BACHELORARBEIT

C(X) as dual space of a Banach space

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Contents

Introduction

The famous *Riesz-Markov representation* theorem gives us a special characterization of the dual space of $C_0(X)$.

Definition 0.0.1. Let X be a non-empty locally compact Hausdorff space. $C_0(X)$ denotes the subset of all functions $f \in C(X)$, for which the set $\{x \in X \mid |f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$. If we endow this space with the supremum norm

$$\|f\|_X = \sup_{x \in X} |f(x)|,$$

it is a Banach space.

Definition 0.0.2. Let X be a non-empty, locally compact space. Then we denote by $M(X)$ the space of complex-valued, regular Borel measures on X and we set

$$\|\mu\| = |\mu|(X).$$

With respect to this norm, called the *total variational norm*, it is a Banach space.

Theorem 0.0.3 (Riesz-Markov)**.** *Let X be a locally compact Hausdorff space. Then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, as*

$$\Phi(\mu)f = \int_X f \; d\mu,$$

for every $f \in C_0(X)$. More precisely, Φ is an isometric isomorphism from $C_0(X)'$ to $M(X)$.

A proof of this theorem can be found in, e.g., [5, Theorem 6.19, p.130].

In this bachelor thesis we deal with the following question:

> When is $C_0(X)$ (isometrically) isomorphic to a dual space and if a predual exists, how does it look like?

The thesis is mainly based on [2].

Existence of a predual of a Banach space is not always guaranteed.

Example 0.0.4. Let Z be a Banach space with $Ext(B_1^Z(0)) = \emptyset$, where $Ext(B_1^Z(0))$ denotes the set of extreme points in $B_1^Z(0) = \{z \in Z \mid \|z\| \leq 1\}$, then there is no Banach space Y with $Y' \cong Z$.
To show this, assume that Z is isometrical isomorphic to the dual of a space Y. If we endow Z with the weak*-topology $(Z, \sigma(Z, Y))$, then, by the *Banach-Alaoglu* theorem the unit ball is weak*-compact. So $B_1^Z(0)$ is a non-empty, compact and convex subset of a locally convex space. By *Krein-Milman*, $Ext(B_1^Z(0)) \neq \emptyset$, a contradiction. //

Proposition 0.0.5. *For a non-empty, locally compact space X, $f \in Ext(B_1^{C_0(X)}(0))$ if and only if $|f(x)| = 1$ for $x \in X$.*

Proof. Take $f \in B_1^{C_0(X)}(0)$ and suppose that there exists $x_0 \in X$ such that $|f(x_0)| < 1$. Set $\epsilon = \frac{1-|f(x_0)|}{2}$. Then there exists a neighbourhood U of x_0 with $|f(x_0)| < 1 - \epsilon$, for $x \in U$. Take $g \in C_{\mathbb{R}}(X)$ such that $0 \leq g \leq \mathbb{1}_U$ and $g(x_0) = 1$. Then $f \pm \epsilon g \in B_1^{C_0(X)}(0)$ and

$$f = \frac{1}{2}(f + \epsilon g) + \frac{1}{2}(f - \epsilon g),$$

and so $f \notin Ext(B_1^{C_0(X)}(0))$. On the other hand, if we have $|f(x)| = 1$ for all $x \in X$ and $1 \neq |g|, |h|$ with $g, h \in B_1^{C_0(X)}(0)$, then there is x_0 with $|h(x_0)| < 1$. We get

$$1 = |f(x_0)| = |(1-t)g(x_0) + th(x_0)| \leq (1-t)|g(x_0)| + t|h(x_0)| < 1 - t + t = 1,$$

a contradiction. ■

Corollary 0.0.6. *Let X be a non-empty, locally compact space, that is not compact. Then $Ext(B_1^{C_0(X)}(0)) = \emptyset$. Hence, $C_0(X)$ is not isometrically isomorphic to a Banach space.*

Proof. By *Proposition* 0.0.5, it is $|f(x)| = 1$ for all $x \in X$, for $f \in Ext(B_1^{C_0(X)}(0))$. Since X is not compact, $f \notin C_0(X)$ and with *Example* 0.0.4, $C_0(X)$ cannot be isometrically isomorphic to a dual space. ■

In view of *Corollary* 0.0.6 we may restrict our attention to compact spaces X. Moreover, we will always assume X to be Hausdorff.

Let us note that any predual of a space $C(X)$ is isometrically isomorphic to a closed subspace of $M(X)$. This is the consequence of the following theorems that are part of almost every basic functional analysis course. These proofs can be found in [7, Lemma 5.5.2, p.86; Theorem 5.3.3, p.79].

Theorem 0.0.7. *Let Z be a vector space and let Y be a seperating linear subspace of the algebraic dual Z^*. Then $(Z, \sigma(Z,Y))' = Y$.*

Theorem 0.0.8. *Let $(X, \|.\|)$ be a normed space and let ι be the map*

$$\iota : \begin{cases} X \to (X')^* \\ x \mapsto (f \mapsto f(x)). \end{cases}$$

Then ι maps into the topological bidual space $(X', \|.\|_{X'})'$, is linear, and is isometric if we endow X'' with the operator norm $\|.\|_{X''}$.

By means of *Theorem* 0.0.7 and *Theorem* 0.0.8, we can indeed identify a predual Y of $C(X)$ with a subspace of $M(X)$:

$$Y \cong \iota(Y) \subseteq Y'' \cong C(X)' \cong M(X) \tag{0.1}$$

$$(C(X), \sigma(C(X), \iota(Y)))' = \iota(Y) \subseteq M(X). \tag{0.2}$$

In the end we will even get some sort of uniqueness of this predual space. We have to distinguish between types of preduals.

Definition 0.0.9. Let Z be a Banach space. Y is an *isomorphic predual* of Z if Z is isomorphic to Y' (linear homeomorphic) and a Banach space Y is an *isometric predual* of Z if Z is isometrically isomorphic to Y', we will write $Y' \cong Z$.

There are examples of spaces with isomorphic dual spaces, that are not isometrically isomorphic. We will need the following proposition.

Proposition 0.0.10. *Let Z and E be Banach spaces and let T be an isometric isomorphism. Then $T(Ext(B_1^Z(0))) = Ext(B_1^E(0))$.*

Proof. T is a bijective linear map. Now for $z \in Ext(B_1^Z(0))$ the following holds:

$$T(z) = tT(a) + (1-t)T(b) = T(ta + (1-t)b) \Rightarrow z = ta + (1-t)b \Rightarrow z = a = b.$$

Hence z is an extreme point whenever $T(z)$ is and vice versa. ■

Example 0.0.11. Let c be the set of convergent sequences in $\mathbb{R}$ and c_0 the subspace consisting of the sequences with limit 0. We know that $c_0' \cong \ell^1 \cong c'$. It is easy to see that $B_1^c(0)$ has extreme points (e.g. the sequence $(1, 1, 1, \cdots)$), but the unit ball of $B_1^{c_0}(0)$ has no extreme points. Let $x = (x_n)_{n \in \mathbb{N}} \in B_1^{c_0}(0)$. Since x converges to 0 there is an index $N > 0$ for which $|x_N| < \frac{1}{2}$. Now define $y_\pm \in B_1^{c_0}(0)$ as

$$y_{n_\pm} = \begin{cases} x_n & n \neq N \\ x_N \pm \frac{1}{4} & n = N. \end{cases}$$

So we can write $x = \frac{1}{2}(y_+ + y_-)$. So by *Proposition* 0.0.10 there can't be an isometric isomorphism between c_0 and c.
To see that these spaces are isomorphic, set

$$T(x) = (2x_\infty, x_1 - x_\infty, x_2 - x_\infty, \cdots)$$

for $x = (x_n)_{n \in \mathbb{N}} \in c$ with $\lim\limits_{n \to \infty} x_n = x_\infty$. Then $T : c \to c_0$ is a linear map. Further, we know that

$$T(x) = (0, 0, 0, \cdots) \Rightarrow x = 0$$

since $\lim_{n \to \infty} x_n = 0$ and for every sequence $y \in c_0$ we take

$$x = \left(y_2 + \frac{y_1}{2}, y_3 + \frac{y_1}{2}, \cdots \right) \to \frac{y_1}{2} \quad \text{and} \quad T(x) = y.$$

Obviously, $\|T\| = 2$. And as one can see

$$\frac{2}{3} \|x\| \leq \|T(x)\| .$$

It follows that $\|T^{-1}\| \leq \frac{3}{2}$, and so c is isomorphic to c_0. //

1 Stonean spaces and normal measures

1.1 Normal measures

As we have to deal with a subspace of $M(X)$, we should take a closer a look at it.

Definition 1.1.1. Let $(X, \mathcal{T})$ be a topological space. Then the *Borel sets* in X are the members of the σ-algebra $\sigma(\mathcal{T})$ generated by the family $\mathcal{T}$ of open subsets of X; we set $\mathfrak{B}_X = \sigma(\mathcal{T})$.

Identifying $M(X)$ as the dual space of $C(X)$, we define

$$\langle f, \mu \rangle = \int_X f \; d\mu \quad f \in C(X), \; \mu \in M(X).$$

For real-valued measures $\mu, \nu \in M_{\mathbb{R}}(X)$, we define

$$(\mu \vee \nu)(B) = \sup_{\substack{A \in \mathfrak{B}_X \\ A \subseteq B}} \mu(A) + \nu(B \setminus A)$$
$$(\mu \wedge \nu)(B) = \inf_{\substack{A \in \mathfrak{B}_X \\ A \subseteq B}} \mu(A) + \nu(B \setminus A).$$

and further $\mu^+ = \mu \vee 0$ and $\mu^- = \mu \wedge 0$. It is obvious that $|\mu| = \mu^+ + \mu^-$. The set of positive measures in $M(X)$ is denoted by $M(X)^+$.

In the following $C(X)^+ \subseteq C(X)$ denotes the space of real-valued, continuous and positive functions with pointwise order. Since the norm on $C(X)$ is compatible with the lattice structure, the following definition is appropriate.

Definition 1.1.2. Let $(Z, \|.\|)$ be a Banach space and $(Z, \leq)$ an ordered linear space. The norm is a *lattice norm* if $\|y\| \leq \|z\|$ whenever $|y| \leq |z|$, with $|z| = \sup\{z, -z\}$ in the lattice. The space Z is then called a *Banach lattice*.

To find a more concrete characterization of the space $\iota(Y)$ in *Equation* (0.1), we define the space of normal measures:

Definition 1.1.3. Let X be a non-empty, compact space, and let $\mu \in M(X)$. Then μ is *normal* if $\langle f_i, \mu \rangle \to 0$ for each net $(f_i)_{i \in I}$ in $C(X)^+$ with $f_i \searrow 0$. We write $f_i \searrow 0$ if $(f_i)_{i \in I}$ is decreasing and $\inf_{i \in I} f_i = 0$ in the lattice. We denote the subspace of normal measures in $M(X)$ by $N(X)$.

Remark 1.1.4. We want $N(X)$ to be a Banach space, so we have to check if it is a closed linear space with respect to the total variation norm. It is obviously a linear space as we have

$$\langle f_i, \mu + \nu \rangle = \int_X f_i \; d(\mu + \nu) = \int_X f_i \; d\mu + \int_X f_i \; d\nu = \langle f_i, \mu \rangle + \langle f_i, \nu \rangle \to 0$$

and similar with scalar multiplication. To see that $N(X)$ is closed, we again take a net $(f_i)_{i \in I}$ with $f_i \searrow 0$, $\epsilon > 0$ and a sequence $(\mu_n)_{n \in \mathbb{N}}$ with $\mu_n \to \mu$. Then

$$|\langle f_i, \mu \rangle| = |\langle f_i, \mu - \mu_n \rangle + \langle f_i, \mu_n \rangle| \leq |\langle f_i, \mu - \mu_n \rangle| + |\langle f_i, \mu_n \rangle| .$$

Choose i_1 and take n_0 with $\|\mu - \mu_{n_0}\| \leq \frac{\epsilon}{2\|f_{i_1}\|_X}$. For this n_0 we get i_0 with $|\langle f_i, \mu_{n_0} \rangle| \leq \frac{\epsilon}{2}$ for $i \geq i_0$ and because f_i is decreasing, $\|f_i\|_X \leq \|f_j\|_X$ for $j \leq i$. This leads to

$$|\langle f_i, \mu - \mu_{n_0} \rangle| \leq \|f_i\|_X \, \|\mu - \mu_{n_0}\| \leq \frac{\epsilon}{2} \quad , i \geq i_1$$

and to sum up $|\langle f_i, \mu \rangle| \leq \epsilon$, for $i \geq i_1, i_0$. Since ϵ was arbitrary, we get $\langle f_i, \mu \rangle \to 0$. //

Subsequently we will need some basic properties of normal measures:

Theorem 1.1.5. *Let X be a non-empty, compact space. Then:*

(i) $\mu \in M(X)$ is normal if and only if $\Re(\mu)$ and $\Im(\mu)$ are normal;

(ii) $\mu \in M_{\mathbb{R}}(X)$ is normal if and only if $|\mu|$ is normal if and only if μ^+ and μ^- are normal;

(iii) $\mu \in M(X)$ is normal if and only if $|\mu|$ is normal

To proof this theorem we need a corollary of *Urysohn's lemma* [5, Theorem 2.12, p.39]:

Corollary 1.1.6. *Let X be a non-empty, compact space. Suppose that C is compact and U is open in X such that $C \subseteq U$. Then there exists $f \in C(X)^+$ with $\mathbb{1}_C \leq f \leq \mathbb{1}_U$.*

Proof. Since X is compact and Hausdorff, X is a normal space and hence, we can apply Urysohn's lemma to the closed subsets C and U^{c}. It gives us a function

$$f : X \to [0,1] \text{ with } f(C) \subseteq \{1\} \text{ and } f(U^{\mathsf{c}}) \subseteq \{0\}. \tag{1.1}$$

■

Proof of Theorem 1.1.5.
(i) This is trivial.
(ii) Suppose that $\mu^+, \mu^- \in N(X)$. Then certainly $\mu, |\mu| \in N(X)$. Suppose that $|\mu| \in N(X)$ and that $\nu \in N(X)$ with $|\nu| \leq |\mu|$. Then

$$0 \leq \left| \int_X f_i \; d\nu \right| \leq \int_X f_i \; d\,|\mu| \to 0$$

when $f_i \searrow 0 \in C(X)^+$, and so $\nu \in N(X)$. In particular, μ, μ^+ and μ^- are normal whenever $|\mu|$ is normal.
Suppose that $\mu \in M_{\mathbb{R}}(X)$ is normal and that $f_i \searrow 0$ in $B_1^{C(X)^+}(0)$. Let $\{P, N\}$ be a *Hahn decomposition* of X with respect to μ, and take $\epsilon > 0$. Since μ is regular, there exist a compact set C and an open set U in X with $C \subseteq P \subseteq U$ and $|\mu|(U \setminus C) < \epsilon$. Now there exists $g \in C(X)^+$ with $\mathbb{1}_C \leq g \leq \mathbb{1}_U$. Then

$$\int_X f_i \; d\mu^+ = \int_P f_i \; d\mu \leq \int_C g f_i \; d\mu + \int_{U \setminus C} g f_i \; d\mu + 2\epsilon = \int_X g f_i \; d\mu + 2\epsilon.$$

Since $g f_i \searrow 0$ and μ is normal, $\lim_{i \in I} \int_X g f_i \; d\mu = 0$, and so

$$\limsup_{i \in I} \int_X f_i \; d\mu^+ \leq 2\epsilon.$$

This holds true for each $\epsilon > 0$, and so

$$\lim_{i \in I} \int_X f_i \; d\mu^+ = 0.$$

Thus, μ^+ is normal.
(iii) Suppose that $\mu \in N(X)$. Then $|\Re(\mu)| + |\Im(\mu)| \in N(X)$ from (i) and (ii). However, $|\mu| \leq |\Re(\mu)| + |\Im(\mu)|$, and so $|\mu| \in N(X)$. ■

For another characterization of normal measures we will need the following theorem of Dini.

Theorem 1.1.7 (Dini's theorem)**.** *Let X be a non-empty, compact space, and suppose that $(f_i)_{i\in I}$ is a net in $C_{\mathbb{R}}(X)$ such that $f_i(x) \searrow g(x)$, for each $x \in X$, where $g \in C_{\mathbb{R}}(X)$. Then, for each $\epsilon > 0$, there exists $i_0 \in I$ such that $\|f_i - g\|_X < \epsilon$, $i \geq i_0$.*

Proof. Fix $\epsilon > 0$ and $i_1 \in I$, and then take the compact subset C of X such that $f_{i_1}(x) < \epsilon$ for $x \in X \setminus C$. Set

$$X_i = \{x \in X \mid |f_i(x) - g(x)| \geq \epsilon\}$$

and $C_i = X_i \cap C$ for $i \in I$, so that each C_i is a compact subset of C. Assume towards a contradiction that each set C_i is non-empty. The family $(C_i)_{i\in I}$ has the finite intersection property: for $n \in \mathbb{N}$ there is an index j with $i_1, \ldots, i_n \leq j$. Since the net is decreasing, we have $f_{i_1}, \ldots, f_{i_n} \geq f_j$ and this leads to

$$\epsilon \leq f_j(x) - g(x) \leq f_{i_k}(x) - g(x), \quad k \in \{1, \ldots, n\} \Rightarrow x \in \bigcap_{k=1}^{n} C_{i_k}.$$

It follows that $\bigcap_{i\in I} C_i \neq \emptyset$, a contradiction of the fact that $f_i(x) \searrow g(x)$. We get $X_i = \emptyset$, for $i \geq i_0$. ■

The following is a well-known theorem in measure theory. A proof can be found, e.g., in [5, Theorem 2.24, p. 55]. We will need it to prove the next important characterization.

Theorem 1.1.8 (Lusin's theorem)**.** *Let X be a non-empty, compact space, and take $\mu \in M_{\mathbb{R}}(X)$. For each Borel function f on X and each $\epsilon > 0$, there is a compact subset C of X such that $|\mu|(X \setminus C) < \epsilon$ and $f|_C$ is continuous.*

Theorem 1.1.9. *Let X be a non-empty, compact space. Then a measure $\mu \in M(X)$ is normal if and only if $\mu(C) = 0$ for $C \in \mathscr{K}_X$, where $\mathscr{K}_X$ denotes the family of compact subsets C of X such that $C^\circ = \emptyset$.*

Proof.
"$\Rightarrow$" Suppose that $\mu \in N(X)$. We may suppose that $\mu \in N(X)^+$. Now take $C \in \mathscr{K}_X$, and consider the non-empty set

$$\mathscr{F} = \{f \in C_{\mathbb{R}}(X) \mid f \geq \mathbb{1}_C\}.$$

Suppose that $g = \inf \mathscr{F}$ in $C_{\mathbb{R}}(X)$. Then $g(x) = 0$ for $x \in X \setminus C$. If there was $x_0 \in X \setminus C$ with $g(x_0) > 0$, we can apply Urysohn's lemma to the closed sets C and x_0. So we get a function $f \in \mathscr{F}$, with $f(x_0) \leq g(x_0)$, so $g(x_0) = 0$ and since

$$\overline{X \setminus C} = X \setminus C^\circ = X,$$

$g(x) = 0$ for a dense subset. It follows $\inf \mathscr{F} = 0$. Now $(\mathscr{F}, \leq)$ is a directed set and the net $(f)_{f\in\mathscr{F}}$ is decreasing. Since

$$\mu(C) = \int_X \mathbb{1}_C \, d\mu = \lim_{f\in\mathscr{F}} \int_X f \, d\mu = \inf_{f\in\mathscr{F}} \int_X f \, d\mu,$$

we have $\mu(C) = 0$.
"$\Leftarrow$" Conversely, suppose that $\mu \in M(X)$ and $\mu(C) = 0$ for $C \in \mathscr{K}_X$. It suffices to suppose that $\mu \in M(X)^+$. Take $(f_i)_{i\in I}$ in $C(X)^+$ with $f_i \searrow 0$. We may suppose that $f_i \leq 1$ for each i. Set

$$g(x) = \inf_{i\in I} f_i(x) \quad x \in X.$$

Then g is a Borel function, since

$$g(x) < c \Leftrightarrow \exists i_0 : f_{i_0} < c \Rightarrow g^{-1}(-\infty, c) = \bigcup_{i \in I} f_i^{-1}(-\infty, c) \tag{1.2}$$

the right hand side of *Equation* (1.2) is open as a union of open sets and so it is a Borel set. For $n \in \mathbb{N}$, set $B_n = \{x \in X \mid g(x) > \frac{1}{n}\}$, so that $B_n \in \mathfrak{B}_X$. For each compact subset C of B_n, we have $C^\circ = \emptyset$. To see this observe that $C^{\mathsf{c}} \supseteq B_n^{\mathsf{c}}$. If we can show that B_n^{c} is dense the claim follows. Since $B_n^{\mathsf{c}} = \{x \in X \mid g(x) \leq \frac{1}{n}\}$, we have to show that for every open set $U \subseteq X$ there is $x_0 \in U$ with $g(x_0) \leq \frac{1}{n}$. If there was no such x_0, then for all $i \in I$, $f_i(x) > \frac{1}{n}$ for all $x \in U$. Now Urysohn's lemma applies to show that there is a continuous function f_U with $f_U(x) \leq \frac{1}{n}$ for $x \in U$ and $f_U(U^{\mathsf{c}}) = 0$. Now we have

$$f_U \leq f_i, \quad \forall i \in I,$$

a contradiction to $f_i \searrow 0$. So C^{c} is dense and $C^\circ = \emptyset$. According to our condition $\mu(C) = 0$. Thus, since μ is regular, $\mu(B_n) = 0$, and so

$$\mu\left(\{x \in X \mid g(x) > 0\}\right) = \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = 0,$$

whence $\int_X g \, d\mu = 0$. Hence, it suffices to show that

$$\lim_{i \in I} \int_X f_i \, d\mu = \int_X g \, d\mu. \tag{1.3}$$

Take $\epsilon > 0$. By Lusin's theorem, *Theorem* 1.1.8, there is a compact subset K of X with $\mu(X \setminus K) < \epsilon$ and such that $g|_K \in C(K)$. By Dini's theorem, *Theorem* 1.1.7, we know that $\lim_{i \in I} \|f_i|_K - g|_K\|_K = 0$, and so there exists i_0 with $\|f_i|_K - g|_K\|_K < \epsilon$ for $i \geq i_0$. It follows that

$$\left|\int_X f_i - g \, d\mu\right| \leq \int_K |f_i - g| \, d\mu + 2\epsilon < (\|\mu\| + 2)\epsilon, \quad i \geq i_0$$

giving *Equation* (1.3). ■

Corollary 1.1.10. *Let X be a non-empty compact space, and suppose that $\mu \in M(X)$. Then the following are equivalent:*

(i) $\mu \in N(X)$.

(ii) $|\mu|(\overline{B} \setminus B^\circ) = 0$ *for each* $B \in \mathfrak{B}_X$.

(iii) $\mu(B_1) = \mu(B_2)$ *for each* $B_1, B_2 \in \mathfrak{B}_X$ *with* $B_1 \triangle B_2$ *meagre.*

Proof. We may suppose that $\mu \in M(X)^+$.
"$(i) \Rightarrow (ii)$" Take $B \in \mathfrak{B}_X$. For each $\epsilon > 0$, there exists an open set U in X with $B \subseteq U$ and $\mu(U \setminus B) < \epsilon$. Since $\overline{U} \setminus U \in \mathscr{K}_X$, we have $\mu(\overline{U} \setminus U) = 0$. Thus

$$\mu(B) \leq \mu(\overline{B}) \leq \mu(\overline{U}) = \mu(U) \leq \mu(B) + \epsilon,$$

and so $\mu(\overline{B}) = \mu(B)$. By taking complements, it follows that $\mu(B^\circ) = \mu(B)$. Hence, $\mu(\overline{B} \setminus B^\circ) = 0$.
"$(i) \Rightarrow (iii)$" We know that $\mu(B) = 0$ for each nowhere dense set B in $\mathfrak{B}_X$, and so $\mu(B) = 0$ for each meagre set B in $\mathfrak{B}_X$. Thus, $\mu(B_1) = \mu(B_2)$ whenever $B_1, B_2 \in \mathfrak{B}_X$ with $B_1 \triangle B_2$ meagre.
"$(ii), (iii) \Rightarrow (i)$" These are immediate from *Theorem* 1.1.9. ■

There is a connection between measures in $M(X)$.

Definition 1.1.11. Let X be a non-empty, compact space and suppose that $\mu, \nu \in M(X)$. Then we write $\mu \perp \nu$ if μ and ν are *mutually singular*, in the sense that there exists $B \in \mathfrak{B}_X$ with $|\mu|(B) = 0$ and $|\nu|(X \setminus B) = 0$, and $\mu \ll \nu$ if $|\mu|$ is *absolutely continuous* with respect to $|\nu|$, in the sense that $|\mu|(B) = 0$ whenever $B \in \mathfrak{B}_X$ and $|\nu|(B) = 0$.
A family $\mathscr{F}$ of measures in $M(X)^+$ is *singular* if $\mu \perp \nu$ whenever $\mu, \nu \in \mathscr{F}$ and $\mu \neq \nu$.

Remark 1.1.12. The collection of singular families in $M(X)^+$ is ordered by inclusion. With *Zorn's lemma* we see that the collection of singular families of a non-empty subspace $\mathscr{F}$ of $M(X)^+$ has a maximal member that contains any specific singular family in $\mathscr{F}$, a *maximal singular family* in $\mathscr{F}$. //

Definition 1.1.13. Let X be a compact space. A measure $\mu \in M(X)$ is *supported* on a Borel subset B of X if $|\mu|(X \setminus B) = 0$. The support is denoted by supp μ.

As supp μ is the complement of the union of open sets U in X such that $|\mu|(U) = 0$, it is a closed subset of X.

1.2 Stonean spaces

Since our main interest is the space $C(X)$, the topology on X will play an important rule. We will make use of a certain seperation property.

Definition 1.2.1. A topological space X is *extremely disconnected* if the closure of every open set is itself open.

Remark 1.2.2. Equivalently, extremely disconnected means if pairs of disjoint open subsets of X have disjoint closure. To see this let $U \in \mathcal{T}$, then U and $\overline{U}^{\mathsf{c}}$ are disjoint open sets. Since every two disjoint open sets have disjoint closures we get

$$\overline{U} \cap \overline{X \setminus \overline{U}} = \emptyset \Rightarrow \overline{X \setminus \overline{U}} \subseteq X \setminus \overline{U},$$

which shows that $\overline{U}^{\mathsf{c}}$ is closed and $\overline{U}$ is open. Conversely take disjoint open sets U and V. Since $\overline{V}$ is open for any $x \in \overline{V}$, it is an open neighbourhood of x disjoint from U and so $x \notin \overline{U}$. It follows that $\overline{U} \cap \overline{V} = \emptyset$. //

Definition 1.2.3. A compact, extremely disconnected space is a *Stonean space.*

The definition of a Stonean space seems artificial but there are natural examples of topological spaces which do have this seperation property.

Example 1.2.4. Let B be a complete Boolean algebra. The *Stone space* is the family of ultrafilters on B, denoted by $St(B)$. We define a topology on $St(B)$ by taking the sets

$$S_b = \{p \in St(B) \,|\, b \in p\}, \quad b \in B$$

as a base of the topology. With this topology the Stone space is a Hausdorff, compact and extremely disconnected topological space with clopen basis sets S_b.
To see this take $p \neq q \in St(B)$. Now there is $x \in p$ with $x \notin q$. By definition of S_x, we get $q \in St(B) \setminus S_x$, and since these are ultrafilters, there exists $y \in q$ with $x \wedge y = 0$, and so $q \in S_y \subseteq St(B) \setminus S_x$. These are disjoint open neighbourhoods of p respectively q and since S_x is open and its complement is a neighbourhood of every element, S_x is clopen.
For a Boolean algebra we have $St(B) = S_1$. Taking $\Gamma \subseteq B$ such that $\{S_a \,|\, a \in \Gamma\}$ is a cover of S_1

with basic sets, we may suppose that Γ is closed under finite union. We claim that necessarily $1 \in \Gamma$. For otherwise, $a' \neq 0$ for each $a \in \Gamma$. Since

$$\bigwedge_{i=1}^{n} a_i' = \left(\bigvee_{i=1}^{n} a_i\right)' \neq 0, \quad n \in \mathbb{N}$$

the family is contained in some $p \in S_1$. But $p \notin \bigcup_{a\in\Gamma} S_a$, a contradiction. So $1 \in \Gamma$ and S_1 is compact.
Finally, we have to check the seperation property from *Definition* 1.2.1. Take an open set U. Since S_x for $x \in B$ form a base of the open set, we get $U = \bigcup_{b\in\Gamma} S_b$ for a subset Γ of B. Since B is complete, $a = \bigvee_{b\in\Gamma} b$ exists. We claim that $\overline{U} = S_a$. Now take $p \in S_a$. For each $c \in p$, we have $c \wedge a \neq 0$, and hence $c \wedge b \neq 0$ for some $b \in \Gamma$, for otherwise we would have $b \leq c'$ for $b \in \Gamma$, and hence $a \leq c'$. Thus $S_c \cap U \neq \emptyset$. This shows that $S_a \subseteq \overline{U}$. The reverse inclusion is immideate and since S_a is open, $\overline{U}$ is open and $St(B)$ is extremely disconnected. //

Definition 1.2.5. A subset U of a topological space X is *regular-open* if $U = \left(\overline{U}\right)^{\circ}$.

Proposition 1.2.6. *Let X be a Stonean space. Then every regular-open set in X is clopen, and, for every $B \in \mathfrak{B}_X$, there is a unique set $C \in \mathfrak{C}_X$ with $B \triangle C$ is meagre, where $\mathfrak{C}_X$ denotes the family of open and compact subsets of X.*

Proof. Let U be a regular-open set. We have

$$U \in \mathcal{T} \Rightarrow \overline{U} \in \mathcal{T} \Rightarrow \overline{U} = \left(\overline{U}\right)^{\circ} = U.$$

For the second part, let $\mathscr{F}$ be the family of subsets of X that differ from a clopen set by a meagre set and since X is compact these sets are compact and open. If $B \in \mathscr{F}$ and C is a clopen set such that $C \triangle B$ is meagre, then B^{c} and C^{c} differ by this same set. As C^{c} is clopen, $C^{\mathsf{c}} \in \mathscr{F}$. Each open set U lies in $\mathscr{F}$, since $\overline{U}$ is clopen and $\overline{U} \setminus U$ is nowhere dense. If $B_n \in \mathscr{F}$ for $n \in \mathbb{N}$ and C_n is a clopen set such that $B_n \triangle C_n$ is meagre, then

$$\left(\bigcup_{n=1}^{\infty} B_n\right) \triangle \left(\bigcup_{n=1}^{\infty} C_n\right) \subseteq \bigcup_{n=1}^{\infty} \left(B_n \triangle C_n\right).$$

As $\bigcup_{n=1}^{\infty} \left(B_n \triangle C_n\right)$ is meagre and $\bigcup_{n=1}^{\infty} C_n$ is open, $\bigcup_{n=1}^{\infty} B_n \in \mathscr{F}$. Hence $\mathfrak{B}_X \subseteq \mathscr{F}$ and $\mathscr{F}$ contains the Borel subsets of X. ■

The second part of the proof of *Proposition* 1.2.6 is taken from [4, Lemma 5.2.10, p.322].

Definition 1.2.7. A set U is *regular-closed* if its complement is regular-open.

Remark 1.2.8. Equivalently the equality $U = \overline{U^{\circ}}$ holds:

$$\left(\overline{U^{\circ}}\right)^{\mathsf{c}} = \left(\overline{\left(\overline{U^{\mathsf{c}}}\right)^{\mathsf{c}}}\right)^{\mathsf{c}} = \left(\overline{U^{\mathsf{c}}}\right)^{\circ} = U^{\mathsf{c}}.$$

It is sometimes easier to work with this property. //

The properties of the topological space X have also effect on the measures on this space:

Proposition 1.2.9. *Let X be a non-empty, compact space and suppose that $\mu \in N(X)$. Then supp μ is a regular-closed set.*

Proof. Since supp μ = supp $|\mu|$, we may suppose that $\mu \in N(X)^+$. Set $A = \text{supp } \mu$, a closed set, and set $U = A^\circ$, so that $\overline{U} \subseteq A$. Since $A \setminus \overline{U}$ is nowhere dense, $\mu(A \setminus \overline{U}) = 0$. Thus $\mu(X \setminus \overline{U}) = 0$, and so, by the definition of supp μ, we have $X \setminus \overline{U} \subseteq X \setminus A$. Hence $\overline{U} = A$, and A is regular-closed. ■

Corollary 1.2.10. *Let X be a Stonean space, and suppose that $\mu \in N(X)^+ \setminus \{0\}$. Then:*

(i) The space supp μ is clopen in X, and hence Stonean.

(ii) For each $B \in \mathfrak{B}_X$, there is a unique set $C \in \mathfrak{C}_X$ with $C \subseteq$ supp μ and $\mu(B \triangle C) = 0$.

Proof.
(*i*) In a Stonean space, every regular-closed set is clopen. Since the closure of a set in the subspace topology is just

$$\overline{U}^{\mathcal{T}|_{\text{supp } \mu}} = \overline{U}^{\mathcal{T}} \cap \text{supp } \mu$$

and an open set is obtained in the same way, $\overline{U}^{\mathcal{T}|_{\text{supp } \mu}}$ is open in supp μ.
(*ii*) By (*i*) supp μ is a clopen subset of X and $\mu(X \setminus \text{supp } \mu) = 0$, and so we may suppose that $X = \text{supp } \mu$. Take $B \in \mathfrak{B}_X$. By *Proposition* 1.2.6, there is a unique $C \in \mathfrak{C}_X$ with $B \triangle C$ meagre, and then $\mu(B \triangle C) = 0$. Suppose that $C_1, C_2 \in \mathfrak{C}_X$ are such that $\mu(B \triangle C_1) = \mu(B \triangle C_2) = 0$. Then $C_1 \triangle C_2 \subseteq (B \triangle C_1) \cup (B \triangle C_2)$, so that $\mu(C_1 \triangle C_2) = 0$. Since $C_1 \triangle C_2$ is an open set in $X = \text{supp } \mu$ and $\mu(U) > 0$ for all non-empty open subsets U of X, it follows that $C_1 \triangle C_2 = \emptyset$, i.e., $C_1 = C_2$. ■

Proposition 1.2.11. *Let X be a Stonean space, and suppose that $\mu, \nu \in N(X)$. Then:*

(i) supp $\nu \subseteq$ supp μ if and only if $\nu \ll \mu$.

(ii) $\mu \perp \nu$ if and only if supp $\mu \cap$ supp $\nu = \emptyset$.

Proof.
(i) Always supp $\nu \subseteq$ supp μ when $\nu \ll \mu$. For the converse, we may suppose that $\mu, \nu \in N(X)^+$. By *Proposition* 1.2.6, for each $B \in \mathfrak{B}_X$, there exists $C \in \mathfrak{C}_X$ with $B \triangle C$ meagre. Now suppose that B is a μ-nullset. Then by *Corollary* 1.2.10, C is also a μ-nullset, and so $C \cap \text{supp } \nu = \emptyset$, whence $\nu(B) = \nu(C) = 0$. This shows that $\nu \ll \mu$.
(ii) Clearly $\mu \perp \nu$ when supp $\mu \cap$ supp $\nu = \emptyset$. Next suppose that $\mu \perp \nu$, and set $U =$ supp $\mu \cap$ supp ν, so that U is an open set. Then $\nu|_U \perp \mu$ and $\nu|_U \ll \mu$. Thus $\nu|_U = 0$, and hence $U = \emptyset$. ■

Definition 1.2.12. A lattice is *Dedekind complete* if every non-empty subset which is bounded above has a supremum and every non-empty subset which is bounded below has an infimum.

If the space $C(X)$ satisfies this completeness property, we can infer that the space X has our required seperation property.

Theorem 1.2.13. *Let X be a non-empty, compact space. Then X is Stonean if and only if $C(X)$ is Dedekind complete.*

Proof.
"$\Rightarrow$" Suppose that $C_{\mathbb{R}}(X)$ is Dedekind complete, and let U be an open set in X. Take $\mathscr{F}$ to be the family of functions $f \in C_{\mathbb{R}}(X)$ such that

$$\mathscr{F} = \{f \in C_{\mathbb{R}}(X) \mid f(x) = 0 \text{ for } x \in X \setminus U \ , \ 0 \leq f \leq 1\}.$$

Then since $C(X)$ is Dedekind complete, $\mathscr{F}$ has a supremum, say $f_0 \in C_{\mathbb{R}}(X)$. To determine this supremum we make use of Urysohn's lemma. We claim that $f_0(x) = 1$ for $x \in U$ and $f_0(x) = 0$ for $x \in X \setminus U$. To see this take the closed sets $\{x\}$ and U^{c}. Then there is $f_x \in C(X)$ with $f_x(x) = 1$, $f_x(X \setminus U) = 0$ and $0 \le f_x \le 1$. Now $f_x \in \mathscr{F}$ and $f_x \le f_0$. Next take $x \in X \setminus \overline{U}$. Again with Urysohn's lemma we get $g \in C(X)$ with $g(\{x\}) = 0$ and $g(\overline{U}) = 1$. This leads to

$$f \in \mathscr{F} \Rightarrow f = fg \le g$$

and $\sup \mathscr{F} \le g$. Hence we get $f_0 = \mathbb{1}_{\overline{U}}$. As $X \setminus \overline{U}$ is closed as the preimage of $\{0\}$ under the continuous function f_0, it follows that $\overline{U}$ is open and X is Stonean.
"$\Leftarrow$" Conversely, suppose that X is Stonean, and let $\mathscr{F}$ be a family in $C(X)^+$ which is bounded above, say by 1. For $r \in [0,1]$, define

$$U_r = \bigcup_{f \in \mathscr{F}} \{x \in X \mid f(x) > r\}.$$

Then U_r is open in X, and so $V_r := \overline{U_r}$ is also open in X. Clearly $V_1 = \emptyset$. Define

$$g(x) = \sup_{x \in U_r} r.$$

If $g(x) \in (r,s)$, then $x \in V_r \setminus V_s$, and, if $x \in V_r \setminus V_s$, then $g(x) \in [r,s]$. Take $x_0 \in X$, and take a neighbourhood V of $g(x_0)$. Then there exist $r, s \in \mathbb{R}$ with $g(x_0) \in (r,s) \subseteq [r,s] \subseteq V$. Since $V_r \setminus V_s$ is an open set and

$$x_0 \in V_r \setminus V_s \subseteq g^{-1}([r,s]) \subseteq g^{-1}(V),$$

we see that g is continuous at x_0. Thus $g \in C_{\mathbb{R}}(X)$.
Now take $h \in C_{\mathbb{R}}(X)$ with $h \ge f$ for $f \in \mathscr{F}$. Assume that there exists $x_0 \in X$ with $h(x_0) < g(x_0)$. Then $h(x_0) < r$ for some r with $x_0 \in V_r$. Let W be a neighbourhood of x_0 with $h(x) < r$ for $x \in W$. Then there exists $x \in W$ with $f(x) > r$ for some $f \in \mathscr{F}$, a contradiction. Thus $h \ge g$, and so $g = \sup \mathscr{F}$. We have shown that $C_{\mathbb{R}}(X)$ is Dedekind complete. ■

We will make use of *Theorem* 1.2.13 in the following:

Example 1.2.14. A *character* on an Banach algebra Z is a homomorphism from Z to $\mathbb{C}$. The set of all characters on Z is denoted by Φ_Z, this is the *character space* of Z.
For a locally compact space Γ and a measure $\mu \in P(\Gamma)$, the character space of the C^*-algebra $L^\infty(\Gamma, \mu)$ is denoted by Φ_μ. Since $L^\infty(\Gamma, \mu)$ is commutative the *Gelfand transform*

$$\Psi : \begin{cases} L^\infty(\Gamma, \mu) \to C(\Phi_\mu) \\ f \mapsto \hat{f} \end{cases}$$

is an isomorphism and moreover, a lattice isometry. Since $L^\infty(\Gamma, \mu)$ is Dedekind complete, it follows that $C(\Phi_\mu)$ is also Dedekind complete. Now *Theorem* 1.2.13 applies to show that Φ_μ is a Stonean space. //

Theorem 1.2.15 (Baire's theorem)**.** *If X is a compact Hausdorff space then the intersection of every countable collection of dense open subsets of X is dense in X.*

Proof. Suppose $(V_n)_{n \in \mathbb{N}}$ are dense open subsets of X. Let U_0 be an arbitrary non-empty open set in X. If $n \ge 1$ and an open non-empty U_{n-1} has been chosen, then there exists an open non-empty U_n since V_n is dense with

$$\overline{U_n} \subseteq V_n \cap U_{n-1}.$$

Since $(\overline{U_n})_{n\in\mathbb{N}}$ has the finite intersection property, the set

$$K = \bigcap_{n=1}^{\infty} \overline{U_n}$$

is non-empty and we have $K \subseteq U_0$ and $K \subseteq V_n$ for each n. Hence U_0 intersects $\bigcap_{n=1}^{\infty} V_n$. ■

Theorem 1.2.16. *Let X be a Stonean space, and let U be dense a or open subspace of X. Take a compact space L and $f \in C(U, L)$. Then there exists $F \in C(\overline{U}, L)$ such that $F|_U = f$.*

Proof. Take $x \in \overline{U}$, and let $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ be nets in U with $\lim_{i\in I} x_i = \lim_{j\in J} y_j = x$. Then the nets $(f(x_i))_{i\in I}$ and $(f(y_j))_{j\in J}$ have accumulation points, say x_1 and x_2, respectively, in L. Assume towards a contradiction that $x_1 \neq x_2$, and take open neighbourhoods N_{x_1} and N_{x_2} of x_1 and x_2, respectively, such that $\overline{N_{x_1}} \cap \overline{N_{x_2}} = \emptyset$. Then the sets

$$\{y \in U \mid f(y) \in N_{x_1}\} \quad \text{and} \quad \{y \in U \mid f(y) \in N_{x_2}\}$$

are disjoint, relatively open subsets of U, and so they have the form $U \cap V$ and $U \cap W$, respectively, for some open subsets V and W in X. Since $\overline{U} = X$, we have $V \cap W = \emptyset$, and since X is Stonean $\overline{V} \cap \overline{W} = \emptyset$. In the case where U is open, $\overline{U \cap V} \cap \overline{U \cap W} = \emptyset$. However $x \in \overline{U \cap V} \cap \overline{U \cap W}$. Thus $x_1 = x_2$. It follows that $(f(x_i))_{i\in I}$ converges to a unique limit $F(x)$, in L, and that the limit is independent of the net $(x_i)_{i\in I}$. Now F is the required extension of f. ■

Corollary 1.2.17. *The complement of a meagre set M is dense in X.*

Proof. M can be written as the countable union of nowhere dense sets $(M_n)_{n\in\mathbb{N}}$. Taking complements we get

$$\overline{M^{\mathsf{c}}} = \overline{\left(\bigcup_{n=1}^{\infty} M_n\right)^{\mathsf{c}}} \supseteq \overline{\left(\bigcup_{n=1}^{\infty} \overline{M_n}\right)^{\mathsf{c}}} = \left(\left(\bigcup_{n=1}^{\infty} \overline{M_n}\right)^{\circ}\right)^{\mathsf{c}}.$$

And since the union of sets with empty interior has empty interior, M^{c} is dense. ■

Remark 1.2.18. Let X be a non-empty, compact space, and define

$$M_X := \{f \in B^b(X) \mid \{x \in X \mid f(x) \neq 0\} \text{ is meagre}\}.$$

Then M_X is a closed ideal in the C^*-algebra $B^b(X)$:
Set

$$m_f := \{x \in X : f(x) \neq 0\} = f^{-1}(\{0\})^{\mathsf{c}}.$$

Take $g \in B^b(X)$ and $f \in M_X$, we have to show that the set m_{fg} is meagre. Now since every subset of a meagre set is meagre and

$$m_f^{\mathsf{c}} = f^{-1}(\{0\}) \subseteq fg^{-1}(\{0\}) = m_{fg}^{\mathsf{c}},$$

it follows that $m_{fg} \subseteq m_f$. So $fg \in M_X$.
Secondly we have to show that the sum of two functions $f, g \in M_X$ is again in M_X. This follows because $m_{f+g} \subseteq m_f \cup m_g$ and the union of meagre sets is meagre.
At last we have to show that M_X is closed. Let $f_n \to f$ with $f_n \in M_X$. We have to show that m_f is meagre. Take $x \in m_f$, then $|f(x)| = \alpha > 0$. Now let n_0 be sufficiently large so that $|f(x) - f_{n_0}(x)| < \frac{\alpha}{2}$. Then $x \in m_{f_n}$ for $n \geq n_0$ and

$$m_f \subseteq \bigcup_{n\in\mathbb{N}} m_{f_n}.$$

The countable union of meagre sets is again meagre and so m_f is meagre. //

Definition 1.2.19. Let X be a non-empty, compact space. Then

$$D(X) = B^b(X)/M_X$$

is the *Dixmier algebra* of X.

Theorem 1.2.20. *Let X be a non-empty, Stonean space. Then $D(X)$ and $C(X)$ are C^*-isomorphic.*

Proof. First consider a simple bounded Borel function f of the form $f = \sum_{i=1}^n \alpha_i \mathbb{1}_{B_i}$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $B_1, \ldots, B_n \in \mathfrak{B}_X$ are pairwise disjoint. As we already know, there exist $C_1, \ldots, C_n \in \mathfrak{C}_X$ such that $B_i \triangle C_i$ is meagre. Clearly, the sets $C_1, \ldots, C_n$ are pairwise disjoint. We define

$$g = \sum_{i=1}^n \alpha_i \mathbb{1}_{C_i}.$$

We have $g \in C(X)$ since

$$C_i \in \mathfrak{C}_X \Rightarrow \exists\ g_i \in C(X) : g_i(C_i) \subseteq \{1\},\ g_i(C_i^{\mathfrak{c}}) \subseteq \{0\} \Rightarrow g_i \equiv \mathbb{1}_{C_i}$$

and so the set $\{x \in X \mid f(x) \neq g(x)\}$ is meagre.
Now consider a general function $f \in B^b(X)$. There is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple, bounded Borel functions that converges uniformly to f on X. For each $n \in \mathbb{N}$, choose $g_n \in C(X)$ such that $M_n := \{x \in X \mid f(x) \neq g_n(x)\}$ is a meagre subset of X. The set

$$M := \bigcup_{n\in\mathbb{N}} M_n$$

is also meagre in X, and $g_n(x) = f_n(x)$ for all $n \in \mathbb{N}$ and $x \in X \setminus M$, and so $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C(X \setminus M), \|\cdot\|_{X\setminus M})$. The sequence converges uniformly to a function, say g, in $C(X \setminus M)$. Now by *Theorem* 1.2.15, $X \setminus M$ is dense in X and by *Theorem* 1.2.16, $\overline{g}$ has an extension in $C(X)$.
For each $f \in B^b(X)$, take $\pi(f)$ to be the unique $\overline{g} \in C(X)$ and consider the map

$$\pi : B^b(X) \to C(X).$$

Clearly the restriction of π to the simple functions is a $*$-homomorphism; since the simple functions are dense in $B^b(X)$ and $\pi(f) = f, f \in C(X)$, the map π is a C^*-homomorphism that is a bounded projection from $B^b(X)$ to $C(X)$. Clearly ker $\pi = M_X$, and so the map

$$\overline{\pi} : D(X) = B^b(X)/M_X \to C(X).$$

is a C^*-isomorphism. ■

Corollary 1.2.21. *Let X be a Stonean space, and suppose that $\mu \in N(X) \cap P(X)$ is a strictly positive measure. Then every equivalence class in $L^\infty(X, \mu)$ contains a continuous function, the C^*-algebras $(L^\infty(X, \mu), \|.\|_\infty)$ and $(C(X), \|.\|_X)$ are C^*-isomorphic.*

Proof. By *Theorem* 1.2.20, there is a C^*-isomorphism $\overline{\pi} : D(X) \to C(X)$. However $\mu(B) = 0$ for each meagre set $B \in \mathfrak{B}_X$ by *Corollary* 1.1.10 and so ker $\overline{\pi}$ is exactly the kernel of the projection of $B^b(X)$ onto $L^\infty(X, \mu)$. ■

1.3 The complexification of $C_{\mathbb{R}}(X)$

Remark 1.3.1. Often it is easier to work with real Banach spaces. Since we are interested in the complex Banach space $C(X)$, we want to infer from the real to the complex case.
We give a sketch of how the complexification transfers to the dual space: Let $(Z, \|.\|)$ be a real Banach lattice with dual Z' and $Z_{\mathbb{C}} = Z \oplus iZ$ its complexification. If we want to endow this complexification with a fitting norm that respects the order, for $z = x + iy$, define

$$|z| = |x+iy| = \sup_{0\leq\theta\leq 2\pi} x\cos\theta + y\sin\theta.$$

Then the norm

$$\|z\| = \||z|\|$$

makes $Z_{\mathbb{C}}$ to a Banach space. At first we can identify Z' as a real-linear subspace of $Z'_{\mathbb{C}}$ if we define $\lambda(x+iy) = \lambda(x) + i\lambda(y)$ for $\lambda \in Z', x, y \in Z$. And for each $\lambda \in Z'_{\mathbb{C}}$, there exist λ_1 and λ_2 in Z' such that $\lambda(x) = \lambda_1(x) + i\lambda_2(x)$ for $x \in Z$ and so $Z'_{\mathbb{C}}$ is isomorphic as a complex Banach space to the complexification $Z' \oplus iZ'$. //

In the following section we will deal with this complexification. We want to show that $C(X)$ is a dual space of a Banach space if and only if $C_{\mathbb{R}}(X)$ is a dual space.

Lemma 1.3.2. *Let X be a compact space, and let $\mu \in M(X)^+$. Take $f, g \in L^1_{\mathbb{R}}(\mu)$ and $\epsilon > 0$. Suppose that $\|f+ig\|_1 = 1$ and that $1-\epsilon < \|f\|_1 \leq 1$. Then $\|g\|_1 \leq \sqrt{2\epsilon}$.*

Proof. Take $a, b > 0$. Since

$$\sqrt{1+t} \leq 1 + \frac{t}{2}, \quad t \geq 0,$$

we have

$$a^2 + b^2 \geq a^2\sqrt{1+\frac{b^2}{a^2}} + \frac{b^2}{2} = a\sqrt{a^2+b^2} + \frac{b^2}{2} \Leftrightarrow \sqrt{a^2+b^2} \geq a + \frac{b^2}{2\sqrt{a^2+b^2}}.$$

Set $h = \frac{g^2}{\sqrt{f^2+g^2}}$. It follows that

$$1 = \int_X \sqrt{f^2+g^2}\ d\mu \geq \int_X |f|\ \ d\mu + \frac{1}{2}\int_X h\ d\mu,$$

and so $\int_X h\ d\mu < 2\epsilon$. We then have

$$\int_X |g|\ \ d\mu = \int_X \frac{|g|\,(f^2+g^2)^{\frac{1}{4}}}{(f^2+g^2)^{\frac{1}{4}}}\ d\mu \leq \left(\int_X h\ d\mu\right)^{\frac{1}{2}} \left(\int_X \sqrt{f^2+g^2}\ d\mu\right)^{\frac{1}{2}}$$

and so $\|g\|_1 \leq \sqrt{2\epsilon}$. ∎

Corollary 1.3.3. *Let X be a compact space, and let $\mu, \nu \in M_{\mathbb{R}}(X)$. Take $\epsilon > 0$, and suppose that $\|\mu + i\nu\| = 1$ and that $1-\epsilon < \|\mu\| \leq 1$. Then $\|\nu\| \leq \sqrt{2\epsilon}$.*

Proof. Consider the measure

$$\lambda = |\mu| + |\nu| \in M(X)^+.$$

Then $\mu = f\ d\lambda$ and $\nu = g\ d\lambda$ for some $f, g \in L^1_{\mathbb{R}}(\lambda)$ such that $\|\mu\| = \|f\|_1$ and $\|\nu\| = \|g\|_1$ and the claim follows from *Lemma* 1.3.2. ∎

Proposition 1.3.4. *Let Z be a Banach space. Then $\iota(Z)$ is weak*-dense in Z''.*

Proof. Since Z'' is endowed with the weak*-topology we have $(Z'', \sigma(Z'', \iota'(Z')))' = \iota'(Z')$. We know from a corollary of the *Hahn-Banach* theorem [7, Corollary 5.2.6, p.79] that

$$\overline{\iota(Z)} = \bigcap_{\substack{f \in \iota'(Z') \\ \iota(Z) \subseteq \ker f}} \ker f.$$

We have to show that 0 is the only element with $\iota(Z) \subseteq \ker f$.

$$\iota'(f)[\iota(z)] = \iota(z)[f] = f(z) = 0, \ \forall z \in Z$$

Hence $f \equiv 0$. ■

Proposition 1.3.5. *Let X be a non-empty, compact space. Then the Banach space $C(X)$ is isometrically the dual of a Banach space if and only if the real Banach space $C_{\mathbb{R}}(X)$ is isometrically the dual of a real Banach space.*

Proof.
"$\Leftarrow$" Suppose $C_{\mathbb{R}}(X)$ is isometrically isomorphic to Y' for a real Banach space Y, and regard Y as a closed subspace of $C_{\mathbb{R}}(X)'$. Now set

$$Y_{\mathbb{C}} = Y \oplus iY$$

so that $Y_{\mathbb{C}}$ is a closed subspace of $C(X)'$ and we have

$$C(X)' \cong Y'' \oplus iY'' = Y''_{\mathbb{C}}$$

and $Y_{\mathbb{C}}$ is a Banach space. It must yet be shown that $Y'_{\mathbb{C}} \cong C(X)$:
Take $f \in C(X)$ and set

$$\lambda(y) = \langle f, y \rangle, \ y \in Y_{\mathbb{C}}.$$

Then $\lambda \in Y'_{\mathbb{C}}$ with $\|\lambda\| \leq \|f\|$, and the map

$$S : \begin{cases} C(X) \to Y'_{\mathbb{C}} \\ f \mapsto \lambda \end{cases}$$

is a linear contraction. Take $\lambda \in Y'_{\mathbb{C}}$, and set

$$\lambda_1 = \Re(\lambda)|_Y, \ \lambda_2 = \Im(\lambda)|_Y$$

so that λ_1 and λ_2 are bounded real-linear functionals on Y with $\lambda = \lambda_1 + i\lambda_2$. Thus there exist unique elements x and z in $C_{\mathbb{R}}(X)$ such that

$$\lambda_1(g) = \langle x, g \rangle, \lambda_2(g) = \langle z, g \rangle$$

for $g \in Y$. Set $h = x + iz \in C(X)$. Then for each $g_1, g_2 \in Y$, we have

$$\begin{aligned} \lambda(g_1 + ig_2) &= (\lambda_1 + i\lambda_2)(g_1 + ig_2) = \langle x, g_1 \rangle - \langle z, g_2 \rangle + i(\langle z, g_1 \rangle + \langle x, g_2 \rangle) \\ &= \langle x + iz, g_1 + ig_2 \rangle = \langle h, g_1 + ig_2 \rangle \end{aligned}$$

and so $\lambda = S(h)$. Thus S is a surejection.
Now fix $\epsilon > 0$. By *Proposition* 1.3.4 we see, that $Y_{\mathbb{C}}$ is weak*-dense in $Y''_{\mathbb{C}}$ and there exists $k \in Y_{\mathbb{C}}$ with $\|k\| = 1$ and $|\langle f, k \rangle| > \|f\| - \epsilon$, and hence $\|\lambda\| > \|f\| - \epsilon$. This holds for each $\epsilon > 0$,

and so $\|\lambda\| \geq \|f\|$. So S is an isometric isomorphism.
"$\Rightarrow$" Now suppose $C(X) \cong Y'$ where Y is a Banach space. We regard Y as a closed subspace of $Y'' = M(X)$. Define

$$Y_{\mathbb{R}} = \{\Re(\mu) \in M_{\mathbb{R}}(X) \,|\, \mu \in Y\}.$$

Then $Y_{\mathbb{R}}$ is a real-linear subspace of $M_{\mathbb{R}}(X)$, and $\Re(\mu), \Im(\mu) \in Y_{\mathbb{R}}$ whenever $\mu \in Y$, so that $Y = Y_{\mathbb{R}} \oplus iY_{\mathbb{R}}$. For each $\lambda \in Y'_{\mathbb{R}}$, define

$$\overline{\lambda}(\mu + i\nu) = \lambda(\mu) + i\lambda(\nu), \quad \mu, \nu \in Y_{\mathbb{R}}.$$

Then $\overline{\lambda}$ is a continuous, complex-linear functional on Y with

$$\|\lambda\| \leq \left\|\overline{\lambda}\right\| \leq \sqrt{2}\,\|\lambda\|\,.$$

Thus there exist unique elements $f, g \in C_{\mathbb{R}}(X)$ with

$$\overline{\lambda}(\mu + i\nu) = \langle f + ig, \mu + i\nu\rangle\,, \quad \mu + i\nu \in Y.$$

It follows that

$$\lambda(\mu) = \langle f, \mu\rangle - \langle g, \nu\rangle \text{ and } \lambda(\nu) = \langle f, \nu\rangle + \langle g, \mu\rangle\,.$$

Define

$$T : \begin{cases} Y'_{\mathbb{R}} \to C_{\mathbb{R}}(X) \\ \lambda \mapsto f. \end{cases}$$

Then T is a continuous, real-linear map such that

$$\|T(\lambda)\|_X \geq \|\lambda\|\,. \tag{1.4}$$

Take $f \in C_{\mathbb{R}}(X)$ and define

$$\lambda(\mu) = \langle f, \mu\rangle\,, \quad \mu \in Y_{\mathbb{R}}.$$

Then $\lambda \in Y'_{\mathbb{R}}$ is such that $\|\lambda\| \leq \|f\|_X$ and $T(\lambda) = f$. This shows T is a surjection. To show injectivity we take $\lambda \in Y'_{\mathbb{R}}$ with $T(\lambda) = 0$, and assume towards a contradiction that $\lambda \neq 0$. Then $\overline{\lambda} \neq 0$, and so we may suppose that $\left\|\overline{\lambda}\right\| = 1$. Now there exists $g \in C_{\mathbb{R}}(X)$ with $\|g\|_X = 1$ such that

$$\lambda(\mu) = -\langle g, \nu\rangle \text{ and } \lambda(\nu) = \langle g, \mu\rangle\,, \quad \mu + i\nu \in Y.$$

Choose $x \in X$ with $|g(x)| = 1$, without loss of generality $g(x) = 1$. Since the closed unit ball $B_1^Y(0)$ is weak*-dense in $B_1^{M(X)}(0)$, and so for each $\epsilon > 0$, there exists $\mu_0 + i\nu_0 \in B_1^Y(0)$ with $|\langle g, \delta_x - \mu_0 + i\nu_0\rangle| < \epsilon$. Thus,

$$|1 - \langle g, \mu_0\rangle| \leq |1 - \langle g, \mu_0 + i\nu_0\rangle| < \epsilon.$$

Since

$$1 - \epsilon < \|\mu_0\| \leq 1,$$

it follows from *Corollary* 1.3.3 that

$$1 - \epsilon \leq |\langle g, \mu_0\rangle| = |\lambda(\nu_0)| \leq \|\nu_0\| \leq \sqrt{2\epsilon},$$

a contradiction for some $\epsilon > 0$. Thus $\lambda = 0$ and T is injective. Finally we have to show that T is an isometry and since *Theorem* 0.0.8, it remains to show that

$$\|T(\lambda)\|_X \leq \|\lambda\|\,, \quad \lambda \in Y'_{\mathbb{R}}.$$

Take $f \in C_{\mathbb{R}}(X)$. Since X is compact there is $x_0 \in X$ with $|f(x_0)| = \|f\|_X$. For each $\epsilon > 0$, there exists $\mu + i\nu \in B_1^Y(0)$ with

$$|f(x_0) - \langle f, \mu + i\nu \rangle| < \epsilon.$$

We have $\mu \in Y_{\mathbb{R}}$ with $\|\mu\| \leq 1$. Take the unique λ with $T(\lambda) = f$, so that, as above, $\lambda(\mu) = \langle f, \mu \rangle$. Then

$$\|\lambda\| \geq |\langle f, \mu \rangle| > |f(x_0)| - \epsilon = \|f\|_X - \epsilon = \|T(\lambda)\|_X - \epsilon,$$

and so $\|T(\lambda)\|_X \leq \|\lambda\| + \epsilon$. This holds true for each $\epsilon > 0$, and so $\|T(\lambda)\|_X \leq \|\lambda\|$ and so T is an isometry. ■

1.4 Hyper-Stonean spaces

Definition 1.4.1. Let X be a non-empty, compact space. Then

$$W_X := \bigcup_{\mu \in N(X)} supp\ \mu.$$

The space X is *hyper-Stonean* if X is Stonean and W_X is dense in X.

Since the restriction of a normal measure to a Borel set is a normal measure, for each non-empty, open subset U of X, there exists $\mu \in N(X) \cap P(X)$ with supp $\mu \subseteq U$.

The following theorem will characterize Hyper-Stonean spaces by a certain measure:

Definition 1.4.2. A positive measure μ on the Borel sets of a Stonean space X is a *category measure* if

(i) μ is regular on closed subsets of finite measure;

(ii) every non-empty, clopen set in X contains a clopen set U with $0 < \mu(U) < \infty$;

(iii) every nowhere dense Borel set has measure zero.

Proposition 1.4.3. *Let X be a Stonean space. Then X is hyper-Stonean if and only if there exists a category measure on X.*

Proof.
"⇒" Suppose that X is hyper-Stonean. Consider a maximal family $(\mu_i)_{i \in I}$ of measures in $N(X)^+$ with pairwise-disjoint supports, and set

$$\mu = \sum_{i \in I} \mu_i,$$

so that μ is a positive measure on $\mathfrak{B}_X$. Take C to be a clopen subset of X. Then

$$C_0 := C \cap \text{supp } \mu_{i_0} \neq \emptyset$$

for some i_0 because of the maximality of the family $(\mu_i)_{i \in I}$ and the assumption that X is hyper-Stonean. Since X is Stonean, supp μ_{i_0} is clopen, and so C_0 is a clopen subset of C with

$$0 < \mu(C_0) = \mu_{i_0}(C_0) < \infty.$$

Clearly $\mu(B) = 0$ for each nowhere dense Borel set B because $\mu_i(B) = 0$ for each such B and each i. Thus, μ is a category measure.

"⇐" Conversely, suppose that μ is a category measure on X. For an arbitrary clopen set C in X, take some clopen $C_0 \subseteq C$ with $0 < \mu(C_0) < \infty$, and set

$$\mu_C = \mu|_{C_0}.$$

By our characterization of normal measures, we have $\mu_C \in N(X)^+$ and supp $\mu_C \subseteq C$. Since C was arbitrary, X is hyper-Stonean. ■

Remark 1.4.4. As we have seen in *Example* 1.2.14, the character space of a C^*-algebra is an interesting tool. To describe the character space of $C(X)$, let us remark that the kernel of a character is a maximal modular ideal and on the other hand every maximal modular ideal is the kernel of a character. Now in this case there is an easy description of those sets. Define

$$\epsilon_x : \begin{cases} C(X) \to \mathbb{C} \\ f \mapsto f(x) \end{cases}$$

called the *evaluation character* at x, and

$$M_x := \{ f \in C(X) \mid f(x) = 0 \} = \ker \epsilon_x.$$

It can be shown that these are all characters. Finally we can identify the character space of $C(X)$ with X:

$$\Phi_{C(X)} = X.$$

So if X is Stonean and we take a normal measure μ, we get by *Corollary* 1.2.21, that $\Phi_\mu = \Phi_{C(X)}$ is homeomorphic to X. //

Definition 1.4.5. Let $(Z_i, \|.\|_i)_{i\in I}$ be a family of Banach spaces, defined for each i in a non-empty index set I. Then set

$$\bigoplus_\infty Z_i = \{ (z_i)_{i\in I} \mid \|(z_i)_{i\in I}\| = \sup_{i\in I} \|z_i\|_i < \infty \}$$

and

$$\bigoplus_p Z_i = \{ (z_i)_{i\in I} \mid \|(z_i)_{i\in I}\| = \left(\sum_{i\in I} \|z_i\|_i^p \right)^{\frac{1}{p}} < \infty \}.$$

These are Banach spaces.

Remark 1.4.6. Let q be the conjugate index to p, then similar to the L^p-spaces the duality

$$\left(\bigoplus_p Z_i \right)' = \bigoplus_q Z_i',$$

holds. //

Remark 1.4.7. As a preparation for *Theorem* 2.1.1 we want to sum up: Let X be a Stonean space such that $N(X) \neq \{0\}$, and take $(\mu_i)_{i\in I}$ to be a maximal singular family in $N(X) \cap P(X)$, where the measures μ_i are distinct. For each $i \in I$, set $S_i = \text{supp } \mu_i$, so that, each S_i is Stonean, and hence by *Corollary* 1.2.21, $\Phi_{\mu_i} = \Phi_{C(S_i)}$ is homeomorphic to S_i. $(S_i)_{i\in I}$ is a pairwise-disjoint family of clopen subsets of X. We set

$$U_{\mathscr{F}} = \bigcup_{i\in I} \text{supp } \mu_i.$$

Then $U_{\mathscr{F}}$ is an open subset of X. In the case where X is hyper-Stonean, $U_{\mathscr{F}}$ is dense in X. For the family of compact spaces $(S_i)_{i\in I}$ set

$$\mathscr{A} = \bigoplus_{\infty} C(S_i).$$

Take $j \in I$, and write δ_j for the element $(f_i)_{i\in I}$ in $\mathscr{A}$ such that $f_j = \mathbb{1}_{S_j}$ and $f_i = 0$ for $j \neq i$. Take $j \in I$ and $x \in S_j$. Then the map

$$\phi_x : \begin{cases} \mathscr{A} \to \mathbb{C} \\ (f_i)_{i\in I} \mapsto f_j(x) \end{cases}$$

is a character on $\mathscr{A}$, and the map

$$\psi : \begin{cases} S_i \to \Phi_{\mathscr{A}} \\ x \mapsto \phi_x \end{cases}$$

is a homeomorphism onto a compact subspace of $\Phi_{\mathscr{A}}$, which we identify with S_i. Clearly $S_i \cap S_j = \emptyset$ when $i, j \in I$ with $i \neq j$. For each $i \in I$, we have $S_i = \{\phi \in \Phi_{\mathscr{A}} \mid \phi(\delta_i) = 1\}$, and so S_i is clopen in $\Phi_{\mathscr{A}}$. Further, $U_{\Phi_{\mathscr{A}}} = \bigcup_{i\in I} S_i$ and $U_{\Phi_{\mathscr{A}}}$ is a dense, open subspace of $\Phi_{\mathscr{A}}$. //

We have to consider a generalization of σ-finite measures:

Definition 1.4.8. A measure space $(\Gamma, \mathfrak{B}, \mu)$ is *decomposable* if there is a subfamily $\mathcal{U}$ of $\mathfrak{B}$ that partitions X such that:

(i) $0 \leq \mu(U) < \infty, \quad U \in \mathcal{U}$.

(ii) $\mu(B) = \sum_{U\in\mathcal{U}} \mu(U \cap B)$ for each $B \in \mathfrak{B}$ with $\mu(B) < \infty$.

(iii) $B \in \mathfrak{B}$ for each $B \subseteq \Gamma$ such that $B \cap U \in \mathfrak{B}$ for $U \in \mathcal{U}$.

Not all properties that are true for σ-finite measures hold true for decomposable measures. The duality of the spaces L^1 and L^∞, thus, still applies. The proof of the following can be found in [3, Theorem 20.19, p. 351].

Theorem 1.4.9. *Let $(\Gamma, \mathfrak{B}, \mu)$ be a decomposable measure space. Then $\left(L^1(\Gamma, \mu), \|.\|_1\right)'$ is isometrically isomorphic to $(L^\infty(\Gamma, \mu), \|.\|_\infty)$.*

Example 1.4.10. Let X be a non-empty, Stonean space and let $(\mu_i)_{i\in I}$ be a maximal singular family in $N(X) \cap P(X)$ and set $S_i = \text{supp } \mu_i$. Now take Γ to be the union of the family $(S_i)_{i\in I}$ and set

$$\mu = \sum_{i\in I} \mu_i.$$

Then μ is a decomposable measure as in *Definition* 1.4.8:
(*i*) Since $\mu_i(X) = 1$ for all $i \in I$, it follows that $0 \leq \mu(S_{i_0}) = \mu_{i_0}(S_{i_0}) \leq \mu_{i_0}(X) < \infty$.
(*ii*) The family $(S_i)_{i\in I}$ consits of pairwise disjoint sets, so

$$\mu(B) = \sum_{i\in I} \mu(B \cap S_i) = \sum_{i\in I} \mu_i(B).$$

(*iii*) This is trivial. //

2 $C(X)$ as dual space of a Banach space

2.1 Dual space theorem

Theorem 2.1.1. *Let X be a non-empty compact space. Then the following statements are equivalent.*

(i) $C(X)$ is isometrically a dual space;

(ii) there is a C^*-isomorphism

$$T : \begin{cases} f \mapsto f|_{S_i} \\ C(X) \to \bigoplus_{\infty} L^\infty(S_i, \mu_i) \end{cases}$$

where $(\mu_i)_{i \in I}$ is a maximal singular family in $N(X) \cap P(X)$ and we are setting $S_i = \text{supp}\ \mu_i$, $i \in I$;

(iii) the map $T : C(X) \to N(X)'$ defined by

$$(Tf)(\mu) = \langle f, \mu \rangle = \int_X f\ d\mu$$

is an isometric isomorphism, and so $C(X) \cong N(X)'$.

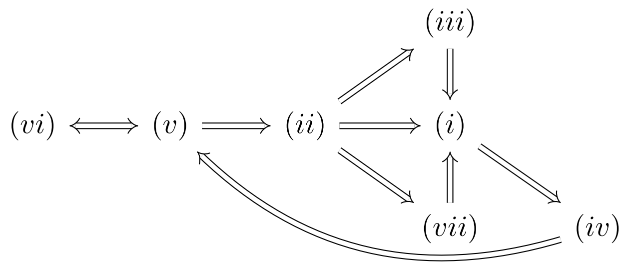
(iv) X is Stonean and, for each $f \in C(X)^+$ with $f \neq 0$, there exists $\mu \in N(X)^+$ with $\langle f, \mu \rangle \neq 0$;

(v) X is hyper-Stonean;

(vi) X is Stonean and there exists a category measure on X;

(vii) there is a locally compact space Γ and a decomposable measure μ on Γ such that $C(X)$ is C^*-isomorphic to $L^\infty(\Gamma, \mu)$.

Proof.
We are going to establish the following implications:



"$(ii) \Rightarrow (i)$" This is trivial.
"$(iii) \Rightarrow (i)$" This is trivial.
"$(i) \Rightarrow (iv)$" By *Proposition* 1.3.5, there exists a real-linear subspace Y of $M_{\mathbb{R}}(X)$ with $Y' = C_{\mathbb{R}}(X)$. The space $(B_1^{C_{\mathbb{R}}(X)}(0), \sigma(C_{\mathbb{R}}(X), Y))$ is compact. Since the map

$$\psi : \begin{cases} f \mapsto \frac{1}{2}(1 + f) \\ B_1^{C_{\mathbb{R}}(X)}(0) \to B_1^{C(X)}(0)^+ \end{cases}$$

is a bijection which is a homeomorphism with respect to the weak*-topology and so $B_1^{C(X)}(0)^+$ is compact as the continuous image of a compact set. By the *Krein-Šmulian* theorem [7, Theorem 6.3.4, p.121], the positive cone is closed. Take $f \in C_{\mathbb{R}}(X) \setminus C(X)^+$. Then, by the Hahn-Banach theorem, there exists

$$\lambda \in (C_{\mathbb{R}}(X), \sigma(C_{\mathbb{R}}(X), Y))' = Y$$

with

$$\int_X f \ d\lambda < \inf_{g \in C_{\mathbb{R}}(X)^+} \int_X g \ d\lambda.$$

It cannot be that

$$\int_X g \ d\lambda < 0$$

for some $g \in C_{\mathbb{R}}(X)^+$: indeed this would imply that

$$\int_X ng \ d\lambda < \int_X f \ d\lambda$$

for some $n \in \mathbb{N}$, a contradiction, and so

$$\inf_{g \in C_{\mathbb{R}}(X)^+} \int_X g \ d\lambda = 0.$$

Thus $\lambda \in Y^+$. It follows that, for each $f \in C_{\mathbb{R}}(X)$, we have $f \geq 0$ if and only if

$$0 \leq \int_X f \ d\lambda, \quad \lambda \in Y^+.$$

Let $(f_i)_{i \in I}$ be an increasing net in $B_1^{C_{\mathbb{R}}(X)^+}(0)$. Then $(f_i)_{i \in I}$ has an accumulation point, say f_0, in the unit ball of $C_{\mathbb{R}}(X)^+$ endowed with $\sigma(C_{\mathbb{R}}(X), Y))$. By passing to a subnet we may suppose that $\lim_{j \in J} f_{i_j} = f_0$ with respect to the weak*-topology. For each $\lambda \in Y^+$, the net $(\int_X f_i \ d\lambda)_{i \in I}$ is increasing and bounded. So it converges to the limit of the subnet, and hence to $\int_X f_0 \ d\lambda$, and so

$$\int_X f_i \ d\lambda \leq \int_X f_0 \ d\lambda, \quad i \in I.$$

It follows that $f_i \leq f_0$ for $i \in I$. Suppose that $g \in C(X)^+$ with $f_i \leq g$ for all $i \in I$. Then

$$\int_X f_i \ d\lambda \leq \int_X g \ d\lambda, \quad \lambda \in Y^+,$$

for each $i \in I$, and so

$$\int_X f_0 \ d\lambda \leq \int_X g \ d\lambda, \quad \lambda \in Y^+.$$

This implies that $f_0 \leq g$ and hence that $f_0 = \sup_{i \in I} f_i$. Thus $C(X)$ is Dedekind complete, thus X is a Stonean space.
Next suppose that $\mu \in Y$ and $g_i \searrow 0$ in $C_{\mathbb{R}}(X)$. Then

$$1 = \sup_{i \in I} (1 - g_i)$$

and we know from the first part of the proof that $1 - g_i \stackrel{w^*}{\to} 1$, hence we get

$$\lim_{i \in I} \int_X g_i \, d\mu = 0.$$

This shows that μ is normal. Thus, $Y \subseteq N(X)$. For each $f \in C(X)^+$ with $f \neq 0$, there exists $\mu \in Y^+$ with

$$\int_X f \, d\mu \neq 0,$$

since $Y^+ \subseteq N(X)^+$.
"$(iv) \Rightarrow (v)$" Let U be a non-empty, open subset of the Stonean space X. Then there exists $f \in C(X)^+$ with $f \neq 0$ such that $\operatorname{supp} f \subseteq U$. By (iv), there exists $\mu \in N(X)^+$ with

$$\int_X f \, d\mu \neq 0.$$

Clearly $\operatorname{supp} \mu \cap U \neq \emptyset$. This shows that W_X is dense in X, and so X is hyper-Stonean.
"$(v) \Leftrightarrow (ii)$" Since X is Stonean and $U_{\mathscr{F}}$, from *Remark* 1.4.7, is dense in X, the map

$$\psi : \begin{cases} f \mapsto f|_{U_{\mathscr{F}}} \\ C(X) \to C^b(U_{\mathscr{F}}) \end{cases}$$

is a unital C^*-isomorphism. The map

$$\phi : \begin{cases} f \mapsto f|_{S_i} \\ C^b(U_{\mathscr{F}}) \to \bigoplus_{\infty} C(S_i) \end{cases}$$

is clearly a unital C^*-isomorphism. For each $i \in I$, the measure μ_i is normal, and so $L^\infty(S_i, \mu_i) = C(S_i)$.
"$(ii) \Rightarrow (iii)$" Since $(ii) \Rightarrow (i) \Rightarrow (iv)$, the space X is Stonean, and the spaces S_i are pairwise disjoint. Set $Y = \bigoplus_1 L^1(S_i, \mu_i)$, so that $Y' = \bigoplus_\infty L^\infty(S_i, \mu_i)$. The map

$$T' : Y'' \to M(X)$$

is an isometric isomorphism. We will show that T' maps Y onto $N(X)$. Take $y = (y_i)_{i \in I}$ in Y and set

$$\lambda = T'y \in M(X).$$

Take $f \in C(X)$, and, for each i, set $f_i = f|_{S_i}$, so that

$$\int_X f \, d\lambda = \langle f, \lambda \rangle = \langle Tf, y \rangle = \sum_{i \in I} \int_{S_i} f_i y_i \, d\mu_i, \tag{2.1}$$

where we note that

$$\int_{S_i} f_i y_j \, d\mu_i = 0, \quad i \neq j.$$

Take $C \in \mathscr{K}_X$. Then, for each $i \in I$, we have $C \cap S_i \in \mathscr{K}_X$ and $\mu_i \in N(X)$, and so $\mu_i(C \cap S_i) = 0$. By *Equation* (2.1) with $f = \mathbb{1}_C$, we have $\lambda(C) = 0$. By *Theorem* 1.1.9, $\lambda \in N(X)$.
Conversely, take $\lambda \in N(X)$. Then $|\lambda| \left(X \setminus \bigcap_{i \in I} S_i\right) = 0$. For each $i \in I$, it follows that $\lambda|_{S_i} \ll \mu_i$,

and so, by the Radon-Nikodym theorem, there exists $y_i \in L^1(S_i, \mu_i)$ with $\lambda|_{S_i} = y_i \; d\mu_i$ and $\|y_i\|_1 = \|\lambda|_{S_i}\|$. Set $y = (y_i)_{i \in I}$. Then

$$\sum_{i \in I} \|y_i\|_1 = \sum_{i \in I} \|\lambda|_{S_i}\| = \|\lambda\| \,,$$

so that $y \in Y$, and

$$\int_X f \; d\lambda = \sum_{i \in I} \int_{S_i} f_i y_i \; d\mu_i,$$

whence $T'y = \lambda$. It follows that $C(X) \cong N(X)'$. When we identify Y and $N(X)$, we obtain the formula.
"$(v) \Leftrightarrow (vi)$" This follows from *Proposition* 1.4.3.
"$(vii) \Rightarrow (i)$" This follows from *Theorem* 1.4.9 because $L^\infty(\Gamma, \mu) \cong L^1(\Gamma, \mu)'$.
"$(ii) \Rightarrow (vii)$" We take Γ to be the pairwise disjoint union of the family $(S_i)_{i \in I}$, and set $\mu = \sum_{i \in I} \mu_i$. We have seen in *Example* 1.4.10 that μ is decomposable. It is clear that $C(X)$ is C^*-isomorphic to $L^\infty(\Gamma, \mu)$. ■

Definition 2.1.2. A C^*-algebra Z is a *von Neumann algebra* if there is a Hilbert space H such that Z is a C^*-subalgebra of $\mathcal{B}(H)$ and Z closed in the weak operator topology.

Theorem 2.1.1 will help us proving, that every commutative C^*-algebra that is isometrically isomorphic to a dual space is a von Neumann algebra.

Definition 2.1.3. Let Z be a subset of $\mathcal{B}(H)$, for a Hilbert space H. Then the *commutant* of Z is

$$Z^{\complement} = \{T \in \mathcal{B}(H) \,|\, TS = ST, \; S \in Z\}.$$

Theorem 2.1.4. *Let H be a Hilbert space, and let Z be a C^*-subalgebra of $\mathcal{B}(H)$. Then* $\overline{Z}^{wo} = Z^{\complement\complement}$.

A proof of this can be found, e.g., in [1, Theorem 3.2.32].

Example 2.1.5. Let Z be a commutative C^*-algebra which is isometrically a dual space. As we have already seen Z is isometrically isomorphic to $C(X)$ for a compact space X. Now by *Theorem* 2.1.1, there is a locally compact space Γ and a decomposable measure μ on Γ such that $C(X)$ is C^*-isomorphic to $L^\infty(\Gamma, \mu)$. We show that $L^\infty(\Gamma, \mu)$ is a von Neumann algebra. Take H to be the Hilbert space $L^2(\Gamma, \mu)$, and for $f \in L^\infty(\Gamma, \mu)$, define

$$M_f(g) = fg, \quad g \in L^2(\Gamma, \mu).$$

Then $M_f \in \mathcal{B}(L^2(\Gamma, \mu))$ and the set $N := \{M_f \,|\, f \in L^\infty(\Gamma, \mu)\}$ is a C^*-subalgebra of $\mathcal{B}(L^2(\Gamma, \mu))$. The map

$$\psi : \begin{cases} L^\infty(\Gamma, \mu) \to N \\ f \mapsto M_f \end{cases}$$

is a C^*-isomorphism. N is a C^*-subalgebra and if N is closed in the weak operator topology it is even a von Neumann algebra. To show this we will make use of *Theorem* 2.1.4. We even show that $N = N^{\complement}$.
Let $T \in N^{\complement}$ with $T \neq 0$ and let $f = T(\mathbb{1}_\Gamma) \in L^2(\Gamma, \mu)$. We have to show that f belongs to $L^\infty(\Gamma, \mu)$ and T is M_f. We claim that the essential supremum of $|f|$ is less than $\|T\|$. Assume the contrary, then there exists a measureable set $A \subseteq \Gamma$ of positive measure such that $|f| > \|T\|$ on A. Define a function

$$g : \begin{cases} \Gamma \to \mathbb{C} \\ x \mapsto \mathbb{1}_A \frac{1}{f(x)} \end{cases}$$

Then $g \in L^\infty(\Gamma, \mu)$, so we have

$$T(g) = T(M_g(\mathbb{1}_\Gamma)) = M_g(T(\mathbb{1}_\Gamma)) = gT(\mathbb{1}_\Gamma) = gf. \tag{2.2}$$

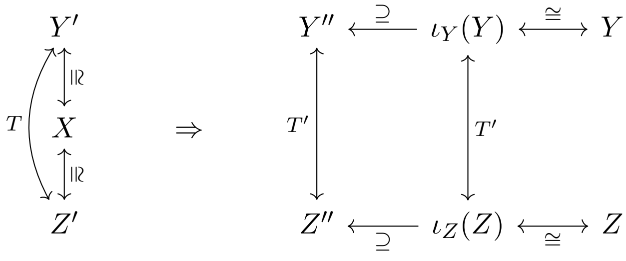
Now since $gf \equiv \mathbb{1}_A$, we have

$$\mu(A) = \|gf\|^2_{L^2(\Gamma,\mu)} = \|T(g)\|^2_{L^2(\Gamma,\mu)} \le \|T\|^2 \|g\|^2_{L^2(\Gamma,\mu)} < \|T\|^2 \frac{\mu(A)}{\|T\|^2} = \mu(A).$$

A contradiction and so $f \in L^\infty(\Gamma, \mu)$. Moreover, *Equation* (2.2) shows that $T(g) = gf$ for g in the dense subset $L^\infty(\Gamma, \mu)$ of $L^2(\Gamma, \mu)$ and we get $T = M_f$.
So we see that $L^\infty(\Gamma, \mu)$ satisfies the conditions of *Definition* 2.1.2 and is a von Neumann algebra. //

2.2 Uniqueness of the predual

Definition 2.2.1. Let X be a Banach space with an isometric predual Y. Then X has a *strongly unique* predual Y if, whenever Z is also a Banach space and $T : Z' \to Y'$ is an isometric isomorphism, the map $T' : Y'' \to Z''$ carries $\iota_Y(Y)$ onto $\iota_Z(Z)$.



Lemma 2.2.2. *Let Y and Z be Banach spaces. A linear map $T : Z' \to Y'$ is weak*-weak*-continuous if and only if $T = S'$ for some bounded operator $S : Y \to Z$.*

Proof.
"⇒" Since T is weak*-weak*-continuous, take $y \in Y$, then $\iota(y) \circ T$ is weak*-continuous on Z and so it is of the form $\iota(S(y))$ for a unique $S(y) \in Y$. Since $S(y)$ is uniquely determined, it follows that S is linear. Now S is continuous by the closed graph theorem. If $y_n \to y$ and $Sy_n \to z$ then for each z' on Z' we have

$$\langle z, z' \rangle_{Z,Z'} = \lim_{n\to\infty} \langle Sy_n, z' \rangle_{Z,Z'} = \lim_{n\to\infty} \langle y_n, Tz' \rangle_{Y,Y'} = \langle y, Tz' \rangle_{Y,Y'} = \langle Sy, z' \rangle_{Z,Z'}$$

and thus $z = Sy$. Hence S is bounded.
"⇐" Conversely, to see that the dual of a bounded operator is weak*-weak*-continuous, we take a net $z'_i \overset{w^*}{\to} z'$. Then we get

$$\langle y, Tz'_i - Tz' \rangle_{Y,Y'} = \langle Sy, z'_i - z' \rangle_{Z,Z'} \to 0$$

and hence the claim follows.

■

The following proposition can be useful to see when isometric isomorphisms have dual operators that take $\iota_Z(Z)$ onto $\iota_Y(Y)$.

Proposition 2.2.3. *The dual of an isometric isomorphism T is weak*-weak*-continuous if and only if T' maps $\iota_Y(Y)$ onto $\iota_Z(Z)$.*

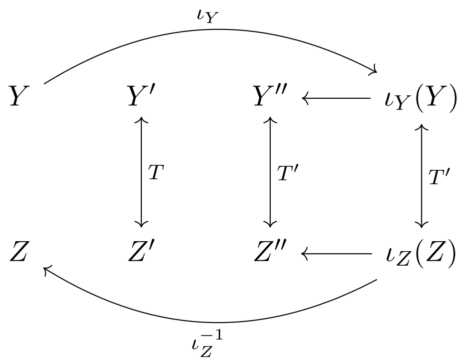
Proof.
"$\Rightarrow$" We have

$$\langle z', T' \circ \iota_Y(y) \rangle_{Z',Z''} = \langle Tz', \iota_Y(y) \rangle_{Y',Y''} = \langle y, Tz' \rangle_{Y,Y'} .$$

By *Lemma* 2.2.2 there is a bounded operator S with $S' = T$. This leads to

$$\langle z', T' \circ \iota_Y(y) \rangle_{Z',Z''} = \langle y, S'z' \rangle_{Y,Y'} = \langle Sy, z' \rangle_{Z,Z'} = \langle z', \iota_Z(Sy) \rangle_{Z',Z''} .$$

Since S is bijective, T' is a bijection between $\iota_Y(Y)$ and $\iota_Z(Z)$.
"$\Leftarrow$" We define a map S by the diagram below, $S = \iota_Z^{-1} \circ T' \circ \iota_Y$.



If we can show that $S' = T$ the statement follows by *Lemma* 2.2.2. We compute

$$\begin{aligned} \langle y, S'z' \rangle_{Y,Y'} &= \langle Sy, z' \rangle_{Z,Z'} = \left\langle \iota_Z^{-1} \circ T' \circ \iota_Y(y), z' \right\rangle_{Z,Z'} \\ &= \langle z', T' \circ \iota_Y(y) \rangle_{Z',Z''} = \langle Tz', \iota_Y(y) \rangle_{Y',Y''} = \langle y, Tz' \rangle_{Y,Y'} . \end{aligned}$$

And T is weak*-weak*-continuous. ■

Theorem 2.2.4. *Let X be a non-empty, hyper-Stonean space. Then $N(X)$ is a strongly unique predual of $C(X)$.*

Proof. Suppose that Y is an isometric predual of $C(X)$. Then we can regard Y as a closed linear subspace of $M(X)$, and we have noted in the proof of *Theorem* 2.1.1 in implication "$(i) \Rightarrow (iv)$" that $Y \subseteq N(X)$. Now assume that there is $\mu \in N(X) \setminus Y$. With the Hahn-Banach theorem we get

$$\exists f \in N(X)' : f(Y) \leq \gamma_1 < \gamma_2 \leq f(\mu) \Rightarrow f(Y) = 0$$

but Y operates seperating on $C(X)$ and so $f = 0$. Thus, we have $f(\mu) \neq 0$, a contradiction. Hence, $Y = N(X)$.
Next suppose that Z is a Banach space and that

$$T : N(X)' \to Z'$$

is an isometric isomorphism. By *Lemma* 2.2.2, we know that T' is weak*-weak*-continuous. Now we endow Z' with $\sigma(Z', Z)$ and $C(X)$ with $\sigma(C(X), N(X))$. It now follows that $T'(\iota_Z(Z)) \subseteq \iota_{N(X)}(N(X))$. The first part of the proof now applies to show that $T'(\iota_Z(Z)) = \iota_{N(X)}(N(X))$. Thus $N(X)$ is the strongly unique predual of $C(X)$. ■

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