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Approximation of Approximation Numbers of Linear Bounded Operators

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Contents

1 Introduction

In this paper we deal with the approximation of approximation numbers of bounded linear operators on a infinite dimensional separable Hilbert space. The kth approximation number $s_k(T)$ of an operator T describes the distance from T to the subset of all bounded linear operators with an at most $k-1$-dimensional image. A central fact for this paper is a theorem, that proves the convergence of a sequence $s_k(T_n)$, $n \in \mathbb{N}$ to the kth approximation number $s_k(T)$. The proof of this statement is rather easy and the natural question occurs, if there may be a weakening of the requirements without loosing the results. In this paper we discuss the following two generalized settings:

- First we consider the following situation: Let $T : X \to Y$ be a linear and bounded Operator, where X is a separable and Y is a reflexive vector space.
- Second we assume that the codomain Y is not reflexive but a dual space of a separable normed linear space, $Y = X'$.

2 Approximation numbers on separable Hilbert spaces

In this chapter we discuss the approximation of the approximation numbers on separable Hilbert spaces. Let $\mathcal{B}_L(X, Y)$ denote the set of all bounded linear operators $T : X \to Y$. Our sources for this chapter are Chapter 1 in [1], Lemma 4.12 in [2] and Chapter 1 in [3].

2.1 The approximation of approximation numbers on separable Hilbert spaces

First we introduce the term of an approximation number:

Definition 2.1 *Let X, Y be normed linear spaces. For $T \in \mathcal{B}_L(X, Y)$ and $k \in \mathbb{N}$ the **kth approximation number** $s_k(T)$ is defined by*

$$s_k(T) := \inf\{||T - A|| : A \in \mathcal{B}_L(X, Y), \operatorname{rank} A \leqslant k - 1\}. \tag{2.1}$$

Hence $s_1(T) = ||T||$.

As $\{A \in \mathcal{B}_L : \operatorname{rank} A \leqslant k - 1\} \subseteq \{A \in \mathcal{B}_L : \operatorname{rank} A \leqslant k\}$ for $k \in \mathbb{N}$ it follows that $s_1(T) \geqslant s_2(T) \geqslant ... \geqslant 0$. Let us now formulate the first version of the Theorem of approximation numbers.

Theorem 2.2 *Let H_1, H_2 be separable Hilbert spaces and $P_n : H_1 \to H_1$, $Q_n : H_2 \to H_2$ be linear projections with $||P_n|| = ||Q_n|| = 1$ and $\forall x \in H_1 : P_n x \to x$, $\forall y \in H_2 : Q_n y \to y$. Let $T \in \mathcal{B}_L(H_1, H_2)$ and $T_n := Q_n T P_n$ for all $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} s_k(T_n) = s_k(T). \tag{2.2}$$

Before we prove this theorem let us formulate a lemma, that will help us with that:

Lemma 2.3 *Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(H_1, H_2)$ be a sequence of uniformly bounded Operators. For all $n \in \mathbb{N}$ let $\operatorname{rank} A_n \leqslant k$ for a $k \in \mathbb{N}$. Then there exists an operator $A \in \mathcal{B}_L(H_1, H_2)$ with $\operatorname{rank} A \leqslant k$ and for all $x \in H_1$, $y \in H_2$ the sequence $(y, A_n x)_{n\in\mathbb{N}}$ has a convergent subsequence such that $\lim_{l\to\infty}(y, A_{n_l} x) = (y, Ax)$.*

Proof:
As $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded, there exists a $M > 0$ with $||A_n|| \leqslant M$. As for all $n \in \mathbb{N}$ rank $A_n \leqslant k$ there exist orthonormal sets $\mathcal{E}_n = \{e_1^{(n)}, ..., e_k^{(n)}\} \subseteq H_1$, $\mathcal{F}_n = \{f_1^{(n)}, ..., f_k^{(n)}\} \subseteq H_2$ and numbers $\psi_1^{(n)}, ..., \psi_k^{(n)}$ such that we can express the operator A_n for all $x \in H_1$ as

$$A_n x = \sum_{i=1}^{k} \psi_i^{(n)}(x, e_i^{(n)}) f_i^{(n)}.$$

It follows for $n \in \mathbb{N}$, that

$$M \geqslant ||A_n|| = \sup_{||x|| \leqslant 1} ||A_n x|| \geqslant ||A_n e_i^{(n)}|| =$$

$$\sqrt{\left(\sum_{j=1}^{k} \psi_j^{(n)}(e_i^{(n)}, e_j^{(n)}) f_j^{(n)}, \sum_{j=1}^{k} \psi_j^{(n)}(e_i^{(n)}, e_j^{(n)}) f_j^{(n)}\right)} = \sqrt{\left(\psi_i^{(n)} f_i^{(n)}, \psi_i^{(n)} f_i^{(n)}\right)} = |\psi_i^{(n)}|. \tag{2.3}$$

Let $\mathcal{K}_1^{H_1} := \{x \in H_1 : ||x|| \leqslant 1\}$, $\mathcal{K}_1^{H_2} := \{x \in H_2 : ||x|| \leqslant 1\}$ and $\mathcal{K}_M^{\mathbb{C}} := \{\lambda \in \mathbb{C} : |\lambda| \leqslant M\}$. We now consider the elements $x \in H_1$ as elements of a dual space $x \in G' = H_1$ according to the mapping $\phi : H_1 \to G', x \mapsto \hat{x}$ with $\hat{x}(y) := (x, y)$. As a subset of the normed space G', $\mathcal{K}_1^{H_1}$ is according to the Theorem of Banach-Alaoglu (see Theorem 5.5.6 in [6]) compact in respect to the weak-$*$-Topology. The same way follows, that $\mathcal{K}_1^{H_2}$ is compact with respect to the weak-$*$-topology. As $\mathcal{K}_M^{\mathbb{C}}$ is compact in $\mathbb{C}$ according to the Theorem of Tychonoff (see Theorem 1.3.1 in [6]) the set

$$D := \prod_{i=1}^{k} \mathcal{K}_1^{\mathbb{C}} \times \prod_{j=1}^{k} \mathcal{K}_M^{H_1} \times \prod_{l=1}^{k} \mathcal{K}_M^{H_2}$$

is compact. Therefore the sequence $(\psi_1^{(n)}, ..., \psi_k^{(n)}, e_1^{(n)}, ..., e_k^{(n)}, f_1^{(n)}, ..., f_k^{(n)})_{n\in\mathbb{N}}$ has a convergent subsequence, that converges to $(\psi_1, ..., \psi_k, e_1, ..., e_k, f_1, ..., f_k) \in D$. We now define

$$A : H_1 \to H_2, A(x) := \sum_{i=1}^{k} \psi_i(x, e_i) f_i.$$

It is clear, that A is bounded. As $(e_i^{(n_l)})_{n\in\mathbb{N}}$ and $(f_i^{(n_l)})_{n\in\mathbb{N}}$ are convergent with respect to the weak-$*$-topology, it follows that for $x \in H_1$, $y \in H_2$

$$(y, e_i^{(n)}) \to (y, e_i) \text{ and } (x, f_i^{(n)}) \to (x, f_i).$$

Since $\psi_i^{(n_l)} \to \psi_i$ it follows that

$$(y, A_{n_l}x) = \big(y, \textstyle\sum_{i=1}^{k} \psi_i^{(n_l)}(x, e_i^{(n_l)}) f_i^{(n_l)}\big) \to \big(y, \textstyle\sum_{i=1}^{k} \psi_i(x, e_i) f_i\big) = (y, Ax).$$

■

Now we can prove Theorem 2.2 with help of Lemma 2.3

Proof 2.4 *of Theorem 2.2:*

First we will show, that $\limsup\limits_{n\to\infty} s_k(T_n) \leqslant s_k(T)$.
As $s_k(T) := \inf\{||T - A|| : A \in \mathcal{B}_L(H_1, H_2), \operatorname{rank} A \leqslant k-1\}$ there exists for every $\epsilon > 0$ an $A_\epsilon \in \mathcal{B}_L(H_1, H_2)$ with $\operatorname{rank} A_\epsilon \leqslant k-1$ such that $||T - A_\epsilon|| < s_k(T) + \epsilon$. Since for all $n \in \mathbb{N}$ $\operatorname{rank} Q_n A_\epsilon P_n \leqslant k-1$ and $||Q_n|| = ||P_n|| = 1$ it follows that

$$s_k(T_n) \leqslant ||T_n - Q_n A_\epsilon P_n|| = ||Q_n T P_n - Q_n A_\epsilon P_n|| \leqslant ||Q_n|| \cdot ||T - A_\epsilon|| \cdot ||P_n|| = ||T - A_\epsilon|| < s_k(T) + \epsilon.$$

As $\epsilon > 0$ was arbitrary it follows that

$$\limsup_{n\to\infty} s_k(T_n) \leqslant s_k(T). \tag{2.4}$$

Now we show $\liminf\limits_{n\to\infty} s_k(T_n) \geqslant s_k(T)$.
Let us assume, that $\liminf_{n\to\infty} s_k(T_n) < s_k(T)$. As $s_k(T_n) \geqslant 0$ we can assume, that $s_k(T) > 0$. Hence there has to exist a $\delta \in (0, s_k(T))$ such that for every $n_0 \in \mathbb{N}$ there exists a $n \in \mathbb{N}$ with $n \geqslant n_0$, such that

$$s_k(T_n) \leqslant s_k(T) - \delta.$$

According to the definition of $s_k(T_n)$ for every $\epsilon \in (0, s_k(T) - \delta)$ we can find a $A_\epsilon \in \mathcal{B}_L(H_1, H_2)$ with $\operatorname{rank} A_\epsilon \leqslant k-1$ such that

$$||T_n - A_\epsilon|| = ||P_n T P_n - A_\epsilon|| < s_k(T) - \delta + \epsilon.$$

Now according to Lemma 2.3 there exists an Operator $A \in \mathcal{B}_L(H)$ with $\operatorname{rank} A \leqslant k-1$ such that A is a limit of a subsequence of $(A_\epsilon)_{\epsilon\in(0,s_k(T)-\delta)}$ that enjoys the following property

$$\forall x \in H_1 \forall y \in H_2 \lim_{\epsilon_j \to 0} (y, A_{\epsilon_j} x) = (y, Ax). \tag{2.5}$$

Let $x \in H_1$, $y \in H_2$ such that $||x|| = ||y|| = 1$. Then follows that

$$|(y, Q_n T P_n x) - (y, A_\epsilon x)| \leqslant ||Q_n T P_n - A_\epsilon|| < s_k(T) - \delta + \epsilon.$$

As the inner product of H_2 is continuous, this implies, that

$$|(y,Tx)-(y,Ax)| = \lim_{\epsilon_j\to 0} |(y,T_nx)-(y,A_{\epsilon_j}x)| \leqslant s_k(T)-\delta+\epsilon.$$

Hence $||T-A|| \leqslant s_k(T)-\delta+\epsilon$ and therefore $s_k(T) \leqslant s_k(T)-\delta+\epsilon < s_k(T)$. This is a contradiction, so the assumption $\liminf_{n\to\infty} s_k(T_n) < s_k(T)$ was wrong. Hence

$$\liminf_{n\to\infty} s_k(T_n) \geqslant s_k(T). \tag{2.6}$$

Together with equation (2.4) it follows that

$$\forall k \in \mathbb{N} : \lim_{n\to\infty} s_k(T_n) = s_k(T).$$

■

3 Approximation under reflexivity assumption

As we now got a first answer of how we can approximate the approximation numbers of a linear bounded operator, we now want to evolve these results to a more general setting. Instead of Hilbert spaces we want to consider normed spaces X, Y and a linear bounded mapping $T : X \to Y$. In addition we want to have weaker requirements to the sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(X,Y)$ that converges to T for approximating $s_k(T)$. As we will see, we need to assume one more condition to the codomain of the Operator T. In this chapter we will assume, that X is a separable and Y is a reflexive normed space. First we will introduce terms of topology, in order to formulate the final theorem of this chapter. Second we will prove some lemmas, that will help us for our further conclusions and finally we will prove the main theorem of this chapter.

3.1 The weak and the weak-* topology

We now will introduce the terms of the weak and the weak-$*$-topology. This section follows the thoughts of Chapter 5.3 in [6].

Definition 3.1 *Let X be a vector space and Y be a point separating linear subspace of the algebraic Dual space X^*. The* ***weak topology*** *with respect to Y on X is defined as the initial topology with respect to all functions $y \in Y$. This means, that the weak topology on X is the coarsest topology such that all $y \in Y$ are continuous. The weak topology on X denotes the weak topology on X with respect to the topological dual space X'.*

Definition 3.2 *Let X and Y are normed spaces. We consider the set*

$$Z := \operatorname{span}\{T \mapsto \phi(T(x)) : x \in X, \phi \in Y'\}.$$

It is clear, that Z is a point separating subspace of the algebraic Dual space of $\mathcal{B}_L(X,Y)$. The weak topology on $\mathcal{B}_L(X,Y)$ with respect to Z is called the ***weak operator topology*** *and is denoted by $\mathcal{T}_\omega$.*

Remark 3.3 *Hence for a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}_L(X,Y)$ holds that*

$$(T_n)_{n\in\mathbb{N}} \to T \ in \ \mathcal{T}_\omega \Leftrightarrow \forall x \in X \forall \phi \in Y : \phi(T_n x) \to \phi(Tx).$$

Let X be a vector space and Y be a subspace of X^*. Let us consider the mapping $\iota : X \to Y^*$ with $\iota(x) : Y \to \mathbb{C}, \iota(x)(y) := y(x)$. Then $\iota(X)$ is a point separating subspace of Y^*. Therefore the following definition is well defined:

Definition 3.4 *Let X be a vector space and ι be defined as in Remark 3.3. The weak topology on X' with respect to $\iota(X)$ is called the **weak-*-topology** on X' and is denoted by $\mathcal{T}_{\omega^*}$.*

Remark 3.5 *Let X and Y be normed linear spaces. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence with $T_n \in \mathcal{B}_L(X, Y')$. The sequence $(T_n)_{n\in\mathbb{N}}$ converges to T with respect to the weak-$*$ operator topology if*

$$\forall x \in X \forall y \in Y : T_n x(y) \to Tx(y) \ as \ n \to \infty.$$

It follows that strong operator convergence implies weak operator convergence and weak operator convergence implies weak-$*$ operator convergence.

3.2 The approximation under reflexivity assumption

Now we will formulate the theorem of approximation under the reflexivity assumption. Before we prove this theorem, we will discuss two lemmas, that will help us with that. This section follows Chapter 2 of [3].

Theorem 3.6 *Let X be a separable normed linear space and Y be a reflexive Banach space. In addition let $T \in \mathcal{B}_L(X, Y)$ and $(P_n)_{n\in\mathbb{N}} \subset \mathcal{B}_L(X)$, $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{B}_L(Y)$ be sequences of operators with $\forall n \in \mathbb{N}\ ||P_n|| \leqslant 1, ||Q_n|| \leqslant 1$. We define $T_n := Q_n T P_n$. If $T_n \to T$ in $\mathcal{T}_\omega$ as $n \to \infty$, then holds that*

$$\forall k \in \mathbb{N} \lim_{n\to\infty} s_k(T_n) = s_k(T).$$

The first of the two lemmas is the following:

Lemma 3.7 *Let X, Y be normed linear spaces and $T \in \mathcal{B}_L(X, Y)$ with $\operatorname{rank} T = k$ for a $k \in \mathbb{N}$. Then holds that there exists a basis $\{b_1, ..., b_k\}$ of $T(X) \subseteq Y$ and a set $\{\phi_1, ..., \phi_k\} \subset X'$ with $\forall j \in \{1, ..., k\} : ||b_j|| = 1$ and $||\phi_j|| \leqslant ||T||$, such that*

$$\forall x \in X : Tx = \sum_{j=1}^{k} \phi_j(x) b_j.$$

Proof:
As $T(X)$ is the image of a linear space under a linear mapping with rank $= k$ we can find a basis $\{a_1, ..., a_k\}$ of $T(X)$ with $\forall j : ||a_j|| = 1$. Let $\mathcal{K}_1^{T(X)} := \{y \in T(X) : ||y|| \leqslant 1\}$. As $\mathcal{K}_1^{T(X)}$ is closed, it follows that

$$K := \prod_{i=1}^{k} \mathcal{K}_1^{T(X)}$$

is closed. Let us consider the mapping $\det : K \to \mathbb{C}$. According to linear algebra we know, that this mapping is continuous and

$$\det(x_1, ..., x_k) > 0 \Leftrightarrow (x_1, ..., x_k) \text{ is linear independent.}$$

Since K is closed and det is continuous there exists a maximum in K. Let this maximum be denoted by $(b_1, ..., b_k)$. As $(b_1, ..., b_k) \in K$ it holds that $\forall j \in \{1, ..., k\} : ||b_j|| \leqslant 1$. Being the maximum on K in follows that $\det(b_1, ..., b_k) \geqslant 1$, because $\det(a_1, ..., a_k) = 1$. In particular follows, that all $b_j \neq 0$. It holds that

$$\forall j \in \{1, ..., k\} : \det(b_1, ..., b_k) \geqslant \det(b_1, ..., \tfrac{b_j}{||b_j||}, ..., b_k) = \tfrac{\det(b_1, ..., b_k)}{||b_j||}$$

and therefore $\forall j \in \{1, ..., k\} : ||b_j|| \geqslant 1$. Together follows, that

$$\forall j \in \{1, ..., k\} : ||b_j|| = 1. \tag{3.1}$$

Let for all i $U_i := \text{span}(\{b_1, ..., b_k\} \backslash \{b_i\})$. Then $b_i \notin U_i$ and we see, that $dist(b_i, U_i) \leqslant 1$ and for every $u \in U_i$, that $||b_i - u|| > 0$. Considering

$$1 = \det(b_1, ..., b_k) \geqslant \det(b_1, ..., b_{i-1}, \tfrac{b_i - u}{||b_i - u||}, b_{i+1}, ..., b_k) = \tfrac{\det(b_1, ..., b_k)}{||b_i - u||}$$

we see, that $||b_i - u|| \geqslant 1$ for all $u \in U_i$. In conclusion follows, that $dist(b_i, U_i) = 1$. Let us for $j \in \{1, ..., k\}$ define $f_j : \text{span}\{b_j\} \to \mathbb{C}, f_j(\gamma b_j) := \gamma$. Then follows, that $||f_j|| = 1$. According to the Theorem of Hahn-Banach (see Theorem 5.2.3 in [6]) there exist $\forall j \in \{1, ..., k\}$ an extension $F_j : X \to \mathbb{C}, F_j|_{\text{span}\{b_j\}} = f_j$ with $F_j \in X'$, $||F_j|| = 1$ and $F_j|_{U_j} = 0$. Since F_j is an extension of f_j it follows that

$$\forall i, j \in \{1, ..., k\} : F_j(b_i) = \delta_{ij}.$$

As $(b_1, ..., b_k)$ is a basis of $T(X)$ we can describe every $x \in T(X)$ as $x = \sum_{i=1}^{k} \gamma_i b_i$. In conclusion it holds, that

$$\forall y \in T(X) : y = \sum_{j=1}^{k} F_j(y) b_j.$$

Let us now for all $j \in \{1, ..., k\}$ define

$$\phi_j : X \to \mathbb{C}, \phi_j(x) := F_j(Tx). \tag{3.2}$$

Then $\phi_j \in X'$ and it follows, that $||\phi_j|| = ||F_j \circ T|| \leqslant ||F_j|| \cdot ||T|| = ||T||$ and

$$Tx = \sum_{j=1}^{k} \phi_j(x) b_j. \tag{3.3}$$

■

Let us now formulate a second lemma, that corresponds to Lemma 2.3 before we will start with the proof of Theorem 3.6 (see Lemma 2.4 in [3]).

Lemma 3.8 *Let X be a separable normed linear space, Y be a reflexive Banach space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}_L(X,Y)$ a sequence of uniformly bounded operators with $\forall n \in \mathbb{N} : \operatorname{rank}(A_n) \leqslant k$. Then there exists an operator $A \in \mathcal{B}_L(X,Y)$ with $\operatorname{rank} A \leqslant k$ and a subsequence $(A_{n_j})_{j\in\mathbb{N}}$ such that*

$$A_{n_j} \to A \text{ in } \mathcal{T}_\omega$$

Proof:

Since $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded, there exists a $M > 0$ with $\forall n \in \mathbb{N} : ||A_n|| \leqslant M$. As for all $n \in \mathbb{N} : A_n \in \mathcal{B}_L(X,Y)$ and $\operatorname{rank} A_n \leqslant k$ by a consequence of Lemma 2.3 there exist $b_1^{(n)}, ..., b_k^{(n)} \in A_n(X)$ and $\phi_1^{(n)}, ..., \phi_k^{(n)} \in X'$ with

$$\forall n \in \mathbb{N} \forall i \in \{1, ..., k\} : ||b_i^{(n)}|| = 1 \text{ and } ||\phi_i^{(n)}|| \leqslant ||A_n|| \leqslant M \text{ such that}$$
$$\forall x \in X : A_n x = \sum_{i=1}^{k} \phi_i^{(n)}(x) b_i^{(n)}.$$

According to the Theorem Banach-Alaoglu (see Theorem 5.5.6 in [6]) for every i there exists a subsequence $(\phi_i^{(n_j)})_{n_j\in\mathbb{N}}$, that converges in $\mathcal{T}_{\omega*}$ to $\phi_i \in X'$. Since Y is a reflexive Banach space by consequence of the Eberlein-Smulian Theorem for every i there exists a subsequence $(b_i^{(n_j)})_{n_j\in\mathbb{N}}$, that converges in $\mathcal{T}_\omega$ to $b_i \in Y$ (see Theorem 8.25 in [5]). Let us define

$$A : X \to Y, \forall x \in X : Ax = \sum_{i=1}^{k} \phi_i(x) b_i. \tag{3.4}$$

It is clear, that $\operatorname{rank} A \leqslant k$ and $A \in \mathcal{B}_L(X,Y)$. Hence $\forall f \in Y'$ holds that

$$\lim_{j\to\infty} f(A_{n_j}x) = \lim_{j\to\infty} \sum_{i=1}^{k} \phi_i^{(n_j)}(x) f(b_i^{(n_j)}) = \sum_{i=1}^{k} \phi_i(x) f(b_i) = f(Ax).$$

■

Let us now prove Theorem 3.6 with help of Lemma 3.8. This proof is inspired of theorem 2.8 in [3]. The main idea is similar to the proof of Theorem 2.2, that corresponds with Theorem 3.6.

Proof *of Theorem 3.6:*
As in proof 2.4 we will first show, that $\limsup_{n\to\infty} s_k(T_n) \leqslant s_k(T)$ and then with a contradiction, that $\liminf_{n\to\infty} s_k(T_n) \geqslant s_k(T)$.
Similar to proof 2.4 for all $k \in \mathbb{N}$ and every $\epsilon > 0$ there exists an $A_\epsilon \in \mathcal{B}_L(X, Y)$ with $\operatorname{rank} A_\epsilon \leqslant k-1$ and $||T - A_\epsilon|| < s_k(T) + \epsilon$. Then for all $n \in \mathbb{N}$ holds that

$$s_k(T_n) \leqslant ||Q_n T P_n - Q_n A_\epsilon P_n|| \leqslant ||Q_n|| \cdot ||T - A_\epsilon|| \cdot ||P_n|| \leqslant ||T - A_\epsilon|| < s_k(T) + \epsilon.$$

Since ϵ was arbitrary, we get

$$\limsup_{n\to\infty} s_k(T_n) \leqslant s_k(T). \tag{3.5}$$

Now we proof the other inequality. The conclusion holds if $s_k(T) = 0$. Again as in proof 2.4 we now assume, that $s_k(T) > 0$ and $liminf_{n\to\infty} s_k(T_n) < s_k(T)$. Therefore it follows, that

$$\exists \epsilon \in (0, s_k(T)) \forall n \in \mathbb{N} \exists n_l \geqslant n : s_k(T_{n_l}) < s_k(T) - \epsilon.$$

Hence for every $l \in \mathbb{N}$ there exists an $A_{n_l} \in \mathcal{B}_L(X, Y)$ with $\operatorname{rank} A_{n_l} \leqslant k-1$ such that

$$||T_{n_l} - A_{n_l}|| < s_k(T) - \epsilon. \tag{3.6}$$

Therefore holds that

$$||A_{n_l}|| \leqslant ||A_{n_l} - T_{n_l}|| + ||T_{n_l}|| < s_k(T) + ||T||.$$

Thus the sequence $(A_{n_l})_{l\in\mathbb{N}}$ is uniformly bounded and the assumptions of Lemma 3.8 are fulfilled. Therefore there exists a subsequence $(A_j)_{j\in\mathbb{N}} \subseteq (A_{n_l})_{l\in\mathbb{N}}$ and an operator $A : X \to Y$ such that $\operatorname{rank} A \leqslant k-1$ and $A_j \to A$ in $\mathcal{T}_\omega$ as $j \to \infty$. Let $x \in X$ and $f \in Y'$ be arbitrary with $||x|| \leqslant 1$ and $||f|| \leqslant 1$. We want to get the following term small:

$$\forall j \in \mathbb{N} : |f(Tx) - f(Ax)| \leqslant |f(Tx) - f(T_j x)| + |f(T_j x) - f(A_j x)| + |f(A_j x) - f(Ax)|. \tag{3.7}$$

First we see, that

$$|f(T_j x) - f(A_j x)| \leqslant ||f|| \cdot ||T_j - A_j|| < s_k(T) - \epsilon.$$

As $(A_j)_{j\in\mathbb{N}} \to A$ it follows that there exists a $j_0 \in \mathbb{N}$ such that for all $j > j_0$

$$|f(A_jx) - f(Ax)| < \tfrac{\epsilon}{3}.$$

Since $(T_n)_{n\in\mathbb{N}} \to T$ in $\mathcal{T}_\omega$, there exists a $j_1 \in \mathbb{N}$ such that for all $j > j_1$

$$|f(Tx) - f(T_jx)| < \tfrac{\epsilon}{3}.$$

In conclusion follows, as $x \in X$ with $||x|| \leqslant 1$ and $f \in Y'$ with $||y|| \leqslant 1$ were arbitrary

$$s_k(T) \leqslant ||T - A|| \leqslant s_k(T) - \tfrac{\epsilon}{3}.$$

This is a contradiction, hence

$$\liminf_{n\to\infty} s_k(T_n) \geqslant s_k(T). \tag{3.8}$$

Together with (3.5) it follows, that

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

■

Remark 3.9 *According to Eberlein's Theorem (see Theorem 16.5 in [4]) and proposition 2.7 in [3] if Y is a non-reflexive space, then there has not to be a subsequence of $(T_n)_{n\in\mathbb{N}}$, that is convergent in the weak operator topology. Therefore the demand, that Y is a reflexive space, is necessary.*

3.3 Additional thoughts

In this section we want to discuss whether the approximation number of an operator $s_k(T)$ is a minimum or not. At the very end of this chapter we add one last version of the approximation under reflexivity assumption. This section follows 2.5 and 2.9 in [3].

Corollary 3.10 *Let X be a separable normed linear space ,Y be a reflexive Banach space and $T \in \mathcal{B}_L(X,Y)$, $k \in \mathbb{N}$. Then there exists an operator $A \in \mathcal{B}_L(X,Y)$,* $\operatorname{rank} A \leqslant k-1$ *with*

$$s_k(T) = ||T - A||$$

Proof:
According to the definition of $s_k(T)$ for every $n \in \mathbb{N}$ there exists an $A_n \in \mathcal{B}_L(X,Y)$ with $\operatorname{rank} A_n \leqslant k-1$ and

$$||T - A_n|| \leqslant s_k(T) + \frac{1}{n} \tag{3.9}$$

and therefore $||A_n|| \leqslant ||T|| + s_k(T) + \frac{1}{n} < ||T|| + s_k(T) + 1$. As $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded by consequence of Lemma 3.8 there exists an $A \in \mathcal{B}_L(X,Y)$, $\operatorname{rank} A \leqslant k-1$ and a subsequence $(A_{n_l})_{l\in\mathbb{N}}$ with

$$(A_{n_l})_{l\in\mathbb{N}} \to A \text{ in } \mathcal{T}_\omega.$$

Let $\epsilon > 0$, $x \in X$ with $||x|| \leqslant 1$ and $f \in Y'$ with $||f|| \leqslant 1$ be arbitrary. Let $n_\epsilon \in \mathbb{N}$ be such that for all $n_l \geqslant n_\epsilon$ holds

$$\tfrac{1}{n_l} < \tfrac{\epsilon}{2} \text{ and } |f(Ax) - f(A_{n_l}x)| < \tfrac{\epsilon}{2}.$$

Together it follows that

$$|f(Tx) - f(Ax)| \leqslant |f(Tx) - f(A_{n_l}x)| + |f(A_{n_l}x) - f(Ax)| \leqslant$$
$$||f|| \cdot ||x|| \cdot ||T - A_{n_l}|| + \tfrac{\epsilon}{2} < s_k(T) + \tfrac{1}{n_\epsilon} + \tfrac{\epsilon}{2} < s_k(T) + \epsilon.$$

As x, f and ϵ were arbitrary, it follows

$$||T - A|| \leqslant s_k(T). \tag{3.10}$$

By consequence of the definition of $s_k(T)$ follows, that $||T - A|| \geqslant s_k(T)$ and together with (3.10) we get

$$||T - A|| = s_k(T).$$

■

Last we add a corollary, that gives one more version of the approximation of the approximation numbers under reflexivity assumption.

Corollary 3.11 *Let X be a separable normed linear space ,Y be a reflexive Banach space and $T \in \mathcal{B}_L(X,Y)$, $k \in \mathbb{N}$. Let $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(X)$ and $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(Y)$ such that $||P_n|| \leqslant 1$ with $\forall x \in X : P_n x \to x$ and $||Q_n|| \leqslant 1$ with $\forall y \in Y : Q_n y \to y$ in $\mathcal{T}_\omega$. Then holds that*

$$\forall k \in \mathbb{N} : s_k(T_n) \to s_k(T) \textit{ for } n \to \infty.$$

Proof:
We have to show, that $T_n := Q_n T P_n \to T$ in $\mathcal{T}_\omega$. Therefore let $x \in X$ and $f \in Y'$ be arbitrary. As $\forall x \in X : P_n x \to x$ and $\forall y \in Y : Q_n y \to y$ in $\mathcal{T}_\omega$ follows

$$|f(T_n x) - f(Tx)| \leqslant |f(Q_n T P_n x) - f(Q_n T x)| + |f(Q_n T x) - f(Tx)| \leqslant$$
$$||f|| \cdot ||Q_n|| \cdot ||T|| \cdot ||P_n x - x|| + |f(Q_n T x) - f(Tx)| \to 0 \text{ as } n \to \infty.$$

Hence $T_n \to T$ in $\mathcal{T}_\omega$ and therefore the conclusion follows from Theorem 3.6.
■

4 Approximation under duality assumption

In the last chapter we discussed the approximation of approximation numbers of a linear bounded operator $T : X \to Y$ on normed spaces under the assumption, that Y is a reflexive space. We now want to show a second possibility to proof the approximation on normed spaces under slightly different conditions. The main difference will be, that we assume, that the codomain Y of T is not necessarily reflexive but the dual space of a separable normed linear space. In addition we will slightly change the assumptions on the sequence of operators $(T_n)_{n\in\mathbb{N}}$. In corollary 3.11, our final version of the approximation in the last chapter, we asked for the two sequences $(P_n)_{n\in\mathbb{N}}$, $(Q_n)_{n\in\mathbb{N}}$ for $(P_n)_{n\in\mathbb{N}}$ point wise convergence and for $(Q_n)_{n\in\mathbb{N}}$ convergence in the weak operator topology. Now we even want to weaken that, namely that $T_n := Q_nTP_n$ is just convergent in the weak-* operator topology. In the first section of this chapter we will first prove a lemma, similar to Lemma 3.8, that will help us to prove the main theorem of this chapter. That we will do right after discussing this lemma. In the second section of this chapter we will discuss, if the approximation number of an operator $s_k(T)$ is a minimum or not under the duality assumption. Finally, we will give one very last version of the approximation and sum up all three versions we proofed.

4.1 The approximation under duality assumption

Let us now formulate a lemma, that corresponds to Lemma 3.8. This section follows Lemma 3.1 and Theorem 3.3 in [3].

Lemma 4.1 *Let X, Z be separable normed linear spaces and $Y := Z'$, $k \in \mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(X,Y)$ be a sequence of uniformly bounded operators such that $\forall n \in \mathbb{N} : \operatorname{rank} A_n \leqslant k$. Then there exists a subsequence $(A_{n_l})_{l\in\mathbb{N}}$ and an operator $A \in \mathcal{B}_L(X,Y)$ with $\operatorname{rank} A \leqslant k$ such that*

$$A_{n_l} \to A \text{ in } \mathcal{T}_{\omega*} \text{ as } n \to \infty.$$

Proof:
By consequence of Lemma 3.7 for every $n \in \mathbb{N}$ there exist a basis $(b_1^{(n)}, ..., b_k^{(n)})$

of $A(X) \subseteq Y$ with $\forall j \in \{1, ..., k\} : ||b_j|| = 1$ and $(\phi_1^{(n)}, ..., \phi_k^{(n)}) \subseteq X'$ with $\forall j \in \{1, ..., k\} : ||\phi_j^{(n)}|| \leqslant ||T||$ such that

$$\forall x \in X : A_n x = \sum_{j=1}^{k} \phi_j^{(n)}(x) b_j^{(n)}.$$

Since X' and Y are dual spaces of separable normed spaces, according to the Theorems of Banach-Alaoglu and Tychonoff (see Theorem 5.5.6 and Theorem 1.3.1 in [6]), the sequences $(b_1^{(n)}, ..., b_k^{(n)})_{n\in\mathbb{N}}$ and $(\phi_1^{(n)}, ..., \phi_k^{(n)})_{n\in\mathbb{N}}$ have weak-* convergent subsequences. Hence there exist $(b_1, ..., b_k)$ and $(\phi_1, ..., \phi_k)$ such that

$$\forall x \in X : (\phi_1^{(n_l)}(x), ..., \phi_k^{(n_l)}(x)) \to (\phi_1(x), ..., \phi_k(x)) \text{ and}$$
$$\forall z \in Z : (b_1^{(n_l)}(z), ..., b_k^{(n_l)}(z)) \to (b_1(z), ..., b_k(z)) \text{ as } n \to \infty.$$

let us now define, as in Lemma 3.8

$$\forall x \in X : Ax := \sum_{j=1}^{k} \phi_j(x) b_j.$$

It is clear, that $A \in \mathcal{B}_L(X, Y)$ with $\operatorname{rank} A \leqslant k$. Hence for every $x \in X$ and $z \in Z$ follows, that

$$\lim_{l\to\infty} A_{n_l} x(z) = \lim_{l\to\infty} \sum_{j=1}^{k} \phi_j^{(n_l)}(x) b_j^{(n_l)}(z) = \sum_{j=1}^{k} \phi_j(x) b_j(z) = Ax(z).$$

■

Let us now formulate the main theorem of this chapter. It proves the approximation of the approximation numbers under the duality assumption.

Theorem 4.2 *Let X, Z be separable normed linear spaces and $Y := Z'$, $k \in \mathbb{N}$. Let $T \in \mathcal{B}_L(X, Y)$ and $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(X)$, $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(Y)$ be sequences of operators with $\forall n \in \mathbb{N} : ||P_n|| \leqslant 1$, $||Q_n|| \leqslant 1$ such that $T_n := Q_n T P_n \to T$ in $\mathcal{T}_{\omega*}$ as $n \to \infty$. Then holds*

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

Proof:
As in the proof of Theorem 3.3 we again show first $\limsup_{n\to\infty} s_k(T_n) \leqslant s_k(T)$ and then $\liminf_{n\to\infty} s_k(T_n) \geqslant s_k(T)$. The first inequality can be absolutely similarly shown as in Theorem 3.3. So we have

$$\limsup_{n\to\infty} s_k(T_n) \leqslant s_k(T). \tag{4.1}$$

Again we want to prove $\liminf_{n\to\infty} s_k(T_n) \geqslant s_k(T)$ by contradiction. So we assume, that $\liminf_{n\to\infty} s_k(T_n) < s_k(T)$. In consequence there exist an $\epsilon > 0$ and infinitely many $n_l \in \mathbb{N}$ with $s_k(T_{n_l}) < s_k(T) - \epsilon$. Therefore we find a sequence of operators $(A_{n_l})_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(X,Y)$ with $\forall l \in \mathbb{N}$: $\operatorname{rank} A_{n_l} \leqslant k-1$ such that $||A_{n_l} - T_{n_l}|| < s_k(T) - \epsilon$. It follows, that

$$\forall l \in \mathbb{N} : ||A_{n_l}|| \leqslant ||A_{n_l} - T_{n_l}|| + ||T_{n_l}|| < s_k(T) - \epsilon + ||T||.$$

As $(A_{n_l})_{n\in\mathbb{N}}$ is uniformly bounded by consequence of Lemma 4.1 there exist a subsequence $(A_j)_{j\in\mathbb{N}} \subseteq (A_{n_l})_{l\in\mathbb{N}}$ and an operator $A \in \mathcal{B}_L(X,Y)$ such that

$$(A_j)_{j\in\mathbb{N}} \to A \text{ in } \mathcal{T}_{\omega *}.$$

Let $x \in X$ and $z \in Z$ with $||x|| \leqslant 1$ and $||z|| \leqslant 1$. We want to get the following term small:

$$|Tx(z) - Ax(z)| \leqslant |Tx(z) - T_jx(z)| + |T_jx(z) - A_jx(z)| + |A_jx(z) - Ax(z)|. \quad (4.2)$$

Now first we see, that

$$\forall j \in \mathbb{N} : |T_jx(z) - A_jx(z)| \leqslant ||T_j - A_j|| < s_k(T) - \epsilon.$$

Second, by weak-*-convergence of $(T_j)_{j\in\mathbb{N}}$ and $(A_j)_{j\in\mathbb{N}}$ there exists a j_ϵ such that

$$|Tx(z) - T_{j_\epsilon}x(z)| < \tfrac{\epsilon}{3} \text{ and}$$
$$|Ax(z) - A_{j_\epsilon}x(z)| < \tfrac{\epsilon}{3}.$$

Hence $|Tx(z) - Ax(z)| < s_k(T) - \frac{\epsilon}{3}$. As $x \in X$ and $z \in Z$ with $||x|| \leqslant 1$ and $||z|| \leqslant 1$ were arbitrary it follows that

$$s_k(T) \leqslant ||T - A|| \leqslant s_k(T) - \tfrac{\epsilon}{3}.$$

We have reached a contradiction. Hence

$$\liminf_{n\to\infty} s_k(T_n) \geqslant s_k(T). \quad (4.3)$$

Together with (4.1) we have

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

■

4.2 Additional remarks

Similar to section 3.3 we now want to show, that under the duality assumption the approximation numbers are minima, so for every $k \in \mathbb{N}$ there exists an operator $A : X \to Y$ such that $s_k(T) = ||T - A||$. Last we give one final version of the approximation of the approximation numbers under the duality assumption. This section follows Corollary 3.2 and Corollary 3.4 in [3].

Corollary 4.3 *Let X, Z be separable normed linear spaces and $Y := Z'$, $k \in \mathbb{N}$. Let $T \in \mathcal{B}_L(X, Y)$. Then there exists an operator $A \in \mathcal{B}_L(X, Y)$ with* $\operatorname{rank} A \leqslant k-1$ *such that*

$$s_k(T) = ||T - A||.$$

Proof:
Similar to the proof of Corollary 3.10, according to the definition of $s_k(T)$ for every $n \in \mathbb{N}$ there exists an $A_n \in \mathcal{B}_L(X, Y)$ with $\operatorname{rank} A_n \leqslant k - 1$ and

$$||T - A_n|| \leqslant s_k(T) + \frac{1}{n} \tag{4.4}$$

and therefore $||A_n|| \leqslant ||T|| + s_k(T) + \frac{1}{n} < ||T|| + s_k(T) + 1$. As $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded by consequence of Lemma 4.1 there exists an $A \in \mathcal{B}_L(X, Y)$, $\operatorname{rank} A \leqslant k - 1$ and a subsequence $(A_{n_l})_{n\in\mathbb{N}}$ with

$$(A_{n_l})_{n\in\mathbb{N}} \to A \text{ in } \mathcal{T}_{\omega^*}.$$

Let $\epsilon > 0$, $x \in X$, $z \in Z$ with $||x|| \leqslant 1$ and $||z|| \leqslant 1$. Let $n_\epsilon \in \mathbb{N}$ be such that for all $n_l \geqslant n_\epsilon$ holds that

$$\tfrac{1}{n_l} < \tfrac{\epsilon}{2} \text{ and } |Ax(z) - A_{n_l}x(z)| < \tfrac{\epsilon}{2}.$$

Together follows that

$$|Tx(z) - Ax(z)| \leqslant |Tx(z) - A_{n_l}x(z)| + |A_{n_l}x(z) - Ax(z)| \leqslant$$
$$||x|| \cdot ||z|| \cdot ||T - A_{n_l}|| + \tfrac{\epsilon}{2} < s_k(T) + \tfrac{1}{n_\epsilon} + \tfrac{\epsilon}{2} < s_k(T) + \epsilon.$$

As x, z and ϵ were arbitrary, it follows

$$||T - A|| \leqslant s_k(T). \tag{4.5}$$

By consequence of the definition of $s_k(T)$ follows, that $||T - A|| \geqslant s_k(T)$ and together with (4.5) we get

$$||T - A|| = s_k(T).$$

■

Last we name a corollary, that gives one very last version of the approximation of the approximation numbers under duality assumption.

Corollary 4.4 *Let X, Z be separable normed linear spaces and $Y := Z'$, $k \in \mathbb{N}$. Let $T \in \mathcal{B}_L(X,Y)$ and $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(X)$, $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(Y)$ be sequences of operators with $\forall n \in \mathbb{N} : ||P_n|| \leqslant 1$ such that $\forall x \in X : P_n x \to x$ and $||Q_n|| \leqslant 1$ with $\forall y \in Y : Q_n y \to y$ in $\mathcal{T}_{\omega*}$ as $n \to \infty$. Then holds that*

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

Proof:
We want to use Theorem 4.2. Therefore we have to show, that $T_n := Q_n T P_n \to T$ in $\mathcal{T}_{\omega*}$ as $n \to \infty$. As $\forall x \in X : P_n x \to x$ and $\forall y \in Y : Q_n y \to y$ in $\mathcal{T}_{\omega*}$ it follows

$$|T_n x(z) - Tx(z)| \leqslant |Q_n T P_n x(z) - Q_n T x(z)| + |Q_n T x(z) - Tx(z)| \leqslant$$
$$||Q_n|| \cdot ||T|| \cdot ||P_n x - x|| \cdot ||z|| + |Q_n T x(z) - Tx(z)| \to 0 \text{ as } n \to \infty.$$

Hence $T_n \to T$ in $\mathcal{T}_{\omega*}$ and therefore the conclusion follows from Theorem 4.2. ■

4.3 Conclusion

Let us now formulate a theorem that sums up all three versions of the approximation of the approximation numbers.

Theorem 4.5 (Approximation of approximation numbers) *Let X and Y be normed spaces and $T \in \mathcal{B}_L(X,Y)$. Furthermore let $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(X)$, $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_L(Y)$ be sequences of Operators with $||P_n|| \leqslant 1$, $||Q_n|| \leqslant 1$ for all $n \in \mathbb{N}$. Let $\forall n \in \mathbb{N} T_n := Q_n T P_n$. If one of the following statements*

- *X and Y are Hilbertspaces. P_n and Q_n are linear projections with $P_n \to id_X$ and $Q_n \to id_Y$.*
- *X is a separable, Y a reflexive Banachspace and $T_n \to T$ in $\mathcal{T}_\omega$.*
- *X and Z are separable normed spaces, $Y := Z'$ and $T_n \to T$ in $\mathcal{T}_{\omega*}$.*

is true, then holds that

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

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