sum_{n=1}^{L}\lambda_n f(x_n)| : f \in K_1^{X'}(0), L \in \mathbb{N}, \lambda_1,\dots,\lambda_L \in K_1^{\mathbb{C}}(0)\}\\
&\overset{(1.1)}{=} \sup\{\|\sum_{n=1}^{L}\lambda_n x_n\| \ : L \in \mathbb{N}, \lambda_1,\dots,\lambda_L \in K_1^{\mathbb{C}}(0)\}\\
&\overset{(3.6)}{=} \sup\{|\sum_{n=1}^{L}\lambda_n f(x_n)| : f \in N, L \in \mathbb{N}, \lambda_1,\dots,\lambda_L \in K_1^{\mathbb{C}}(0)\}\\
&\overset{(3.5)}{=} \sup\{\sum_{n=1}^{L}|f(x_n)| : f \in N, L \in \mathbb{N}\} = \sup\{\sum_{n=1}^{\infty}|f(x_n)| : f \in N\}.
\end{aligned}$$

□

Proposition 3.21. *Let K be a compact Hausdorff space and μ be a positive Borel measure on K, i.e. μ is a positive measure defined on the Borel subsets of K satisfying $\mu(C) < +\infty$ for every compact $C \subseteq K$. In particular, $\mu(K) < +\infty$.*

(i) For each $\phi \in L^1(\mu)$ the multiplication operator

$$\begin{aligned} M_\phi \colon C(K) &\longrightarrow L^1(\mu) \\ g &\longmapsto g \cdot \phi \end{aligned}$$

is absolutely summing with $\pi(M_\phi) = \|\phi\|_{L^1(\mu)}$.

(ii) The canonical embedding

$$\begin{aligned} \iota \colon C(K) &\longrightarrow L^1(\mu) \\ g &\longmapsto g \end{aligned}$$

is absolutely summing with $\pi(\iota) = \mu(K)$.

Proof. (i): For each $x \in K$ the point evaluation $\delta_x \in C(K)'$ is defined by $\delta_x(g) \coloneqq g(x)$. Since for any $g \in C(K)$ we have

$$\|g\|_\infty = \sup_{x \in K} |g(x)| = \sup_{x \in K} |\delta_x(g)|,$$

the point evaluations form a norming subset of $C(K)'$. By Lemma 3.20 we have

$$\|(g_j)_{j\in\mathbb{N}}\|_w = \sup\{\sum_{j=1}^{\infty} |\delta_x(g_j)| : x \in K\} = \sup\{\sum_{j=1}^{\infty} |g_j(x)| : x \in K\}$$

for all $(g_j)_{j\in\mathbb{N}} \in l^1_w(C(K))$. Therefore, given $g_1, \dots, g_n \in C(K)$ we conclude

$$\begin{aligned} \sum_{j=1}^{n} \|M_\phi(g_j)\|_{L^1(\mu)} &= \sum_{j=1}^{n} \int_K |\phi(x) g_j(x)| \,\mathrm{d}\mu(x) = \int_K |\phi(x)| \cdot \sum_{j=1}^{n} |g_j(x)| \,\mathrm{d}\mu(x) \\ &\le \left\|(g_j)_{j=1}^n\right\|_w \int_K |\phi(x)| \,\mathrm{d}\mu(x) = \left\|(g_j)_{j=1}^n\right\|_w \|\phi\|_{L^1(\mu)}. \end{aligned}$$

Hence, M_ϕ is absolutely summing with $\pi(M_\phi) \le \|\phi\|_{L^1(\mu)}$. Furthermore,

$$\pi(M_\phi) \ge \|M_\phi\| \ge \|M_\phi(\mathbb{1}_K)\|_{L^1(\mu)} = \|\phi\|_{L^1(\mu)}$$

gives $\pi(M_\phi) = \|\phi\|_{L^1(\mu)}$.
(ii): Since $\iota = M_{\mathbb{1}_K}$, we conclude from (i) that ι is absolutely summing and $\pi(M_{\mathbb{1}_K}) = \|\mathbb{1}_K\|_{L^1(\mu)} = \mu(K)$.

$\square$

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