BACHELORARBEIT

Some Applications of K-Analytic Spaces to Topology and Functional Analysis

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Institut für
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Ao.Univ.Prof. Dr. Harald Woracek

durch

Johanna Brunar

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1 Introduction

The following thesis is concerned with K-analytic topological spaces, and their applications in topology and functional analysis. A completely regular topological space is said to be K-analytic if it is the image of the Baire space $\mathcal{N}$ under an upper semi-continuous compact-valued map (see Definitions 3.1 and 4.1). The class of K-analytic spaces contains the class of compact spaces as well as the class of Polish spaces.

In every K-analytic space X there exists a family of subsets $(A_\alpha)_{\alpha\in\mathcal{N}}$ satisfying

(A) A_α is compact for every $\alpha \in \mathcal{N}$,

(B) $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, and

(C) $X = \bigcup\limits_{\alpha\in\mathcal{N}} A_\alpha$.

A central theorem of this paper examines under which conditions the existence of such a family implies that a space is K-analytic (Theorem 5.5). This is certainly the case for the class of angelic spaces (Definition 6.1), in which all notions of compactness coincide (Proposition 6.4).

The main strength of K-analytic spaces in their application to topology and functional analysis lies in the fact they are Lindelöf (Corollary 4.7). Thus implied is the separability of metrisable K-analytic spaces. Moreover, a metrisable space is K-analytic if and only if it is analytic (Theorem 7.2.7). The following diagram summarises these relations, wherein the dashed arrows only hold true for metrisable spaces.

$$
\begin{tikzcd}
 & \text{Polish} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{analytic} \arrow[r, Rightarrow] \arrow[dl, dashed, leftrightarrow] & \text{separable} \arrow[ddll, dashed, leftrightarrow] \\
\text{compact} \arrow[r, Rightarrow] & \text{K-analytic} \arrow[d, Rightarrow] & & \\
 & \text{Lindelöf} & &
\end{tikzcd}
$$

After a thorough study of K-analytic spaces and their properties, we will show their applicability regarding the metrisability of compact topological spaces (Section 8), weakly compactly generated Banach spaces (Section 9), Fréchet-Montel spaces (Section 10), and inductive limits of separable Fréchet spaces (Section 11).

The majority of this thesis is based on Chapter 2 of "A biased view of topology as a tool in functional analysis" by Bernardo Cascales and José Orihuela [4]. Knowledge of the basic concepts of topology and functional analysis is tacitly assumed up to the extent of [2] and [3].

2 Basic Definitions and Observations

For the sake of completeness we will state some basic definitions and properties of topological spaces. As in the following sections, we will not prove most of these commonly known results, but refer to the lecture notes "Analysis 3" by Martin Blümlinger [2], and "Topologie" by Harald Woracek [20] instead.

2.1 Underlying Topological Concepts and Conventions

We want to recall the definitions of some classes of topological spaces that we are going to use throughout this thesis.

Definition 2.1.1. Let $(X, \mathcal{T})$ be a topological space. It is said to be

- *metrisable* if there exists a metric d on X so that the topology $\mathcal{T}_d$ induced by d fulfils $\mathcal{T}_d = \mathcal{T}$.
- *completely metrisable*, if there exists a complete metric d so that $\mathcal{T}_d = \mathcal{T}$.
- *separable* if it contains a countable, dense subset.
- *Polish* if it is separable and completely metrisable.
- *first-countable* if every element $x \in X$ has a countable basis of its neighbourhood filter $\mathcal{U}(x)$.
- *second-countable* if X has a countable basis of its topology.
- *Lindelöf* if every cover of X consisting of open sets has a countable subcover.

For metrisable spaces we have the following equivalence whose proof may be found in [6, Satz 1.32].

Proposition 2.1.2. *Let X be a metrisable topological space. The following statements are equivalent:*

(i) X is separable.

(ii) X is second-countable.

(iii) X is Lindelöf.

Let us agree on the following notations:

By $\mathcal{K}(X)$ we denote the class of all non-empty compact subsets of a topological space X. $C(X)$ is the set of all continuous real-valued functions on X. For any $x \in X$ its neighbourhood filter is denoted by $\mathcal{U}(x)$. The powerset of a set S is being referred to as $\mathcal{P}(S)$. If $(X, \|.\|)$ is a normed space, we write for the open and the closed unit ball $U_1(0) := \{x \in X : \|x\| < 1\}$ and $K_1(0) := \{x \in X : \|x\| \leq 1\}$, respectively.

2.2 Separation Axioms

In topological spaces points and subsets may be separated by open sets, or by continuous functions. The following definitions represent a not exhaustive list of *separation axioms*:

Definition 2.2.1. A topological space $(X, \mathcal{T})$ is said to satisfy the separation axiom

(T_1) if

$$\forall x, y \in X, x \neq y \exists O_x, O_y \in \mathcal{T} : (x \in O_x \wedge y \in O_y) \wedge (y \notin O_x \wedge x \notin O_y)$$

(T_2), or is called *Hausdorff* if

$$\forall x, y \in X, x \neq y \exists O_x, O_y \in \mathcal{T} : (x \in O_x \wedge y \in O_y) \wedge (O_x \cap O_y = \emptyset)$$

$(T_{3\frac{1}{2}})$ if

$$\begin{gathered} \forall x \in X, B \subseteq X \text{ closed}, x \notin B \ \exists f : X \to [0,1] \text{ continuous :} \\ f(x) = 1 \wedge f(B) = \{0\} \end{gathered}$$

(T_4) if

$$\begin{gathered} \forall A, B \subseteq X \text{ closed}, A \cap B = \emptyset \ \exists O_A, O_B \in \mathcal{T} : \\ (A \subseteq O_A \wedge B \subseteq O_B) \ \wedge \ (O_A \cap O_B = \emptyset) \end{gathered}$$

$(T_{4\frac{1}{2}})$ if

$$\begin{gathered} \forall A, B \subseteq X \text{ closed}, A \cap B = \emptyset \ \exists f : X \to [0,1] \text{ continuous :} \\ f(A) = \{1\} \wedge f(B) = \{0\} \end{gathered}$$

A topological space is called *completely regular* if it satisfies both (T_1) and $(T_{3\frac{1}{2}})$, and *normal* if it satisfies both (T_1) and (T_4).

Lemma 2.2.2. *A topological space X is (T_1) if and only if for every $x \in X$ the set $\{x\}$ is closed.*

Proof. Take $x \in X$. Assuming that X is T_1, we find an open neighbourhood $O_y \in \mathcal{U}(y)$ with $x \notin O_y$ for every $y \neq x$. Now, $X \setminus \{x\} = \bigcup \{O_y : y \neq x\}$ is open. Conversely, $O_x := X \setminus \{y\}$, and $O_y := X \setminus \{x\}$ are separating open subsets for any $x \neq y$. $\square$

Remark 2.2.3. Note that every completely regular space is Hausdorff: Because of (T_1), singleton sets are closed. By $(T_{3\frac{1}{2}})$ we therefore find a continuous function f with $f(x) = 1$ and $f(y) = 0$ for every $x \neq y$. We obtain Hausdorff by using $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ as separating open sets.

Compact subsets K of a Hausdorff space X are closed: If we fix $x \in X \setminus K$ and take any $y \in K$ we find disjoint open sets O_y, U_y so that $x \in O_y$ and $y \in U_y$. Since $\{U_y : y \in K\}$ is an open cover of K, we find finitely many $y_i \in K$ with $K \subseteq \bigcup_{i=1}^n U_{y_i}$. The set $U := \bigcap_{i=1}^n U_{y_i}$ is an open neighbourhood of x, which is disjoint from K. Therefore, $X \setminus K$ must be open.

The following theorems characterise (T_4)-spaces.

Proposition 2.2.4. *A topological space X is (T_4) if and only if for subsets $E \subseteq O \subseteq X$ with E closed and O open, there exists an open set $U \subseteq X$ so that $E \subseteq U \subseteq \overline{U} \subseteq O$.*

Theorem 2.2.5 (Urysohn's Lemma)**.** *A topological space is (T_4) if and only if it is $(T_{4\frac{1}{2}})$.*

For the proofs we refer to [2, Proposition 1.6.1 and Satz 1.6.2].

3 The Baire Space

The Baire space is a powerful object in set theory, and will play an important role in our disquisition on K-analytic sets. We will cover some of its numerous interesting properties.

Definition 3.1. The *Baire space* is the space $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ of all sequences of natural numbers endowed with the product topology of the discrete topology on $\mathbb{N}$.

Note that sets of the form

$$\left(\prod_{i=1}^{N}\{n_i\}\right) \times \left(\prod_{i=N+1}^{\infty} \mathbb{N}\right) \text{ with } n_i,\ N \in \mathbb{N} \tag{1}$$

form a basis of the topology on $\mathcal{N}$.

For $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$, and $m \in \mathbb{N}$ we will write

$$\alpha|_m := (a_1, a_2, \ldots, a_m).$$

Given sequences $\alpha = (a_n)_{n\in\mathbb{N}}$ and $\beta = (b_n)_{n\in\mathbb{N}}$ in $\mathcal{N}$ we define

$$\alpha \leq \beta :\Leftrightarrow a_n \leq b_n \text{ for all } n \in \mathbb{N}.$$

Proposition 3.2. *The Baire space $\mathcal{N}$ is homeomorphic to the countable product of copies of itself.*

Proof. Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N} : n \mapsto (g_1(n), g_2(n))$ be homeomorphisms so that $h \circ g = \mathrm{id}_{\mathbb{N}}$. We define $\Phi : \mathcal{N}^{\mathbb{N}} \to \mathcal{N}$ by

$$\Phi(f) : \begin{cases} \mathbb{N} & \to \mathbb{N} \\ n & \mapsto f(g_1(n))(g_2(n)) \end{cases}$$

Φ is injective: For any $f \neq \widetilde{f} \in \mathcal{N}^{\mathbb{N}}$ there exist $n, k \in \mathbb{N}$ so that $f(n)(k) \neq \widetilde{f}(n)(k)$. Since g is surjective, we find $m \in \mathbb{N}$ with $g(m) = (g_1(m), g_2(m)) = (n, k)$ and therefore $\Phi(f) \neq \Phi(\widetilde{f})$.

Φ is surjective: Consider the map $\Psi : \mathcal{N} \to \mathcal{N}^{\mathbb{N}}$ with

$$\Psi(a)(n) : \begin{cases} \mathbb{N} \to \mathbb{N} \\ m \mapsto a \circ h(n, m) \end{cases}$$

For $a \in \mathcal{N}$ and $n \in \mathbb{N}$ we get

$$\Phi \circ \Psi(a)(n) = \Psi(a)(g_1(n))(g_2(n)) = a \circ h(g_1(n), g_2(n)) = a(n).$$

Thus, Φ has a right inverse.

Φ is a homeomorphism: Since $\mathbb{N}$ is endowed with the discrete topology, any sequence is continuous. Therefore, both Φ and the inverse Ψ are continuous, being compositions of continuous functions. □

Lemma 3.3. *The Baire space $\mathcal{N}$ is Polish.*

Proof. $\mathbb{N}$ with the discrete topology is metrisable, and

$$d(n, m) := \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

metrises its topology. The metric d is complete as a sequence in $\mathbb{N}$ is Cauchy if and only if it is eventually constant. Trivially, $\mathbb{N}$ is separable. Therefore, the product space $\mathcal{N}$ is separable, and completely metrisable. □

Remark 3.4. It can be shown that the Baire space $\mathcal{N}$ is homeomorphic to the set of irrational numbers with the subspace topology inherited by the Euklidian topology on $\mathbb{R}$ [9, Exercise 3.4]. However, note that the set of irrationals is not completely metrisable. This points out that complete metrisability is a quality of the metric, rather than of the topology.

Theorem 3.5. [14, Theorem 1A.1.] *For every non-empty Polish space X there exists a continuous surjection $f : \mathcal{N} \to X$.*

Proof. Let $Y = \{y_n : n \in \mathbb{N}\}$ be a countable dense subset of X, and d be a compatible complete metric on X . For each $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$ we define a sequence $(x_n^\alpha)_{n\in\mathbb{N}}$ by the recursion

$$x_1^\alpha := y_{a_1}$$
$$x_{n+1}^\alpha := \begin{cases} y_{a_{n+1}} & \text{if } d(x_n^\alpha, y_{a_{n+1}}) < 2^{-n}, \\ x_n^\alpha & \text{else.} \end{cases}$$

By definition, we get that $d(x_n^\alpha, x_{n+1}^\alpha) < 2^{-n}$ for every $n \in \mathbb{N}$. Therefore,

$$d(x_n^\alpha, x_m^\alpha) \leq \sum_{k=n}^{m-1} d(x_k^\alpha, x_{k+1}^\alpha) \leq \sum_{k=n}^{m-1} \left(\frac{1}{2}\right)^k$$

for any $n < m \in \mathbb{N}$. For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n, m \geq N$ it holds that

$$d(x_n^\alpha, x_m^\alpha) \leq \sum_{k=N}^{\infty} \left(\frac{1}{2}\right)^k < \epsilon,$$

thus $(x_n^\alpha)_{n\in\mathbb{N}}$ is Cauchy in X. Because of the completeness of d, it converges, and we can set

$$f : \begin{cases} \mathcal{N} & \to X \\ \alpha & \mapsto \lim\limits_{n\to\infty} x_n^\alpha. \end{cases}$$

For $\alpha, \beta \in \mathcal{N}$, and $n \in \mathbb{N}$ with $\alpha|_n = \beta|_n$ we clearly have $x_n^\alpha = x_n^\beta$. Again, using the triangle inequality, we get

$$d(f(\alpha), f(\beta)) \leq d(f(\alpha), x_n^\alpha) + d(x_n^\beta, f(\beta)) \leq 2\sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{n-2}.$$

Since the sets in (1) are a basis of the topology on $\mathcal{N}$, we have hereby shown the continuity of f.

For an arbitrary $x \in X$ we assign $a_n := \min\left\{k \in \mathbb{N} : d(x, y_k) < \left(\frac{1}{2}\right)^{n+1}\right\}$. This is well-defined, since Y is dense in X. Because of

$$d(y_{a_n}, y_{a_{n+1}}) \leq d(y_{a_n}, x) + d(x, y_{a_{n+1}}) < 2^{-(n+1)} + 2^{-(n+2)} < 2^{-n},$$

the sequence defined by $\alpha := (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$ gives $f(\alpha) = \lim\limits_{n\to\infty} y_{a_n} = x$. □

4 K-Analytic Spaces

This section is devoted to examine some general properties of K-analytic spaces. In particular, we show that every K-analytic space is Lindelöf, which will prove useful in later applications.

In what follows, we will assume that all topological spaces are completely regular.

If $T : X \to \mathcal{P}(Y)$ is a set-valued map, for every $A \subseteq X$ we set

$$T(A) := \bigcup_{a\in A} T(a),$$

Definition 4.1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{V})$ be topological spaces. A set-valued map $T : X \to \mathcal{P}(Y)$ is called *upper semi-continuous* if for every $\alpha \in X$ and $U \in \mathcal{V}$ with $T(\alpha) \subseteq U$ there exists an open neighbourhood $V \in \mathcal{U}(\alpha)$ so that $T(V) \subseteq U$.

Definition 4.2. A topological space $(X, \mathcal{T})$ is said to be *K-analytic* if there exists an upper semi-continuous compact-valued map $T : \mathcal{N} \to \mathcal{K}(X)$ so that $T(\mathcal{N}) = X$.

Remark 4.3. Clearly, every compact topological space X is K-analytic, since we can simply choose $T(\alpha) = X$ for all $\alpha \in \mathcal{N}$.

Lemma 4.4. *Let $X = T(\mathcal{N})$ be a K-analytic space and $K \in \mathcal{K}(\mathcal{N})$. Then $T(K)$ is a compact subset of X.*

Proof. Let $(V_i)_{i\in I}$ be an open cover of $T(K)$. Since for all $\alpha \in K$ we have that $T(\alpha) \subseteq \bigcup_{i\in I} V_i$, and $T(\alpha)$ is compact by assumption, we find $n_\alpha \in \mathbb{N}$ and $i_1 \dots i_{n_\alpha} \in I$ so that $T(\alpha) \subseteq \bigcup_{k=1}^{n_\alpha} V_{i_k} =: W_\alpha$. Because of the upper semi-continuity of T, we can choose an open neighbourhood U_α of α with $T(U_\alpha) \subseteq W_\alpha$. By compactness of K, we get $K \subseteq \bigcup_{j=1}^{m} U_{\alpha_j}$, and therefore obtain a finite subcover

$$T(K) \subseteq \bigcup_{j=1}^{m} T(U_{\alpha_j}) \subseteq \bigcup_{j=1}^{m} W_{\alpha_j} = \bigcup_{j=1}^{m} \bigcup_{k=1}^{n_\alpha} V_{i_k}.$$

□

K-analytic spaces are stable under the following actions:

Proposition 4.5. [18, Theorems 2.5.1, 2.5.5] *Closed subspaces, compact-valued upper semi-continuous images, and countable products of K-analytic spaces are K-analytic.*

Proof.
(i) Let $(X, \mathcal{T})$ with $X = T(\mathcal{N})$, T upper semi-continuous and compact-valued, be a K-analytic space, and A be a closed subset of X. We define

$$\widetilde{T} : \begin{cases} \mathcal{N} \to \mathcal{P}(A) \\ \alpha \mapsto T(\alpha) \cap A. \end{cases}$$

Since compact subsets of a Hausdorff-space are closed, and subsets of a topological space are compact if and only if they are compact with respect to the induced topology, $T(\alpha) \cap A$ is – being a closed subset of the compact set $T(\alpha)$ – itself compact in A. The map $\widetilde{T}$ is therefore compact-valued. We have that

$$\widetilde{T}(\mathcal{N}) = \bigcup_{\alpha\in\mathcal{N}} \widetilde{T}(\alpha) = \bigcup_{\alpha\in\mathcal{N}} (T(\alpha) \cap A) = (\bigcup_{\alpha\in\mathcal{N}} T(\alpha)) \cap A = X \cap A = A.$$

In order to prove the upper semi-continuity of $\widetilde{T}$ as map to A endowed with the induced topology $\mathcal{T}_A$, fix $\alpha \in \mathcal{N}$, and an open subset $U \in \mathcal{T}_A$ that contains $\widetilde{T}(\alpha)$. By definition of the induced topology, we find $O \in \mathcal{T}$ so that $U = O \cap A$. Since T is upper semi-continuous, we find an open neighbourhood $V \in \mathcal{U}(\alpha)$, whose image is contained in O. We obtain $\widetilde{T}(V) = T(V) \cap A \subseteq O \cap A = U$, hence $\widetilde{T}$ is upper semi-continuous.

(ii) Let $X = T(\mathcal{N})$ be a K-analytic space, Y a topological space, and $F : X \to \mathcal{K}(Y)$ a compact-valued upper semi-continuous map. By Lemma 4.4, $F(K)$ is compact for every $K \in \mathcal{K}(X)$. Therefore, as $T(\alpha)$ is compact in X for every $\alpha \in \mathcal{N}$, the composition $T \circ F$ is a compact-valued map from $\mathcal{N}$ to $F(X)$. Since $\bigcup_{\alpha\in\mathcal{N}} F \circ T(\alpha) = F(\bigcup_{\alpha\in\mathcal{N}} T(\alpha)) = F(X)$ we only need to show that $F \circ T$ is upper semi-continuous. Let $\alpha \in \mathcal{N}$, and let U be an open subset of $F(X)$ with $F \circ T(\alpha) \subseteq U$. As F is upper semi-continuous, we find and open neighbourhood $V \in \mathcal{U}(T(\alpha))$ with $F(V) \subseteq U$. As T is upper semi-continuous we can choose an open set W containing α with $T(W) \subseteq V$, and we get $F \circ T(W) \subseteq F(V) \subseteq U$.

(ii) For $i \in \mathbb{N}$ let $X_i = T_i(\mathcal{N})$ be K-analytic spaces, and consider the product $X := \prod\limits_{i\in\mathbb{N}} X_i$ endowed with the product topology. We define

$$T : \begin{cases} \mathcal{N}^{\mathbb{N}} & \to \mathcal{K}(X) \\ (\alpha(i))_{i\in\mathbb{N}} & \mapsto \prod\limits_{i\in\mathbb{N}} T_i(\alpha(i)). \end{cases}$$

By Tychonoff's Theorem, the product $\prod_{i\in\mathbb{N}} T_i(\alpha(i))$ is compact in the product topology, so T is well-defined. We have

$$\begin{aligned} X = \prod_{i\in\mathbb{N}} X_i = \prod_{i\in\mathbb{N}} \bigcup_{\alpha(i)\in\mathcal{N}} T_i(\alpha(i)) = \bigcup_{\alpha\in\mathcal{N}^{\mathbb{N}}} \prod_{i\in\mathbb{N}} T_i(\alpha(i)) = \\ = \bigcup_{\alpha\in\mathcal{N}^{\mathbb{N}}} T((\alpha(i))_{i\in\mathbb{N}}) = T(\mathcal{N}^{\mathbb{N}}). \end{aligned}$$

Since by Proposition 3.2, $\mathcal{N}^{\mathbb{N}}$ is homeomorphic to $\mathcal{N}$, and the composition of an upper semi-continuous function with a continuous function is upper semi-continuous, it suffices to show that T is upper semi-continuous. So, let $(\alpha(i))_{i\in\mathbb{N}}$ be a sequence in $\mathcal{N}$, and $U \subseteq X$ be an open set with $T((\alpha(i))_{i\in\mathbb{N}}) \subseteq U$.

If $T((\alpha(i))_{i\in\mathbb{N}}) = \emptyset$, there exists $j \in \mathbb{N}$ so that $T_j(\alpha(j)) = \emptyset$. Applying the upper semi-continuity of T_j, we find an open neighbourhood $V \subseteq \mathcal{N}$ of $\alpha(j)$ with $T_j(V) = \emptyset$. Using basic open sets as in (1), we can find $m \in \mathbb{N}$ with $V_m := \{\beta \in \mathcal{N} : \beta|_m = \alpha(j)|_m\} \subseteq V$. $W := \{(\beta(i))_{i\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}} : \beta(j) \in V_m\}$ is an open neighbourhood of $(\alpha(i))_{i\in\mathbb{N}}$ in $\mathcal{N}^{\mathbb{N}}$ with $T(W) = \emptyset$.

Now suppose $T((\alpha(i))_{i\in\mathbb{N}}) \neq \emptyset$. Because of its compactness, $T((\alpha(i))_{i\in\mathbb{N}})$ is covered by a finite union of basic open sets that are contained in U. This allows us to pick $N \in \mathbb{N}$, and an open set $U_N \subseteq \prod_{i=1}^N X_i$ with

$$T((\alpha(i))_{i\in\mathbb{N}}) \subseteq U_N \times \left(\prod_{i=N+1}^{\infty} X_i\right) \subseteq U.$$

In particular, we get that $\prod_{i=1}^N T_i(\alpha(i)) \subseteq U_N$. As all $T_i(\alpha(i))$ are compact in X_i, we find open subsets $U_{N,i}$ of X_i with $T_i(\alpha(i)) \subseteq U_{N,i}$ for every $1 \leq i \leq N$, and $\prod_{i=1}^N U_{N,i} \subseteq U_N$. Now, since T_i are upper semi-continuous, we again choose $m_i \in \mathbb{N}$ with $T_i(V_{m_i}) \subseteq U_{N,i}$, where $V_{m_i} := \{\beta \in \mathcal{N} : \beta|_{m_i} = \alpha(i)|_{m_i}\}$ as above. With

$$W := \{(\beta(i))_{i\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}} : \beta(i) \in V_{m_i} \text{ for } 1 \leq i \leq N\}$$

we have found an open neighbourhood of $(\alpha(i))_{i\in\mathbb{N}}$ that fulfils

$$T(W) \subseteq \left(\prod_{i=1}^{N} U_{N,i}\right) \times \left(\prod_{i=1}^{\infty} X_i\right) \subseteq U_N \times \left(\prod_{i=N+1}^{\infty} X_i\right) \subseteq U.$$

□

Proposition 4.6. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{V})$ be topological spaces, and $T : X \to \mathcal{K}(Y)$ be a surjective compact-valued upper semi-continuous map. If $(X, \mathcal{T})$ is Lindelöf, then so is $(Y, \mathcal{V})$.*

Proof. Let $(U_i)_{i\in I}$ be an open cover of $Y = T(X)$. In particular, for every $x \in X$ the compact set $T(x)$ is covered by $(U_i)_{i\in I}$, and we find a finite subcover, say $T(x) \subseteq \bigcup_{i=1}^{n_x} U_i$. As T is upper semi-continuous, we find an open neighbourhood $V_x \in \mathcal{U}(x)$, whose image $T(V_x)$ is still contained in the finite union. As X is Lindelöf, we may choose a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $X = \bigcup_{n\in\mathbb{N}} V_{x_n}$. We obtain

$$Y = T(X) \subseteq \bigcup_{n\in\mathbb{N}} T(V_{x_n}) \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{i=1}^{n_x} U_i,$$

which is a countable subcover as required. □

Corollary 4.7. [18, Theorem 2.7.1] *K-analytic spaces are Lindelöf.*

Proof. $\mathcal{N}$ is a second-countable space because it is the countable product of the countable space $\mathbb{N}$. Every second-countable space is Lindelöf. □

We are now going to single out three properties that every K-analytic space has.

For each $\alpha \in \mathcal{N}$ the set $\{\beta \in \mathcal{N} : \beta \leq \alpha\}$ is a compact subset of $\mathcal{N}$: A subset of a discrete topological space is compact if and only if it is finite. Therefore, the set $\{b_n \in \mathbb{N} : b_n \leq a_n\}$ is compact for every $n \in \mathbb{N}$. By Tychonoff's Theorem, the product $\prod_{n\in\mathbb{N}}\{b_n \in \mathbb{N} : b_n \leq a_n\} = \{\beta \in \mathcal{N} : \beta \leq \alpha\}$ is compact in the product topology.

Recall that by Lemma 4.4, the image of any compact subset of $\mathcal{N}$ under a compact-valued upper semi-continuous map T is always compact. For a K-analytic space $X = T(\mathcal{N})$ and $\alpha \in \mathcal{N}$ we define

$$A_\alpha := T(\{\beta \in \mathcal{N} : \beta \leq \alpha\}),$$

and obtain

(A) A_α is compact for every $\alpha \in \mathcal{N}$,

(B) $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, and

(C) $X = \bigcup_{\alpha \in \mathcal{N}} A_\alpha$.

(2)

5 Construction of K-Analytic Spaces

One may ask if any topological space that contains a family $\{A_\alpha : \alpha \in \mathcal{N}\}$ of compact subsets satisfying conditions (A), (B), and (C) from above is K-analytic. In general, the answer is no. However, in the following section we will examine under which additional requirements the converse still holds.

Definition 5.1. Given a sequence $\chi = (x_n)_{n\in\mathbb{N}}$ in a topological space X, we define the set of all *cluster points* of χ in X as

$$\operatorname{clust}_X(\chi) := \bigcap_{n\in\mathbb{N}} \overline{\{x_m : m \geq n\}}.$$

Lemma 5.2. *Let X be a topological space and $\chi = (x_n)_{n\in\mathbb{N}}$ be a sequence in X. A point $x \in X$ is cluster point of χ if and only if for all $V \in \mathcal{U}(x)$ it holds that*

$$|V \cap \{x_n : n \in \mathbb{N}\}| = \aleph_0.$$

Proof. "$\Rightarrow$" Take any $V \in \mathcal{U}(x)$. If $x \in \operatorname{clust}_X(\chi)$, we have that $x \in \overline{\{x_m : m \geq n\}}$ for all $n \in \mathbb{N}$. Therefore, $V \cap \{x_m : m \geq n\} \neq \emptyset$ for every $n \in \mathbb{N}$. Denote by $m_n \geq n$ the smallest index so that $x_{m_n} \in V \cap \{x_m : m \geq n\}$. Clearly, $|V \cap \{x_{n_V} : n \in \mathbb{N}\}| = \aleph_0$.

"$\Leftarrow$" For every $n \in \mathbb{N}$ and for every $V \in \mathcal{U}(x)$ there exists $m \geq n$ so that $x_m \in V$. Therefore, $x \in \overline{\{x_m : m \geq n\}}$ for every n. Thus, x is a cluster point. $\square$

Definition 5.3. Let A be a subset of a topological space X.

(i) A is said to be *relatively countably compact* if for every sequence χ in A the set $\operatorname{clust}_X(\chi)$ is not empty.

(*ii*) A is called *countably compact* if for every sequence χ in A the set $\text{clust}_X(\chi) \cap A$ is not empty.

Given a sequence $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$, and $n \in \mathbb{N}$, we define

$$C_n^\alpha := \bigcup_{\beta\in\mathcal{N}} \{A_\beta \ : \ \beta|_n = \alpha|_n\}$$

Let us state a practical lemma:

Lemma 5.4. *If $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$, and $\chi = (x_n)_{n\in\mathbb{N}}$ is a sequence in X that satisfies $x_n \in C_n^\alpha$ for every $n \in \mathbb{N}$, it holds that*

$$\emptyset \neq \text{clust}_X(\chi) \subseteq \bigcap_{n\in\mathbb{N}} C_n^\alpha.$$

Proof. By definition of C_n^α, we may choose $\beta_n \in \mathcal{N}$ for each $n \in \mathbb{N}$ that fulfils $x_n \in A_{\beta_n}$ and $\beta_n|_n = \alpha|_n$. Let $\pi_n : \mathcal{N} \to \mathbb{N}$ be the projection onto the n-th coordinate, and set

$$b_n := \max_{k\leq n} \pi_n(\beta_k).$$

We consider the sequence $\beta := (b_n)_{n\in\mathbb{N}}$.

In order to see that $\text{clust}_X(\chi) \neq \emptyset$, note that because of condition (B), and $\beta_n \leq \beta$ for every $n \in \mathbb{N}$, we get $A_{\beta_n} \subseteq A_\beta$, which is compact by condition (A). χ is therefore entirely contained in the compact set A_β. Recall that subset of a topological space is compact if and only if every net in the subset has a cluster point within the subset. Hence, $\text{clust}_X(\chi) \neq \emptyset$.

Observe that $\pi_1(\beta) = a_1$, and consequently $\text{clust}_X(\chi) \subseteq A_\beta \subseteq C_1^\alpha$. Now pick any $m \in \mathbb{N}$, and consider the sequence $\chi_m := (x_n)_{n\geq m}$. Clearly, $\text{clust}_X(\chi) = \text{clust}_X(\chi_m)$ and $\beta_n|_m = \alpha|_m$ for $n \geq m$. If we repeat the argument above, we obtain $\text{clust}(\chi_m) \subseteq C_m^\alpha$ and therefore $\text{clust}_X(\chi) \subseteq \bigcap_{m\in\mathbb{N}} C_m^\alpha$. ☐

We can now show in which circumstances the existence of a family satisfying conditions (A), (B), and (C) from (2) is sufficient to guarantee K-analyticity.

Theorem 5.5. [4, Proposition 1] *Let X be a topological space that contains a family $\{A_\alpha : \alpha \in \mathcal{N}\}$ of subsets satisfying conditions (A), (B), and (C) from* (2). *Let T be the following mapping:*

$$T : \begin{cases} \mathcal{N} \to \mathcal{P}(X) \\ \alpha \mapsto \bigcap_{n\in\mathbb{N}} C_n^\alpha \end{cases}$$

(*i*) *For every $\alpha \in \mathcal{N}$ the set $T(\alpha) \subseteq X$ is countably compact.*

(ii) If $T(\alpha)$ is compact for every $\alpha \in \mathcal{N}$, T makes X a K-analytic space.

Proof.
(i) The statement follows immediately from Lemma 5.4, as any sequence $(x_n)_{n\in\mathbb{N}}$ in $T(\alpha)$ fulfils $x_n \in C_n^\alpha$ for all $n \in \mathbb{N}$. Therefore, for every $\alpha \in \mathcal{N}$ we have that every sequence in $T(\alpha)$ admits cluster points, all of which remain in $T(\alpha)$, and $T(\alpha)$ is countably compact.

(ii) Assume that T is compact-valued. Since $A_\alpha \subseteq T(\alpha)$ for every $\alpha \in \mathcal{N}$ we have $T(\mathcal{N}) \supseteq \bigcup_{\alpha\in\mathcal{N}} A_\alpha = X$, the last equality being justified by condition (C). It is only left to show that T is upper semi-continuous.

So, pick $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$, and an open $U \subseteq X$ that contains the image $T(\alpha)$. If for every $n \in \mathbb{N}$ there existed $x_n \in C_n \setminus U$, the sequence defined by $\chi := (x_n)_{n\in\mathbb{N}}$ would fulfil $\operatorname{clust}(\chi) \subseteq X \setminus U \subseteq X \setminus T(\alpha)$ in contradiction to item (i). Therefore, we are able to choose $m \in \mathbb{N}$ with $C_m^\alpha \subseteq U$. $V := \{\beta \in \mathcal{N} \ : \ \beta|_m = \alpha|_m\}$ is a basic open set that contains α. Since $n \leq k$ implies $C_n^\alpha \supseteq C_k^\alpha$, we get

$$
\begin{aligned}
T(V) &= \bigcup_{\beta\in V} T(\beta) = \bigcup_{\beta\in V} \bigcap_{n\in\mathbb{N}} C_n^\beta = \\
&= \bigcup_{\beta\in V} \left(\left(\bigcap_{n\leq m} C_n^\alpha \right) \cap \left(\bigcap_{n>m} C_n^\beta \right) \right) \subseteq \\
&\subseteq \bigcup_{\beta\in V} \left(\left(\bigcap_{n\leq m} C_n^\alpha \right) \cap C_m^\beta \right) = C_m^\alpha \subseteq U.
\end{aligned}
$$

As X is now the image of $\mathcal{N}$ under the compact-valued upper semi-continuous map T, it is a K-analytic space. $\square$

6 Angelic Spaces

If the mapping T as defined in Theorem 5.5 is compact-valued, it creates a K-analytic structure on the topological space. Additionally, we showed that $T(\alpha)$ is countably compact for every $\alpha \in \mathcal{N}$. In angelic spaces the assumption of T being compact-valued may be omitted, as all concepts of compactness coincide.

Definition 6.1. A topological space X is said to be *angelic* if for every relatively countably compact subset $A \subseteq X$ it holds that

(i) A is relatively compact, and

(ii) for every $x \in \bar{A}$ there exists a sequence in A, which converges to x.

In what follows it will be useful to complete Definition 5.3 with equivalent conditions.

Lemma 6.2. *Let X be a topological space. The following conditions are equivalent:*

(i) Every countable open cover of X has a finite subcover.

(ii) Every countable family of closed subsets that has the finite intersection property has non-empty intersection.

(iii) Every sequence in X has a cluster point in X, i.e. X is countably compact as defined in Definition 5.3.

Proof.
$(i) \Rightarrow (ii)$ Let $(C_n)_{n\in\mathbb{N}}$ be a family of closed subsets, whose finite intersections are non-empty. Suppose $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$. It follows that $\bigcup_{n\in\mathbb{N}} C_n^C = X$. By (i) there exists a finite subset E with $\bigcup_{n\in E} C_n^C = X$, or equivalently $\bigcap_{n\in E} C_n = \emptyset$ in contradiction to the finite intersection property.

$(ii) \Rightarrow (i)$ Let $\{U_n : n \in \mathbb{N}\}$ be an open cover of X. If there did not exist a finite subcover, we would get $\bigcap_{n\in E} U_n^C \neq \emptyset$ for every finite E. Hence, $\bigcap_{n\in\mathbb{N}} U_n^C \neq \emptyset$ in contradiction to $\{U_n : n \in \mathbb{N}\}$ being a cover of X.

$(i) \Rightarrow (iii)$ Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and define $C_n := \overline{\{x_m : m \geq n\}}$. The family $(C_n)_{n\in\mathbb{N}}$ is a family of closed subsets that fulfil the finite intersection property. Using the equivalence $(i) \Leftrightarrow (ii)$ we obtain $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$, hence the existence of a cluster point.

$(iii) \Rightarrow (i)$ Let $\{U_n : n \in \mathbb{N}\}$ be an open cover of X. If there did not exist a finite subcover, we could choose $x_n \in \bigcap_{k=1}^n U_k^C$ for every $n \in \mathbb{N}$. For every $x \in X$ there exists $N \in \mathbb{N}$ so that $x \in U_N$. By our choice of the sequence, we obtain $x_n \in U_N^C$ for all $n \geq N$. Therefore, $\overline{\{x_n : n \geq N\}} \subseteq U_N^C$, so x is not a cluster point. Since x was arbitrary, the proof is finished. $\square$

Recall the following definition:

Definition 6.3. A subset A of a topological space X is *sequentially compact* if every sequence in A has a convergent subsequence with limit in A.

It is well known that compactness, countable compactness, and sequential compactness coincide in metric spaces. The significance of angelic spaces is that the compactness behaviour of metric spaces carries over:

Proposition 6.4. *[7] Let X be an angelic space and A be any subset of X. Then the following statements are equivalent:*

(i) A is compact.

(ii) A is countably compact.

(iii) A is sequentially compact.

Proof. We carry out the proof in two steps. First, we show that every sequence $(x_n)_{n\in\mathbb{N}}$ in X with cluster point x has a subsequence converging to x if $\{x_n : n \in \mathbb{N}\}$ is a relatively compact set. As a second step, we will use this property to prove the equivalence of the statements made in the Proposition.

(1) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, x be a cluster point of the sequence, and suppose that the set $Y := \{x_n : n \in \mathbb{N}\}$ is relatively compact. If there exist infinitely many $n \in \mathbb{N}$ with $x_n = x$, we have found a subsequence converging to x. Hence, without loss of generality we assume that $x_n \neq x$ for all $n \in \mathbb{N}$. As $x \in \bar{Y}$, by the definition of angelic spaces, we obtain a sequence $(y_k)_{k\in\mathbb{N}}$ in Y that converges to x. For every $k \in \mathbb{N}$ pick $n_k \in \mathbb{N}$ so that $y_k = x_{n_k}$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is now a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to x.

(2) $(i) \Rightarrow (ii)$ It generally holds that every compact set is countably compact. This can easily be seen with Lemma 6.2.

$(iii) \Rightarrow (ii)$ This implication is also clear, since the limit of a convergent subsequence is a cluster point of the sequence.

$(ii) \Rightarrow (iii)$ Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A with cluster point $x \in A$. As the set $Y := \{x_n : n \in \mathbb{N}\}$ is a subset of A, and A is countably compact, every sequence Y has a cluster point in A. Therefore, Y is relatively countably compact. As X is angelic, Y is relatively compact. By (1), the sequence admits a subsequence converging to x.

$(ii) \Rightarrow (i)$ Clearly, A being countably compact implies it being relatively countably compact. By the definition of angelic spaces, A is relatively compact, and for every $x \in \bar{A}$ we may take a sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to x. As x is the only cluster point of the sequence, and A is countably compact, x must in fact be an element of A. Hence, $A = \bar{A}$ is compact. □

Combining Proposition 6.4 and Theorem 5.5, we immediately obtain the following corollary:

Corollary 6.5. *An angelic space X is K-analytic if and only if there exists a family $\{A_\alpha : \alpha \in \mathcal{N}\}$ of subsets that satisfy conditions (A), (B), and (C) from (2).*

In general, only the implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (ii)$ in Proposition 6.4 hold. For example, by the Theorem of Banach-Alaoglu 9.1.7, the closed unit ball in $(\ell^\infty)'$ is compact with respect to the weak*-topology, but it is not sequentially compact. Otherwise, if there existed a convergent subsequence $(e_{n_k})_{k\in\mathbb{N}}$ of the sequence of all unit vectors, the sequence

$$(e_{n_k}(\xi))_{k\in\mathbb{N}} = (\xi_{n_k})_{k\in\mathbb{N}}$$

would converge for every $\xi = (\xi_n)_{n\in\mathbb{N}} \in \ell^\infty$ [7, page 8].

An example for a space that is countably compact but not compact is the first uncountable ordinal with the order topology, see [19, Part II, Section 39]. However, in all second countable spaces compactness is equivalent to countable compactness, as are countable compactness and sequential compactness in first countable spaces ([19, Part I, Section 3].

Figure 1 summarises the relations between compactness and the countability axioms.

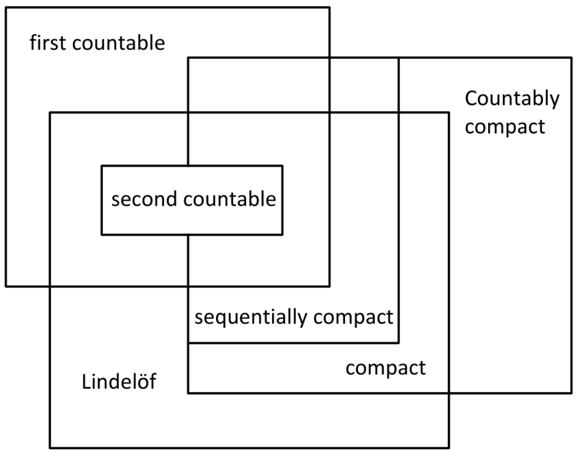

Figure 1: Countability Axioms and Compactness [19]

In [7, Chapter 3] many examples of angelic spaces are given. These include

- metric spaces,
- metrisable locally convex spaces (therefore, in particular Banach spaces) in their weak topology,
- subspaces of $\mathbb{R}^{\mathbb{N}}$ with a locally convex topology finer than the topology of pointwise convergence in their weak topology,
- strict inductive limits of Fréchet spaces in their weak topology (see section 11.1),
- the space $C_p(X, Y)$ of all continuous functions from a separable topological space X into a metric space Y equipped with the topology of pointwise convergence,
- $C_p(X, \mathbb{R})$ for K-analytic (therefore, in particular for compact) spaces X [17],
- subspaces of angelic spaces, and
- the domain X of any continuous injective function $\Phi : X \to Y$ into an angelic space Y if it is completely regular.

7 Analytic Metrisable Spaces

Analytic spaces are continuous images of Polish spaces. We will show that K-analyticity is equivalent to analyticity in all metrisable spaces. In particular, we obtain that the class of K-analytic spaces contains the class of Polish spaces.

7.1 Preliminaries

The following section contains preparatory work for the proof of Theorem 7.2.7.

Definition 7.1.1. Let $(X, \mathcal{T})$ be a topological space. A *compactification* of $(X, \mathcal{T})$ is a triple $(Y, \mathcal{V}, \iota)$, where

(*i*) $(Y, \mathcal{V})$ is a compact topological space,

(*ii*) $\iota : X \to Y$ is an *embedding* i.e. its corestriction $\tilde{\iota} : X \to \iota(X)$ is a homeomorphism of $(X, \mathcal{T})$ onto $(\iota(X), \mathcal{V}|_{\iota(X)})$, and

(*iii*) $\iota(X)$ is dense in $(Y, \mathcal{V})$.

Theorem 7.1.2 (Urysohn's Metrisability Theorem)**.** *Let X be a completely regular topological space. If X is second-countable, then it is metrisable.*

For the proof of Urysohn's Metrisability Theorem we refer to [20, Corollary 3.3.3].

Proposition 7.1.3. *Let X be a separable metrisable space. Then X has a metrisable compactification.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X, and d be a compatible metric on X. Without loss of generality we can assume that $d(x, y) \leq 1$ for all $x, y \in X$. We set

$$\iota : \begin{cases} X & \to [0,1]^{\mathbb{N}} \\ x & \mapsto (d(x, x_n))_{n \in \mathbb{N}} \end{cases}$$

If endowed with the product topology, $[0, 1]^{\mathbb{N}}$ is a compact metrisable Hausdorff space. It's topology is induced by

$$d'(x, y) := \sup_{n \in \mathbb{N}} \frac{1}{n} |\pi_n(x) - \pi_n(y)|,$$

where $\pi_n : [0, 1]^{\mathbb{N}} \to X$ denotes the canonical projection. $\overline{\iota(X)}$ is a compact Hausdorff space with the induced topology, hence normal and in particular completely regular. Being a subspace of the countable product of the second-countable space $[0, 1]$, it is also second-countable. By Urysohn's Metrisation Theorem 7.1.2, $\overline{\iota(X)}$ is metrisable.

It is only left to show that ι is an embedding. For $x \neq y$ there exists $n \in \mathbb{N}$ with $d(x, x_n) \leq \frac{1}{3} d(x, y)$. Therefore, ι is injective. For every $n \in \mathbb{N}$ the composition $\pi_n \circ \iota$ is continuous, hence ι is continuous. Take a convergent sequence $(\iota(x_m))_{m \in \mathbb{N}}$ in $\operatorname{ran}(X)$.

We have $(\iota(x_m))_{m\in\mathbb{N}} \to \iota(x)$ if and only if $d(x_m, x_n) \to d(x, x_n)$ for all $n \in \mathbb{N}$. Given $\epsilon > 0$ choose $n \in \mathbb{N}$ with $d(x, x_n) < \frac{\epsilon}{2}$ and $M \in \mathbb{N}$ with $d(x_m, x_n) < \frac{\epsilon}{2}$ for all $m \geq M$. It follows $d(x_m, x) \leq d(x_m, x_n) + d(x, x_n) < \epsilon$ for all $m \geq M$, hence $x_m \to x$. Thus, ι is homeomorphism onto $\iota(X)$. $\square$

7.2 K-Analyticity of Analytic Metrisable Spaces

In order to proof the equivalence of K-analyticity and analyticity in metrisable spaces, we will make use of different approaches to analytic sets.

Definition 7.2.1. A topological space is called *analytic* if it is the continuous image of a Polish space. A subset of a topological space is analytic if it is analytic with the induced topology.

Remark 7.2.2. Clearly, analytic spaces are separable, being continuous images of separable spaces.

Definition 7.2.3. [11, page 30] Let $R := \{A_{\alpha|_n} : \alpha \in \mathcal{N}, n \in \mathbb{N}\}$ be a systems of sets, which are defined for every finite sequence $\alpha|_n = (a_1, \ldots, a_n)$ of natural numbers. The *$\mathscr{A}$-operation* on R is defined by

$$\mathscr{A}(R) := \bigcup_{\alpha\in\mathcal{N}} \bigcap_{n\in\mathbb{N}} A_{\alpha|_n}$$

Lemma 7.2.4. *Let X be a Polish space, and $A \subseteq X$. The following statements are equivalent:*

(i) A is analytic.

(ii) A is a continuous image of the Baire space $\mathcal{N}$.

(iii) A is the projection of a closed subset of $\mathcal{N} \times X$ onto X.

Proof.
$(i) \Rightarrow (ii)$: This is the conclusion of Theorem 3.5.

$(ii) \Rightarrow (iii)$: Let $f : \mathcal{N} \to A$ be a continuous surjection. As A is Hausdorff, graph(f) is a closed subset of $\mathcal{N} \times X$. It's projection onto X is equal to A.

$(iii) \to (i)$: $\mathcal{N} \times X$ is Polish, so every closed subset of $\mathcal{N} \times X$ is also Polish. Let S be the closed subset $\mathcal{N} \times X$ which projects onto A. The projection onto X is continuous. Thus, A is the continuous image of the Polish space S. $\square$

Lemma 7.2.5. *Let X be a Polish space. If $R := \{A_{\alpha|_n} : \alpha \in \mathcal{N}, n \in \mathbb{N}\}$ consists of closed subsets of X, then $\mathscr{A}(R)$ is analytic.*

Proof. Clearly, $x \in \mathscr{A}(R) \Leftrightarrow \exists \alpha \in \mathcal{N} : x \in A_{\alpha|_n}$ for all $n \in \mathbb{N}$. We therefore define a relation $S \subseteq \mathcal{N} \times X$ by

$$(\alpha, x) \in S :\Leftrightarrow (\forall n \in \mathbb{N} : x \in A_{\alpha_n}).$$

Let $((\alpha_k, x_k))_{k\in\mathbb{N}}$ be a sequence in S converging to (α, x). As $(\alpha_k)_{k\in\mathbb{N}} \to \alpha$, and the topology on $\mathcal{N}$ is the product topology of the discrete topology on $\mathbb{N}$, for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ so that for all $k \geq N$ we have $\pi_n(\alpha_k) = \pi_n(\alpha)$. Therefore, for all $n \in \mathbb{N}$ we have that $x_k \in A_{\alpha|_n}$ for all sufficiently large k. As $A_{\alpha|_n}$ is closed, we obtain $x \in A_{\alpha|_n}$ for every $n \in \mathbb{N}$, hence $(\alpha, x) \in S$, and S is closed. It's projection onto X is equal to $\mathscr{A}(R)$. By Lemma 7.2.4, $\mathscr{A}(R)$ is analytic. □

Remark 7.2.6. Actually, the converse is also true, see for example [11, §38, IX, Theorem 4]. The original definition of analytic sets introduced by Andrei Suslin used this characterisation. The $\mathscr{A}$-operation is also called *Suslin operation*. It is a widely used tool in descriptive set theory. For more information we refer to [9, Chapter III].

Theorem 7.2.7. *[13, Theorem 2.3] A metrisable space is analytic if and only if it is K-analytic.*

Proof.
"$\Rightarrow$" Let X be a metrisable space, and $T : \mathcal{N} \to X$ be a continuous surjection. For every $\alpha \in \mathcal{N}$ we set $A_\alpha := T(\{\beta \in \mathcal{N} : \beta \leq \alpha\})$. As $\{\beta \in \mathcal{N} : \beta \leq \alpha\}$ is a compact subset of $\mathcal{N}$, and T is continuous, A_α is compact for every $\alpha \in \mathcal{N}$, and we have found a family of subsets satisfying conditions (A), (B), and (C) from (2). In a metrisable space every countably compact subset is compact. Theorem 5.5 now gives us that X is K-analytic.

"$\Leftarrow$" Let $\{A_\alpha : \alpha \in \mathcal{N}\}$ be a family of subsets of X satisfying (A), (B), and (C) from (2). By Corollary 4.7, X is Lindelöf, and therefore second-countable with Proposition 2.1.2. By Proposition 7.1.3, there exists a metrisable compactification $(Y, \mathcal{V})$ of X. For every $\alpha \in \mathcal{N}$ and $n \in \mathbb{N}$ we set

$$B_{\alpha|_n} := \overline{\bigcup_{\beta|_n = \alpha|_n} A_\beta}^{\mathcal{V}}.$$

As compact metrisable spaces are always separable and complete, Y is a Polish Space. Note that in particular Y is second-countable. By Lemma 7.2.5,

$$\mathscr{A}(\{B_{\alpha|_n} : \alpha \in \mathcal{N}, n \in \mathbb{N}\})$$

is analytic.

Clearly, $X \subseteq \bigcup_{\alpha\in\mathcal{N}} \bigcap_{n\in\mathbb{N}} B_{\alpha|_n}$. For the converse suppose $x \in \bigcap_{n\in\mathbb{N}} B_{\alpha|_n}$ for some $\alpha \in \mathcal{N}$. Let $\{U_n : n \in \mathbb{N}\}$ be a countable neighbourhood base for $x \in Y$ satisfying $U_{n+1} \subseteq U_n$. By definition, for every $n \in \mathbb{N}$ there exists $\beta_n \in \mathcal{N}$ with $\beta_n|_n = \alpha|_n$ and

$y_n \in A_{\beta_n} \cap U_n$. For all $k \in \mathbb{N}$ we have $\pi_k(\beta_n) \to \pi_k(\alpha)$, hence $\beta_n \to \alpha$. Let $\gamma = (c_n)_{n\in\mathbb{N}}$ be defined by

$$c_n := \max\{\pi_n(\beta_k) : 1 \le k \le n\}.$$

Then $\beta_n \le \gamma$ for all $n \in \mathbb{N}$, which implies $\bigcup_{n\in\mathbb{N}} A_{\beta_n} \subseteq A_\gamma$, as well as $y_n \in A_\gamma$ for every $n \in \mathbb{N}$. A_γ is compact, and therefore closed. Because of $y_n \to x$, we obtain $x \in A_\gamma \subseteq X$. We have now shown that $X = \mathscr{A}(\{B_{\alpha|_n} : \alpha \in \mathcal{N}, n \in \mathbb{N}\})$ is analytic, and the proof is complete. $\square$

Corollary 7.2.8. *Every Polish space is K-analytic.*

As mentioned above, analytic sets form a very rich class in descriptive set theory. It can be shown that analytic sets are Lebesgue measurable. Any Borel set is analytic, but analytic sets need to be Borel. An example for a non-borelian analytic set is the set of all closed uncountable sets in the space of closed subsets of the interval $[0, 1]$, see for example [11, §39].

8 Metrisability of Compact Topological Spaces

Theorem 5.5 gives topological spaces a K-analytic structure provided that the mapping T is compact-valued. Making assumptions on the diagonal of a compact space K, we can show that this is the case for the space $C(K)$ of all real-valued continuous functions on K. As a consequence, given a compact topological space, we will establish a couple of conditions that are equivalent to the property of it being metrisable.

8.1 Preliminaries

The metrisability of a compact topological space K is linked to topological properties of $C(K)$.

Proposition 8.1.1. *Let K be a compact topological space. Then K is metrisable if and only if $(C(K), \|.\|)_\infty$ is separable.*

For the proof we will make use of the following theorem, whose proof can be found in [2, Satz 1.5.2].

Theorem 8.1.2 (Stone-Weierstraß)**.** *Let K be a compact space, and $\mathcal{A}$ a point separating algebra of continuous real-valued functions on K that vanishes nowhere. Then $\mathcal{A}$ is dense in $(C(K), \|.\|)_\infty$.*

Proof of Proposition 8.1.1. Let d be a metric that metrises the topology on K. As all compact metrisable spaces are separable, we may choose a countable dense subset $Y = (y_n)_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$ we define the continuous function

$$f_n : \begin{cases} K & \to \mathbb{R} \\ x & \mapsto d(x, y_n). \end{cases}$$

The family $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is a countable subset of $C(K)$. In order to see that it is point separating, take any $x \neq z \in K$ and choose $n \in \mathbb{N}$ with $d(x, y_n) \leq \frac{1}{3}d(x, z)$. Clearly, $d(x, y_n) = f_n(x) \neq f_n(z) = d(z, y_n)$. If K consists of more than one point, $\mathcal{F}$ vanishes nowhere (if there is only one point, $C(K)$ is equal to $\mathbb{R}$, and therefore separable).

For a finite subset M of $\mathbb{N}$ we write $F_M := \prod_{n\in M} f_n$. Let $\mathcal{A}_n$ be the set of all functions f that can be written as

$$f = \sum_{i=1}^{N} q_i F_{M_i} \quad \text{with } q_i \in \mathbb{Q},\ N \in \mathbb{N},\ M_i \subseteq \{1, \dots, n\}$$

By Stone-Weierstraß, the algebra defined by $\mathcal{A} := \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is dense in $C(K)$. For $n, m \in \mathbb{N}$ and $q_i \in \mathbb{Q},\ M_i \subseteq \mathbb{N}$ as before, the set defined by

$$\mathcal{A}_n^m := \left\{ f \in \mathcal{A}_n : f = \sum_{i=1}^{m} q_i F_{M_i} \right\}$$

is countable. Since $\mathcal{A}_n = \bigcup_{m\in\mathbb{N}} \mathcal{A}_n^m$, we get that for every $n \in \mathbb{N}$ the set $\mathcal{A}_n$ is countable. Thus implied is the countability of $\mathcal{A}$. Therefore, $C(K)$ is separable.

Conversely, let $\mathcal{F}$ be a countable dense subset of $C(K)$. The family

$$\mathcal{B} := \{f^{-1}((a, b)) : f \in \mathcal{F},\ a, b \in \mathbb{Q} \text{ with } a < b\}$$

is countable and consists of open subsets of K. Let $O \subseteq K$ be an arbitrary open set and pick $x \in U$. Since $\{x\}$, as well as $K \setminus O$ are closed, using Urysohn's Lemma 2.2.5 we find a continuous function $g : K \to [0, 1]$ so that $g(x) = 0$ and $g(K \setminus O) = \{1\}$. Clearly, g is also continuous as function from K to $\mathbb{R}$. As $\mathcal{F}$ is dense in $C(K)$, we find $f \in \mathcal{F}$ that satisfies $\|f - g\|_\infty < \frac{1}{4}$. Now, $f^{-1}((-\frac{1}{2}, \frac{1}{2})) \subseteq g^{-1}([0, \frac{3}{4}))$ is an open neighbourhood of x that is contained in U. Thus, $\mathcal{B}$ is a countable basis of the topology on K. Applying Urysohn's Metrisability Theorem 7.1.2, we receive that K is indeed metrisable. □

We will also make use of the well-known theorem of Arzelà-Ascoli.

Definition 8.1.3. Let Φ be a family of real-valued functions on a topological space $(X, \mathcal{T})$.

(i) Φ is called *pointwise bounded* if for all $x \in X$ the set $\{|f(x)| : f \in \Phi\}$ is bounded in $\mathbb{R}$.

(ii) Φ is called *equicontinuous* if for all $x \in X$ and $\epsilon > 0$ there exists a neighbourhood $V \in \mathcal{U}(x)$ so that $|f(x) - f(z)| < \epsilon$ for all $z \in V$ and $f \in \Phi$.

Theorem 8.1.4 (Arzelà-Ascoli)**.** *Let K be a compact topological space. A family of continuous real-valued functions is totally bounded in $(C(K), \|.\|_\infty)$ if and only if it is pointwise bounded and equicontinuous.*

For the proof of the theorem of Arzelá-Ascoli we refer for example to [2, Satz 1.5.1].

8.2 Application of K-Analyticity

The core of our study of metrisable compact spaces is Theorem 8.2.3. Preliminary work includes the provision of conditions under which $C(K)$ is K-analytic.

We want to introduce a new notion:

Definition 8.2.1. We say that a family $\mathcal{F}$ of compact subsets of a topological space X *swallows* all compact subsets of X if for every $K \in \mathcal{K}(X)$ there exists $F \in \mathcal{F}$ with $K \subseteq F$.

Let K be a topological space. We endow the product space $K \times K$ with its product topology, and denote its diagonal by

$$\Delta := \{(x, x) : x \in K\}.$$

Note that the diagonal of a Hausdorff-space K is always closed: For any $x \neq y \in K$ choose disjoint open neighbourhoods $O_x \in \mathcal{U}(x)$, and $O_y \in \mathcal{U}(y)$. Then $O_x \times O_y$ is an open subset of $K \times K$ that is disjoint from Δ. Therefore, $(K \times K) \setminus \Delta$ is open.

If the diagonal of a compact topological space K holds certain properties, the mapping T defined in Theorem 5.5 gives the space $(C(K), \|.\|_\infty)$ a K-analytic structure:

Proposition 8.2.2. *Let K be a compact topological space, and $\{A_\alpha : \alpha \in \mathcal{N}\}$ be a family of compact sets in $(K \times K)$. If*

(i) $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathcal{N}\}$,

(ii) $A_\alpha \subseteq A_\beta$ *for every* $\alpha \leq \beta$, *and*

(iii) $\{A_\alpha : \alpha \in \mathcal{N}\}$ *swallows all compact subsets of* $(K \times K) \setminus \Delta$,

then $(C(K), \|.\|_\infty)$ *is a K-analytic space.*

Proof. For every $\alpha \in \mathcal{N}$ we define $O_\alpha := (K \times K) \setminus A_\alpha$. Since every A_α is compact, and therefore closed, all O_α are open sets. Let $U \in \mathcal{U}(\Delta)$ be an open neighbourhood of the diagonal Δ. Being a closed subset of a compact space, its complement $(K \times K) \setminus U$ is compact, and is therefore swallowed by some A_α. We receive

$$U \supseteq (K \times K) \setminus A_\alpha = O_\alpha.$$

Because of condition (i), O_α contains Δ, and we have shown that $\mathcal{O} := \{O_\alpha : \alpha \in \mathbb{N}\}$ is a basis of the neighbourhood filter of Δ.

If $\alpha \leq \beta$, condition (ii) implies $O_\alpha \supseteq O_\beta$. Given a sequence $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$, and $m \in \mathbb{N}$ we write $\alpha|^m := (a_n)_{n \geq m}$. We name $B_r^{\|\cdot\|_\infty} := \{f \in C(K) \ : \ \|f\|_\infty < r\}$. Now look at the family of functions defined by

$$B_\alpha := \left\{ f \in B_{a_1}^{\|\cdot\|_\infty} \ : \ |f(x) - f(y)| \leq \frac{1}{m} \text{ if } (x, y) \in O_{\alpha|^m} \text{ for } m \in \mathbb{N} \right\}.$$

For each fixed $\alpha \in \mathcal{N}$ the family B_α clearly is pointwise bounded. Pick any $x \in X$, $\epsilon > 0$, and choose $m \in \mathbb{N}$ with $\frac{1}{m} < \epsilon$. Since $O_{\alpha|^m}$ is an open neighbourhood of Δ, we find a neighbourhood $V \in \mathcal{U}(x)$ so that for all $y \in V$ the tuple (x, y) is still contained in $O_{\alpha|^m}$. By definition of B_α, we have $|f(x) - f(y)| \leq \frac{1}{m} < \epsilon$ for all $y \in V$. Thus, B_α is equicontinuous. By the theorem of Arzelà-Ascoli, the family B_α is totally bounded. Since it is closed, and $C(K)$ is complete, it is compact in $(C(K), \|.\|_\infty)$.

We will now show that $C(K) = \bigcup_{\alpha \in \mathcal{N}} B_\alpha$. As the converse is clear, we only show that for every $f \in C(K)$ there exists $\alpha \in \mathcal{N}$ such that $f \in B_\alpha$. Choose $M \in \mathbb{N}$ with $\|f\|_\infty \leq M$. Because $\mathcal{O}$ is a basis of the neighbourhood filter of Δ, for every $m \in \mathbb{N}$ we find a sequence $\alpha_m = (a_n^m)_{n \in \mathbb{N}}$ so that if $(x, y) \in O_{\alpha_m}$, it follows that $|f(x) - f(y)| \leq \frac{1}{m}$. We put

$$a_1 := \max\{a_1^1, M\}, \text{ and}$$
$$a_n := \max\{a_n^1, a_{n-1}^2, \ldots, a_1^n\} \text{ for } n > 1.$$

With $\alpha := (a_n)_{n \in \mathbb{N}}$ we have found a sequence so that $f \in B_\alpha$ because for all $m \in \mathbb{N}$ we get $\alpha_m \leq \alpha|^m$, and therefore $O_{\alpha|^m} \subseteq O_{\alpha_m}$.

We have obtaind that (A) B_α is compact for every $\alpha \in \mathcal{N}$, and (C) $\bigcup\{B_\alpha : \alpha \in \mathcal{N}\} = C(K)$. Clearly, also (B) $\alpha \leq \beta$ implies that $B_\alpha \subseteq B_\beta$. Therefore, the family $\{B_\alpha : \alpha \in \mathcal{N}\}$ meets the assumptions of Theorem 5.5. As in metric spaces the concepts of countably compactness and compactness coincide, by Theorem 5.5 item (ii), $(C(K), \|.\|_\infty)$ is a K-analytic space. □

The metrisability of a compact space K is not only equivalent to $C(K)$ being separable, but also to conditions of its own diagonal. The centrepiece of this section is to examine those equivalences.

Theorem 8.2.3. *[4, Theorem 2] Let K be a compact space. The following statements are equivalent:*

(i) K is metrisable.

(ii) $(C(K), \|.\|_\infty)$ is separable.

(iii) Δ is a G_δ subset of $K \times K$, i.e. Δ is the countable intersection of some open sets.

(iv) There exists a countable basis $\{G_n : n \in \mathbb{N}\}$ of open neighbourhoods of Δ so that $\Delta = \bigcap_{n \in \mathbb{N}} G_n$.

(v) There exists a countable family $\{F_n : n \in \mathbb{N}\}$ of compact subsets of $(K \times K) \setminus \Delta$ with the property that $F_n \subseteq F_m$ for any $n \leq m$, and $(K \times K) \setminus \Delta \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

(vi) There exists a countable family $\{F_n : n \in \mathbb{N}\}$ of compact subsets of $K \times K$ that swallows all the compact subsets of $(K \times K) \setminus \Delta$, and $(K \times K) \setminus \Delta \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

(vii) *There exists a family $\{A_\alpha : \alpha \in \mathcal{N}\}$ of compact subsets of $K \times K$ with the property that it swallows all the compact subsets of $(K \times K) \setminus \Delta$, it holds that $A_\alpha \subseteq A_\beta$ for any $\alpha \leq \beta$, and $(K \times K) \setminus \Delta \subseteq \bigcup_{n\in\mathbb{N}} F_n$.*

(viii) *$(K \times K) \setminus \Delta$ is Lindelöf.*

Proof.
$(i) \Leftrightarrow (ii)$ This equivalence has been proved in Proposition 8.1.1.

$(i) \Rightarrow (iii)$ Let d be a metric on K that induces the topology. For $n \in \mathbb{N}$ we define the open set

$$G_n := \left\{ \{x, y\} \in K \times K : d(x, y) < \frac{1}{n} \right\},$$

and obtain $\Delta = \bigcap_{n\in\mathbb{N}} G_n$.

$(iii) \Rightarrow (iv)$ First, note that every compact completely regular space $(K, \mathcal{T})$ is (T_4): Take two disjoint closed sets $A, B \subseteq K$. For any $a \in A$ the singleton set $\{a\}$ is closed, because of (T_1). With (T_3), we find disjoint open sets O_a, U_a with $a \in O_a$ and $B \subseteq U_a$. Being a closed subset of a compact space, A is itself compact, and we find a finite number of $a_i \in A$ so that $A \subseteq \bigcup_{i=1}^n O_{a_i} =: O_A \in \mathcal{T}$. By $O_B := \bigcap_{i=1}^n U_{a_i}$ we have an open cover of B that satisfies $O_A \cap O_B = \emptyset$.

Assume that $\Delta = \bigcup_{n\in\mathbb{N}} G_n$ with $G_n \subseteq K \times K$ open. The product space $K \times K$ endowed with the product topology is a compact and completely regular space. Using Proposition 2.2.4, for every $n \in \mathbb{N}$ we find an open set $U_n \subseteq K \times K$ so that $\Delta \subseteq U_n \subseteq \overline{U_n} \subseteq G_n$. We define

$$O_n := \bigcap_{k=1}^{n} U_k.$$

Because of $\Delta \subseteq \bigcap_{n\in\mathbb{N}} O_n \subseteq \bigcap_{n\in\mathbb{N}} \overline{O_n} \subseteq \bigcap_{n\in\mathbb{N}} G_n = \Delta$ we have

$$\Delta = \bigcap_{n\in\mathbb{N}} O_n = \bigcap_{n\in\mathbb{N}} \overline{O_n}.$$

Consider an arbitrary open neighbourhood $U \in \mathcal{U}(\Delta)$, and the decreasing sequence $(C_n)_{n\in\mathbb{N}}$ defined by

$$C_n := \overline{O_n} \cap ((K \times K) \setminus U).$$

Clearly, each C_n is closed. Recall that any intersection of a decreasing family of closed subsets of a compact topological space is non-empty (see for instance [2, Satz 1.4.2]). Therefore, if C_n was non-empty for all $n \in \mathbb{N}$, we would have $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$, implying

$$\Delta \cap ((K \times K) \setminus U) = \left(\bigcap_{n\in\mathbb{N}} \overline{O_n} \right) \cap ((K \times K) \setminus U) \neq \emptyset.$$

This, however, contradicts the choice of $U \supseteq \Delta$. Thus, for every open subset U containing Δ there exists $n \in \mathbb{N}$ with $O_n \subseteq \overline{O_n} \subseteq U$. We have hereby shown that $\{O_n : n \in \mathbb{N}\}$ is a basis of the neighbourhood filter of Δ.

$(iv) \Rightarrow (iii)$ Obvious.

$(iii) \Rightarrow (v)$ If $\Delta = \bigcap_{n\in\mathbb{N}} G_n$ with $G_n \subseteq K \times K$ open, the subsets

$$\widetilde{G_n} := \bigcup_{k=1}^{n} G_n$$

form a decreasing family of open subset while still satisfying $\Delta = \bigcap_{n\in\mathbb{N}} \widetilde{G_n}$. As the sets $F_n := (K \times K) \setminus \widetilde{G_n}$ are closed subsets of a compact space, they are themselves compact, and we have found an increasing family of compact sets with $(K \times K) \setminus \Delta \subseteq \bigcup_{n\in\mathbb{N}} F_n$.

$(v) \Rightarrow (iii)$ Obvious by setting $G_n := (K \times K) \setminus F_n$, and using the fact that compact subsets of a Hausdorff-space are closed.

$(iv) \Rightarrow (vi)$ The subsets $F_n := (K \times K) \setminus G_n$ are closed, and therefore compact. Let $C \subseteq (K \times K) \setminus \Delta$ be compact. Then $(K \times K) \setminus C$ is an open subset containing Δ. As $\{G_n : n \in \mathbb{N}\}$ is a basis of open neighbourhoods, we find $n \in \mathbb{N}$ with $G_n \subseteq (K \times K) \setminus C$ or, equivalently, $C \subseteq F_n$.

$(vi) \Rightarrow (iv)$ Same argument as $(iv) \Rightarrow (vi)$.

$(vi) \Rightarrow (vii)$ For $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$ set $A_\alpha := F_{a_1}$.

$(vii) \Rightarrow (ii)$ In Proposition 8.2.2 we have shown that if statement (vii) holds, $(C(K), \|.\|_\infty)$ is K-analytic. By Corollary 4.7, every K-analytic space is Lindelöf. Since $(C(K), \|.\|_\infty)$ is a metric space, and by Lemma 2.1.2 a metric space is Lindelöf if and only if it is separable, it follows that $(C(K), \|.\|_\infty)$ is separable.

We have now shown that all statements from (i) to (vii) are equivalent.

$(v) \Rightarrow (viii)$ By assumption, $(K \times K) \setminus \Delta$ is σ-compact i.e. has a countable cover consisting of compact subsets. Let $\mathcal{O} = \{O_i : i \in I\}$ be an open cover of $(K \times K) \setminus \Delta = \bigcup_{n\in\mathbb{N}} F_n$. In particular, $\mathcal{O}$ covers every compact F_n. For every $n \in \mathbb{N}$ let I_n be a finite subset of I with $F_n \subseteq \bigcup_{k\in I_n} O_k$. We obtain

$$(K \times K) \setminus \Delta \subseteq \bigcup_{n\in N} \bigcup_{k\in I_n} O_k.$$

Thus, $\mathcal{O}$ admits a countable subcover, and $(K \times K) \setminus \Delta$ is Lindelöf (as is every σ-compact topological space).

$(viii) \Rightarrow (iii)$ Take any $x, y \in K$ with $x \neq y$. Since $\{(x,y)\}$ is closed, and contained in the open set $(K \times K) \setminus \Delta$, by Lemma 2.2.4 we find an open set $U_{x,y}$ with

$$(x,y) \in U_{x,y} \subseteq \overline{U_{x,y}} \subseteq (K \times K) \setminus \Delta.$$

As $\{U_{x,y} : (x,y) \in (K \times K) \setminus \Delta\}$ is an open cover of $(K \times K) \setminus \Delta$, which is Lindelöf by assumption, we may choose countably many tuples $(x_n, y_n) := (x,y)_n \in (K \times K) \setminus \Delta$ with

$$(K \times K) \setminus \Delta = \bigcup_{n \in \mathbb{N}} U_{x_n,y_n} = \bigcup_{n \in \mathbb{N}} \overline{U_{x_n,y_n}}$$

Consequently, $\Delta = \bigcap_{n \in \mathbb{N}} (K \times K) \setminus \overline{U_{x_n,y_n}}$ is a G_δ set.

□

9 Weakly Compactly Generated Banach Spaces

According to Section 6, all Banach spaces are angelic in their weak topology. The aim of the following section is to find a family of sets in a weakly compactly generated Banach space so that Corollary 6.5 implies the space being K-analytic in its weak topology.

9.1 Preliminaries

We are going to sketch some of the most fundamental concepts used in functional analysis, including topological vector spaces, special subsets of such, dual spaces, weak topologies, and some powerful properties of the latter. For a more thorough treatise on those topics we refer to [3].

Definition 9.1.1. Let X be a vector space over the field $\mathbb{C}$ of all complex numbers. Let $\mathcal{T}$ be a topology on X. We call $(X, \mathcal{T})$ *topological vector space* if the mapping

$$+ : \begin{cases} X \times X & \to X \\ (x,y) & \mapsto x + y \end{cases}$$

is $\mathcal{T} \times \mathcal{T}$-to-$\mathcal{T}$ continuous, and

$$\cdot : \begin{cases} \mathbb{C} \times X & \to X \\ (\lambda, x) & \mapsto \lambda x \end{cases}$$

is $\mathcal{E} \times \mathcal{T}$-to-$\mathcal{T}$ continuous ($\mathcal{E}$ denotes the Euklidean topology on $\mathbb{C}$). We will furthermore assume that all topological vector spaces are Hausdorff.

In particular, all normed spaces are topological vector spaces if endowed with the topology induced by the norm.

Definition 9.1.2. A subset A of a vector space X is called

(*i*) *convex* if for all $x, y \in A$ and $t \in [0,1]$ it holds that $tx + (1-t)y \in A$.

(*ii*) *balanced* if for all $x \in A$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ it holds that $\lambda x \in A$, or equivalently

$$A = \bigcup_{|\lambda| \leq 1} \lambda A.$$

(*iii*) *absolutely convex* if A is both convex and balanced.

For a subset $A \subseteq X$ we denote the *convex hull* (*balanced hull*), that is the the smallest (with respect to $\subseteq$) convex (balanced) subset of X that contains A, by $\mathrm{co}(A)$ ($\mathrm{bal}(A)$). Clearly, $\mathrm{bal}(A) = \bigcup_{|\lambda| \leq 1} \lambda A$.

The following lemma summarises some straightforward facts about convex and balanced sets.

Lemma 9.1.3.

(i) $A \subseteq X$ is convex if and only if for all $x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ also $\sum_{i=1}^n \lambda_i x_i \in A$.

(ii) $A \subseteq X$ is absolutely convex if and only if for all $m \in \mathbb{N}$, $x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^n |\lambda_i| \leq m$ it holds that $\sum_{i=1}^n \lambda_i x_i \in mA$.

(iii) The closure of a convex (balanced) set it convex (balanced).

(iv) The balanced hull of a compact set is compact.

(v) The convex hull of a balanced set is absolutely convex.

Proof.

(*i*) For $n = 1$ the statement is clear. Now assume that the claim holds for $n-1$ and that $\lambda_i > 0$ for all i. Let $c := \sum_{i=1}^{n-1} \lambda_i$. Since $\sum_{i=1}^{n-1} \frac{\lambda_i}{c} = 1$ and therefore $y := \sum_{i=1}^{n-1} \frac{\lambda_i}{c} x_i \in A$ by assumption, it follows that

$$\sum_{i=1}^{n} \lambda_i x_i = c \sum_{i=1}^{n-1} \frac{\lambda_i}{c} x_i + \lambda_n x_n = cy + (1-c) x_n \in A.$$

As the definition of convexity is the case $n = 2$, the converse of the statement is clear.

Note that it now follows that

$$\mathrm{co}(E) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_i \in A, \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1 \right\}. \tag{3}$$

(*ii*) Let $A \subseteq X$ be an absolutely convex set and $m = 1$. Define $c := 1 - \sum_{i=1}^n \lambda_i \geq 0$ and $c_i := \frac{c}{n}$, i.e. $\sum_{i=1}^n |\lambda_i| + c_i = 1$. Again assuming $\lambda_i \neq 0$ for all i, we have $|\frac{\lambda_i}{|\lambda_i| + c_i}| \leq 1$

and therefore, since A is balanced, $y_i := \frac{\lambda_i}{|\lambda_i|+c_i} x_i \in A$. Because of the convexity of A, we now obtain

$$\sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} (|\lambda_i| + c_i) \frac{\lambda_i}{|\lambda_i| + c_i} x_i = \sum_{i=1}^{n} (|\lambda_i| + c_i) y_i \in A.$$

By induction, $\sum_{i=1}^{n} \lambda_i x_i \in mA$ for all $m \in \mathbb{N}$. The converse of the statement is clear.

(iii) Let A be any subset of X. Because the vector space operations are continuous, we have that $\alpha\bar{A} + \beta\bar{A} \subseteq \overline{\alpha A + \beta A}$ for any $\alpha, \beta \in \mathbb{C}$. If we choose $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ for the convex case, and $|\alpha| \leq 1, \beta = 0$ for the balanced case, we obtain $\alpha\bar{A} + \beta\bar{A} \subseteq \overline{\alpha A + \beta A} \subseteq \bar{A}$, and therefore the desired statement.

(iv) Let A be a compact subset. $K_1 := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ is compact. As $\mathrm{bal}(A) = K_1 \cdot A$ is the image of a compact set under a continuous function, it is itself compact.

(v) Let $A \subseteq X$ be balanced, $x \in \mathrm{co}(A)$, and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. By definition of the convex hull, we find $a, b \in A$, and $t \in [0,1]$ so that $x = ta + (1-t)b$. Since A is balanced, we obtain $\lambda x = t(\lambda a) + (1-t)(\lambda b) \in \mathrm{co}(A)$. Therefore, $\mathrm{co}(A)$ is both convex and balanced, hence absolutely convex.

□

By $\mathrm{cobal}(A)$ we denote the absolutely convex hull of a subset A. Note, that with Lemma 9.1.3 *(v)* we have shown that $\mathrm{cobal}(A) = \mathrm{co}(\mathrm{bal}(A))$. However, $\mathrm{cobal}(A) \neq \mathrm{bal}(\mathrm{co}(A))$ in general.

Definition 9.1.4. Let X be a vector space over the field $\mathbb{C}$ and Y be a point-separating linear subspace of its *algebraic dual space* X^*, that is the space of all linear functionals $f : X \to \mathbb{C}$. The *weak topology* on X induced by Y is the coarsest topology X making all $f \in Y$ continuous. It is denoted by $\sigma(X, Y)$.

A topological vector space is called *locally convex* if there exists a neighbourhood basis of 0 consisting of convex sets. Note that all normed spaces $(X, \|.\|)$ are locally convex, as the open balls $U_\epsilon(0) := \{x \in X : \|x\| < \epsilon\}$ with $\epsilon > 0$ form a basis of the neighbourhood filter of 0. For a locally convex topological vector space X its *dual space* X', that is the space of all continuous linear functionals from X into the base field $\mathbb{C}$, is a point-separating subspace of the algebraic dual space X^* [3, Korollar 5.2.7, *(i)*]. The initial topology on X with respect to X' is called *weak topology*, and is denoted by $\sigma(X, X')$. $(X, \sigma(X, X'))$ is again a locally convex topological vector space. For more information on the weak topology we refer to [3, Chapter 5.3].

Let $(X, \|.\|)$ be a normed space. By $\iota : X \to X'' : x \mapsto \iota(x)$ we denote the canonical embedding defined by

$$\iota(x) : \begin{cases} X' & \to \mathbb{C} \\ f & \mapsto f(x) \end{cases}$$

It's corestriction $\iota : X \to \iota(X)$ is a linear bijection. Analogously, we define $\iota_1 : X' \to X'''$. If X is equipped with its weak topology $\sigma(X, X')$, and $\iota(X)$ is equipped with subspace topology of the weak topology induced by the point-separating linear subspace $\iota_1(X')$ of the algebraic dual space of X'', namely $\sigma(X'', \iota_1(X'))|_{\iota(X)}$,

$$\iota : \big(X, \sigma(X, X')\big) \to \big(\iota(X), \sigma(X'', \iota_1(X')\big)\,|_{\iota(X)} \tag{4}$$

becomes a homeomorphism [3, Lemma 5.5.2 et. seq.].

Remark 9.1.5. The weak topology on X' induced by $\iota(X)$ is also called *weak*-topology*, and denoted by $\sigma(X', X) := \sigma(X', \iota(X))$. In general, it holds that

$$\sigma(X', X) \subseteq \sigma(X', X'') \subseteq \mathcal{T}_{\|\cdot\|_{X'}}.$$

Normed spaces $(X, \|.\|)$ for which $X'' = \iota(X)$ are called *reflexive*.

The proofs of the following two theorems may be found in [3, Satz 5.5.5] and [3, Satz 5.5.6], respectively.

Theorem 9.1.6 (Goldstine)**.** *Let $(X, \|.\|)$ be a normed space. We write $K_1^X(0) := \{x \in X : \|x\| \leq 1\}$ and $K_1^{X''}(0) := \{f \in X'' : \|f\|_{X''} \leq 1\}$ for the closed unit balls in X and X'', respectively. Then*

$$\overline{\iota(K_1^X(0))}^{\sigma(X'', X')} = K_1^{X''}(0).$$

Theorem 9.1.7 (Banach-Alaoglu)**.** *Let $(X, \|.\|)$ be a normed space. Then $K_1^{X'} := \{f \in X' : \|f\|_{X'} \leq 1\}$ is compact with respect to the weak*-topology $\sigma(X', X)$.*

The result of the Theorem of Banach-Alaoglu is particularly interesting, as the closed unit ball of a normed space is $\|.\|$-compact if and only if the space is finite dimensional.

Definition 9.1.8. A subset of a locally convex topological space is called *weakly compact* if it is compact with respect to the weak topology.

Clearly, if A is finite, because of (3) one can show – similarly to Lemma 9.1.3 *(iv)* – that $\mathrm{co}(A)$ is compact, being a continuous image of compact subset. The theorem of Krein-Smulian generalises this observation. For the proof of the theorem we refer to [5, Theorem 13.4].

Theorem 9.1.9 (Krein-Smulian)**.** *Let X be a Banach space. If $K \subseteq X$ is weakly compact, then the closed convex hull $\overline{\mathrm{co}}(K)$ of K is also weakly compact.*

Remark 9.1.10. The closed convex hull $\overline{\mathrm{co}}(K)$ i.e. the smallest closed convex set that contains K, is equal to the closure of the convex hull $\overline{\mathrm{co}(K)}$, as the closure of a convex set is convex by Lemma 9.1.3 *(iii)*.

9.2 Application of K-Analyticity

The theory of K-analytic spaces allows us to deduce topological properties of topological vector spaces. Concretely, we will now show that every weakly compactly generated Banach space is Lindelöf in its weak topology.

Definition 9.2.1. A Banach space $(X, \|.\|)$ is called *weakly compactly generated* if there exists a weakly compact set K in X so that $X = \overline{\operatorname{span}(K)}^{\|\cdot\|}$.

Theorem 9.2.2. *[4, Theorem 4] Every weakly compactly generated Banach space X is Lindelöf with respect to its weak topology.*

Proof. Let $X = \overline{\operatorname{span}(K)}^{\|\cdot\|}$ with K weakly compact. As $(X, \sigma(X, X'))$ is a topological vector space, $\operatorname{bal}(K)$ is weakly compact by Lemma 9.1.3 *(iv)*. The Theorem of Krein-Smulian 9.1.9 now gives us that $\overline{\operatorname{co}}(\operatorname{bal}(K))$ is weakly compact. By Lemma 9.1.3 *(v)*, we have that co(bal(K)) is absolutely convex. As the closure of an absolutely convex set is absolutely convex by Lemma 9.1.3 *(iii)*, we can therefore assume without loss of generality that K is absolutely convex. Using Lemma 9.1.3 *(ii)*, we now obtain $X = \overline{\bigcup_{n\in\mathbb{N}} nK}^{\|\cdot\|}$.

We set $B := \iota^{-1}(K_1^{X''}(0)) \subseteq X$. We will now show that the subsets $A_\alpha \subseteq X$ defined by

$$A_\alpha := \bigcap_{n\in\mathbb{N}} a_n K + \frac{1}{n} B$$

for $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$ satisfy conditions (A), (B), and (C) from (2).

(A) As K is weakly compact, and scaling is a homeomorphism ([3, Lemma 2.1.3, (i)]), the sets $a_n K$ are weakly compact for every $a_n \in \mathbb{N}$. By the Theorem of Banach-Alaoglu 9.1.7, the closed unit ball $K_1^{X''}(0)$ is compact with respect to the weak*-topology $\sigma(X'', X')$. Therefore, $K_1^{X''}(0) \cap \iota(X)$ is compact with respect to the subspace topology $\sigma(X'', X')|_{\iota(X)}$. As ι considered as in (4) is a homeomorphism, we obtain that $\iota^{-1}(K_1^{X''}(0) \cap \iota(X)) = \iota^{-1}(K_1^{X''}(0)) = B$ is weakly compact. Using that the addition is continuous, and that any intersection of compact sets is again compact, we have that A_α is weakly compact for every $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$.

(B) Let $\alpha = (a_n)_{n\in\mathbb{N}}$, $\beta = (b_n)_{n\in\mathbb{N}} \in \mathcal{N}$ with $a_n \leq b_n$ for every $n \in \mathbb{N}$. As K is absolutely convex, by Lemma 9.1.3 *(ii)* we obtain that $a_n K \subseteq b_n K$ for every $n \in \mathbb{N}$. Hence, $A_\alpha \subseteq A_\beta$ for every $\alpha \leq \beta$, as required.

(C) Take any $x \in X$. As $X = \overline{\bigcup_{n\in\mathbb{N}} nK}^{\|\cdot\|}$, for every $m \in \mathbb{N}$ we may choose $n_m \in \mathbb{N}$, and $y \in n_m K$ so that $\|x - y\| \leq \frac{1}{m}$, hence

$$x \in \frac{1}{m} K_1^X(y) \subseteq n_m K + \frac{1}{m} K_1^X(0).$$

The Theorem of Goldstine 9.1.6 now gives us

$$B = \iota^{-1}(K_1^{X''}(0)) = \iota^{-1}(\overline{\iota(K_1^X(0))}^{\sigma(X'',X')}) \supseteq \iota^{-1}(\iota(K_1^X(0))) = K_1^X(0),$$

and therefore $x \in n_m K + \frac{1}{m}B$. Choosing $\alpha := (n_m)_{m\in\mathbb{N}}$, we obtain $x \in A_\alpha$, and $X = \bigcup_{\alpha\in\mathcal{N}} A_\alpha$.

X is angelic in its weak topology. Therefore, $(X, \sigma(X, X'))$ is K-analytic after Corollary 6.5. Since K-analytic spaces are Lindelöf by Corollary 4.7, X is Lindelöf with respect to its weak topology. $\square$

In particular, all separable Banach spaces are weakly compactly generated. Less trivial examples include all reflexive spaces i.e. normed spaces $(X, \|.\|)$ with $X'' = \iota(X)$, as well as the function spaces $L^1(\mu)$ for σ-finite measures μ.

10 Fréchet-Montel Spaces

Frèchet spaces are generalisations of Banach spaces. We show a sufficient condition under which a Fréchet space is separable.

10.1 Preliminaries

For the proof of Theorem 10.2.2 we will have to use certain neighbourhoods of 0.

Definition 10.1.1. Let X be a topological vector space and $A \subseteq X$.

(*i*) A is said to be *absorbing* if for every $x \in X$ there exists $\lambda > 0$ so that $\lambda x \in A$.

(*ii*) A is called a *barrel* if it is absolutely convex, closed and absorbing.

(*iii*) X is a *barrelled space* if every barrel is a neighbourhood of 0.

(*iv*) A is *bounded* if for every $V \in \mathcal{U}(0)$ there exists $\lambda_V > 0$ so that $A \subseteq \lambda_V V$.

Lemma 10.1.2. *[16] Let X be a topological vector space.*

(i) If $\mathcal{V}$ is a neighbourhood base of 0, and $A \subseteq X$ is a nonempty subset, then

$$\bar{A} = \bigcap_{V\in\mathcal{V}} A + V.$$

(ii) If $\mathcal{V}$ is a neighbourhood base of 0, then so is $\{\bar{V} : V \in \mathcal{V}\}$.

(iii) If X is locally convex, it has a neighbourhood basis of 0 consisting of barrels.

Proof.

(i) For every $x \in X$ the set $\{x - V : V \in \mathcal{V}\}$ is a neighbourhood basis of x. Therefore,

$$x \in \bar{A} \Leftrightarrow \forall V \in \mathcal{V} : (x - V) \cap A \neq \emptyset \Leftrightarrow \forall V \in \mathcal{V} : x \in A + V.$$

(ii) Because of the continuity of the addition, and item *(i)*, for any $V \in \mathcal{V}$ there exists $U \in \mathcal{V}$ with $\bar{U} \subseteq U + U \subseteq V$.

(iii) For every convex 0-neighbourhood U there exists an absolutely convex and absorbing open 0-neighbourhood V with $V \subseteq U$ [3, Lemma 2.1.8]. Since X is locally convex, we may choose a basis of the neighbourhood filter of 0 consisting of absolutely convex and absorbing open subsets. As the closure of an absolutely convex and absorbing set is still absolutely convex and absorbing, item *(ii)* implies the existence of a 0-neighbourhood consisting of barrels.

□

10.2 Application of K-Analyticity

Again, we will build a family of subsets as required in Theorem 5.5. We will use that all notions of compactness coincide in metric spaces, as do the concepts of separability and the property of being Lindelöf.

Definition 10.2.1. A topological vector space is called *Fréchet space* if it is locally convex and completely metrisable. A topological space is called *Montel space* if it is barrelled and has the *Heine-Borel property*, that is every closed and bounded subset is compact. A topological vector space is called *Fréchet-Montel space* if it is both Fréchet and Montel.

Theorem 10.2.2. *[4, Theorem 5] Every Fréchet-Montel space X is separable.*

Proof. Let d be a compatible metric, and $U_1(0) := \{x \in X : d(x, 0) < 1\}$. As the set $\{\frac{1}{n}U_1(0) : n \in \mathbb{N}\}$ forms a countable basis of the neighbourhood filter of 0, X is first-countable. Therefore, we may fix a bounded barrel V so that $V_n := \frac{1}{n}V$ for $n \in \mathbb{N}$ defines a family that is a basis of the neighbourhood filter of 0 consisting of bounded barrels. Given $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}$, we set

$$A_\alpha := \bigcap_{n\in\mathbb{N}} a_n V_n.$$

Again, we will show that the family $(A_\alpha)_{\alpha\in\mathcal{N}}$, fulfils the requirements of Theorem 5.5.

(A): Being an intersection of closed and bounded sets, A_α is closed and bounded for every $\alpha \in \mathcal{N}$. Since a Montel space has the Heine-Borel property, A_α is therefore compact.

(B): Since V_n is absolutely convex for every $n \in \mathbb{N}$, we have that $a_n V_n \subseteq b_n V_n$ if $a_n \leq b_n$. It follows that $A_\alpha \subseteq A_\beta$ for every $\alpha \leq \beta$.

(C): Take $x \in X$. As V_n is absorbing, we find $\lambda_n \in \mathbb{N}$ so that $x \in \lambda_n V_n$ for every $n \in \mathbb{N}$. Consequently, $x \in A_\alpha$ for $\alpha := (\lambda_n)_{n \in \mathbb{N}}$, so $X = \bigcup_{\alpha \in \mathcal{N}} A_\alpha$.

In metrisable spaces, the concepts of countable compactness and compactness coincide. Theorem 5.5 gives us that X is K-analytic. By Corollary 4.7, X is Lindelöf, which is equivalent to it being separable after Proposition 2.1.2. $\square$

An example for a Fréchet-Montel space is the space $\mathcal{H}(\Omega)$ of holomorphic functions for any open set $\Omega \subseteq \mathbb{C}$ endowed with the compact-open topology. A subbasis of the compact-open topology is given by sets of the form

$$N(K, \epsilon, x_0) := \{f \in \mathcal{H}(\Omega) : f(K) \subseteq U_\epsilon(x_0)\}$$

for $K \subseteq \Omega$ compact, $\epsilon > 0$, and $x_0 \in \Omega$.

11 Inductive Limits of Separable Fréchet-Spaces

This last section deals with an intersection of concepts in topology, functional analysis, set theory, and algebra. Note that by Theorem 10.2.2, the conclusion of this section's central result, Theorem 11.3.1, holds in particular for strict inductive limits of Fréchet-Montel spaces.

11.1 Inductive Limits

We start with a short foray into category theory. For more detailed information we refer to [8, Chapter 2.2], and [12].

Definition 11.1.1. An *inductive system* $(A_i, e_{i,j})_{i,j \in I}$ is a system consisting of a directed set $(I, \preccurlyeq)$, a family of objects $(A_i)_{i \in I}$ in a certain class, and morphisms $e_{i,j} : A_i \to A_j$ for $i \preccurlyeq j$ that satisfy

(i) $e_{j,k} \circ e_{i,j} = e_{i,k}$ for $i \preccurlyeq j \preccurlyeq k$, and

(ii) $e_{i,i} = \mathrm{id}_{A_i}$.

A *cone* $(B, f_i)_{i \in I}$ is an object B together with a family $(f_i)_{i \in I}$ of morphisms $f_i : A_i \to B$, which satisfy $f_i = f_j \circ e_{i,j}$ for every $i \preccurlyeq j$.

The *inductive limit* of the inductive system is a cone $(\varinjlim A_i, e_i)_{i \in I}$ that has the following universal property: If $(B, f_i)_{i \in I}$ is any cone, then there exists a unique morphism $f : \varinjlim A_i \to B$ so that the diagram below commutes for all indices.

$$
\begin{tikzcd}
A_i \arrow[rr, "e_{i,j}"] \arrow[dr, "e_i"] \arrow[ddr, bend right, "f_i"'] & & A_j \arrow[dl, "e_j"'] \arrow[ddl, bend left, "f_j"] \\
& \varinjlim A_i \arrow[d, dashed, "f"] & \\
& B &
\end{tikzcd}
$$

An inductive system is called *reduced*, if all $e_i : A_i \to \varinjlim A_i$ are injective.

Clearly, if an inductive limit exists, it is unique. In the following, we will consider the category of all locally convex topological spaces with continuous linear mappings serving as morphisms.

Lemma 11.1.2. *[16, Proposition 7.9] Let E be a linear space and $(E_i)_{i\in I}$ be a family of linear subspaces of E with $E = \bigcup_{i\in I} E_i$. For every $i \in I$ there is a topology $\mathcal{T}_i$ given so that $(E_i, \mathcal{T}_i)$ is a locally convex topological space. Suppose that $E_i \subseteq E_j$ for $i \preccurlyeq j$, and that the inclusion mappings $\iota_{i,j} : (E_i, \mathcal{T}_i) \to (E_j \mathcal{T}_j)$ are continuous. Then the inductive limit $\varinjlim E_i$ can be identified with $(E, \mathcal{T})$, where $\mathcal{T}$ is the finest topology on E that makes E a locally convex topological vector space, and that makes all inclusion mappings $\iota_i : E_i \to E$ continuous. The inductive system is reduced.*

Proof. Trivially, all inclusion mappings are injective, thus the inductive system is reduced.

The topology $\mathcal{T}$ always exists, see for example [16, Proposition 6.8]. In order for all inclusion mappings $\iota_i : E_i \to E$ to be continuous, the topology $\mathcal{T}_{\text{ind}}$ on the inductive limit has to be coarser than $\mathcal{T}$. In order for the identity mapping $\text{id} : (E, \mathcal{T}_{\text{ind}}) \to (E, \mathcal{T})$ to be continuous, it must also be finer than $\mathcal{T}$, hence $\mathcal{T}_{\text{ind}} = \mathcal{T}$.

Let $(F, f_i)_{i\in I}$ be a cone. As all f_i are compatible with the inclusion, $f : E \to F$ defined by $f|_{E_i} := f_i$ is the well-defined unique morphism making the diagram commute. $\square$

Note that $\mathcal{T}$ is in general not the final topology with respect to all inclusion mappings $\iota_i : E_i \to E$, as the final topology with respect to a family of linear mappings from topological vector spaces into a linear space does not need to make it a topological vector space. According to [1, page 43], this is however the case if I is countable.

From now on, we will restrict ourselves to countable inductive limits of locally convex spaces. We will furthermore assume that $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of locally convex topological spaces so that each E_n is a subspace of E_{n+1}, and that all inclusion mappings $\iota_{n,m} = E_n \to E_m$ for $n \leq m$ are continuous. An inductive limit of a sequence satisfying these conditions is said to be *strict*.

Lemma 11.1.3. *[15, Theorem 12.1.1] Let $E = \bigcup_{n\in\mathbb{N}} E_n$ be the strict inductive limit of an increasing sequence of locally convex topological spaces $(E_n)_{n\in\mathbb{N}}$. Then the system $\mathcal{U}$ consisting of all sets of the form*

$$U = \operatorname{cobal}(\bigcup_{n\in\mathbb{N}} U_n), \tag{5}$$

where U_n is a neighbourhood of 0 in E_n for every $n \in N$, is a neighbourhood base of 0 in E.

Proof. Any subset of E with the final topology is open if and only if its intersection with E_n is open in E_n for every $n \in \mathbb{N}$. Take $U \in \mathcal{U}$. As $U \cap E_n$ contains U_n, U is absolutely convex neighbourhood of 0 in E. Conversely, for any absolutely convex 0-neighbourhood V in E we find a 0-neighbourhood U_n in E_n with $U_n \subseteq V \cap E_n$. Therefore, $\bigcup_{n\in\mathbb{N}} U_n \subseteq V$. Since V was absolutely convex, we obtain $\operatorname{cobal}(\bigcup_{n\in\mathbb{N}} U_n) \subseteq V$. Thus, $\mathcal{U}$ forms a neighbourhood base of 0. $\square$

Lemma 11.1.4. *[16, Lemma 7.11] Let E be a locally convex topological vector space, M a subspace of E, and U an absolutely convex 0-neighbourhood in M. Then there exists an absolutely convex 0-neighbourhood V in E so that $U = V \cap M$.*

Proof. As M is a subspace of E, we may choose an absolutely convex 0-neighbourhood W in E with $W \cap M \subseteq U$. Set $V := \operatorname{cobal}(W \cup U)$. Clearly, we have

$$V \cap M \supseteq (W \cap M) \cup (U \cap M) \supseteq U.$$

Conversely, take $x \in V \cap M$. As W and U are absolutely convex, there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$, and $w \in W$, $u \in U$ so that $z = \alpha w + \beta u$. Then $\alpha w = z - \beta u \in M$. Therefore, either $\alpha = 0$ or $w \in M$. Both imply $z \in U$. Thus, $U = V \cap M$. $\square$

Proposition 11.1.5. *[16, Theorem 7.12] Let $(E, \mathcal{T})$ be the strict inductive limit of an increasing sequence of locally convex topological spaces $(E_n, \mathcal{T}_n)$. Then every E_n is a topological subspace of E, i.e. $\mathcal{T}$ induces the original topology $\mathcal{T}_n$ on every E_n.*

Proof. As the inclusion $\iota_n : (E_n, \mathcal{T}_n) \to (E, \mathcal{T})$ is continuous, $\mathcal{T}_n$ is finer than $\mathcal{T}|_{E_n}$. Let $U_n \subseteq E_n$ be an absolutely convex 0-neighbourhood with respect to $\mathcal{T}_n$. We show that any absolutely convex 0-neighbourhood U_n in E_n may be written in the form $U \cap E_n$ for some $U \in \mathcal{U}$ as defined in (5). As E_n is a subspace of E_{n+1}, Lemma 11.1.4 gives us an absolutely convex 0-neighbourhood U_{n+1} in E_{n+1} so that $U_n = U_{n+1} \cap E_n$. Inductively, for every $k \in \mathbb{N}$ we choose absolutely convex 0-neighbourhoods U_{n+k} in E_{n+k} with

$$U_{n+k} = U_{n+k+1} \cap E_{n+k}.$$

Then $U := \bigcup_{k\geq 0} U_{n+k}$ is absolutely convex. For every $k \geq 0$ we have that

$$U \cap E_{n+k} = U_{n+k}.$$

For $m < n$ it holds that

$$U \cap E_m = U \cap E_n \cap E_m = U_n \cap E_m,$$

which is a 0-neighbourhood in E_m. It follows $U \in \mathcal{U}$. As $\mathcal{U}$ is a neighbourhood base, $\mathcal{T}_n$ is also coarser than $\mathcal{T}|_{E_n}$. □

Corollary 11.1.6. *If E is the strict inductive limit of a sequence of separable locally convex topological spaces, then E is separable.*

Proof. For every $n \in \mathbb{N}$ fix a countable subset $X_n \subseteq E_n$ with $\overline{X_n}^{\mathcal{T}_n} = E_n$. Since E_n is a subspace of E, we have $\overline{X_n}^{\mathcal{T}_n} = E_n \cap \overline{X_n}^{\mathcal{T}}$. The set $X := \bigcup_{n\in\mathbb{N}} X_n$ is a countable subset of E satisfying

$$\overline{X}^{\mathcal{T}} = \overline{\bigcup_{n\in\mathbb{N}} X_n}^{\mathcal{T}} \supseteq \bigcup_{n\in\mathbb{N}} \overline{X_n}^{\mathcal{T}} \supseteq \bigcup_{n\in\mathbb{N}} \overline{X_n}^{\mathcal{T}_n} = \bigcup_{n\in\mathbb{N}} E_n = E.$$

□

11.2 Topology of Compact Convergence

The weak*-topology is generally not the only topology that makes the dual space of a locally convex space locally convex. To this effect, we will briefly introduce a different topology.

Definition 11.2.1. A *dual pair* (X, Y) consists of a linear vector space X and a point-separating linear subspace Y of the algebraic dual X^*. For $x \in X$ and $y \in Y$ we write

$$(x, y) := y(x).$$

Definition 11.2.2. Let $(X, \mathcal{T})$ be a topological space. A subset $M \subseteq X$ is called *bounded* if for every $V \in \mathcal{U}(0)$ there exists $\lambda_V > 0$ so that $M \subseteq \lambda_V V$.

Let (X, Y) be a dual pair and $\mathcal{M} \subseteq \mathcal{P}(X)$ be a family of $\sigma(X, Y)$-bounded subsets of X. Then for every $M \in \mathcal{M}$ the mapping

$$p_M : \begin{cases} Y & \to [0, \infty) \\ y & \mapsto \sup\limits_{y \in M} |(x, y)| \end{cases}$$

defines a seminorm on Y. If additionally

$$\overline{\operatorname{span}(\bigcup_{M\in\mathcal{M}} M)}^{\sigma(X,Y)} = X,$$

the family $(p_M)_{M\in\mathcal{M}}$ induces a locally convex topology $\mathcal{T}$ on Y [3, Satz 5.1.4]. It holds that

$$\operatorname{span}(\bigcup_{M\in\mathcal{M}} M) \subseteq (X, \mathcal{T})'.$$

The induced topology $\mathcal{T}$ is called *topology of uniform convergence on sets of* $\mathcal{M}$.

If $(E, \mathcal{T})$ is a locally convex topological vector space, the compact subsets of E form a class of $\sigma(X,Y)$-bounded sets, which cover E. Therefore, the topology τ_c of *uniform convergence on the compact subsets of* E is a locally convex topology on the dual space E'.

Definition 11.2.3. Let (X, Y) be a dual pair and $M \subseteq X$. Its *polar* M° is defined by

$$M^\circ := \{y \in Y : |(x,y)| \leq 1,\ x \in M\}.$$

Clearly, $M_1 \subseteq M_2 \subseteq X$ implies $M_2^\circ \subseteq M_2^\circ \subseteq Y$.

The following theorem is a generalisation of the Theorem of Banach-Alaoglu 9.1.7.

Theorem 11.2.4. *Let $(E, \mathcal{T})$ be a locally convex topological vector space, and U be a neighbourhood of* 0 *with respect to $\mathcal{T}$. Then U° is compact with respect to τ_c.*

For the proof we refer to [10, §21.6(3)].

11.3 Application of K-Analyticity

With the background knowledge from above, we may now proceed to show a last application of K-analytic spaces.

Theorem 11.3.1. *Let $E = \varinjlim E_n$ be the strict inducive limit of a sequence $(E_n)_{n\in\mathbb{N}}$ of separable Fréchet spaces. Then (E', τ_c) is analytic.*

Proof. For every $n \in \mathbb{N}$ fix a countable basis $(U_k^n)_{k\in\mathbb{N}}$ of neighbourhoods of 0 consisting of absolutely convex closed sets with $U_{k+1}^n \subseteq U_k^n$ for all $k \in \mathbb{N}$. By Lemma 11.1.3, and Lemma 10.1.2 *(ii)*, the set $\mathcal{U} := \{U_\alpha \mid \alpha = (a_k)_{k\in\mathbb{N}} \in \mathcal{N}\}$ with

$$U_\alpha := \overline{\operatorname{cobal} \bigcup_{k\in\mathbb{N}} U_{a_k}^k}^{\mathcal{T}}$$

is a 0-basis in E. Clearly, $U_\beta \subseteq U_\alpha$ for $\alpha \leq \beta$. By Theorem 11.2.4, the polars U_α° are compact with respect to τ_c. We set $A_\alpha := U_\alpha^\circ$. Then $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$, and $E' = \bigcup_{\alpha\in\mathcal{N}} A_\alpha$ as for every $f \in E'$ we have $f(0) = 0$. By continuity, there exists $\alpha \in \mathcal{N}$ so that $|f(x)| \leq 1$ for all $x \in U_\alpha$. Therefore, the family $(A_\alpha)_{\alpha\in\mathcal{N}}$ satisfies conditions (A), (B), and (C) from (2).

As every E_n is separable, E is itself separable after Corollary 11.1.6. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of E, and consider the linear mapping

$$\iota : \begin{cases} E' & \to \mathbb{C}^{\mathbb{N}} \\ f & \mapsto (f(x_n))_{n\in\mathbb{N}}. \end{cases}$$

We endow $\mathbb{C}^{\mathbb{N}}$ with the product topology $\mathcal{T}_{\Pi}$, and E' with the initial topology $\mathcal{T}_{\iota}$ with respect to ι. Since $\{x_n : n \in \mathbb{N}\}$ is dense in E, $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$ implies $f = g$. Thus, ι is injective. $(E', \mathcal{T}_{\iota})$ is therefore a locally convex topological vector space (see [3, Bemerkung 5.0.3 *(ii)*]). Countable products of metrisable spaces are metrisable. Since $\iota : (E', \mathcal{T}_{\iota}) \to (\mathbb{C}^{\mathbb{N}}, \mathcal{T}_{\Pi})$ is an embedding, we receive that $\mathcal{T}_{\iota}$ is metrisable ([20, Lemma 3.1.7]). Note that $\mathcal{T}_{\iota} \subseteq \sigma(E', E)$, as $\iota : (E', \sigma(E', E)) \to (\mathbb{C}^{\mathbb{N}}, \mathcal{T}_{\Pi})$ is continuous. Furthermore, $\sigma(E', E) \subseteq \tau_c$, as the weak*-topology is determined by the evaluation-map seminorms $\{f \mapsto |f(x)| : x \in E\}$, and singleton-sets are compact.

As the identity mapping $\mathrm{id} : (E', \tau_c) \to (E', \mathcal{T}_{\iota})$ is an injective continuous map into an angelic space, (E', τ_c) is angelic, and thus K-analytic by Theorem 5.5. Furthermore, $(E, \mathcal{T}_{\iota})$ is analytic by Theorem 7.2.7. It follows that (E', τ_c) is analytic[1]. $\square$

A classic example of a strict inductive limit is the space $C_c^{\infty}(\mathbb{R}^n)$ of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. A defining sequence is obtained by $K_n := \{x \in \mathbb{R}^n : \|x\| \leq n\}$, as $C_c^{\infty}(K_n)$ is a Fréchet space for every $n \in \mathbb{N}$.

[1] This fact is stated, but not proved in [4, Theorem 6].

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