REPRESENTATION THEORY OF C*-ALGEBRAS

Contents

1. Introduction

When talking about Banach algebras one often has in mind the common example of bounded operators on a Hilbert space. However these operator algebras have more structure than just that of a Banach algebra - for example the adjoint operation. C*-algebras are Banach algebras that in addition have such an adjoint operation. The major part of this work will be concerned with showing that the Banach algebra structure together with this adjoint operation characterizes operator algebras. In other words we will show that every C*-algebra is isomorphic to an algebra of operators (that is closed under adjoints) on a Hilbert space.

Section 2 reviews basic results and definitions concerning operators on Hilbert spaces, Banach algebras and C*-algebras. Most important the representation theory for commutative unital C*-algebras will be discussed. The results of Section 2 will be given without proofs. For proofs and further results the reader is referred to [2, 3].

Section 3 starts with the unitization construction. Using this we will extend the result of section 2 to non unital commutative C*-algebras. We then go on with introducing positive elements and establishing some of their most important properties. The section ends with a proof for the existence of approximate units.

Section 4 starts with a general treatment of *-representations. We will work out different characterizations of irreducibility using von Neumann's bicommutant theorem. Afterwards we will introduce the notion of positive functionals in order to obtain representations via the Gelfand-Naimark-Segal construction (GNS construction). Ultimately, considering direct sums of GNS-representations, we prove the Gelfand-Naimark theorem which ensures the existence of an irreducible isometric representation for every C*-algebra.

Most of the results and sketches of proofs are taken from [1]. Furthermore some results and proofs are taken from [4].

2. Basics

In this first part we will review important definitions and facts. For proofs cf. [2].

2.1. Operators on Hilbert spaces.

Definition 1. Given a Hilbert space H, we define $B(H)$ as the space of bounded linear operators on H.

Later on, we will have to deal with direct sums of Hilbert spaces and operators on them.

Definition 2. Let $(H_i)_{i\in\,\Omega}$ be a family of Hilbert spaces. The *direct sum* $\underset{i\in\Omega}{\oplus} H_i$ is the space of tuples $(\xi_i)_{i\in\Omega}$, where $\sum\limits_{i\in\Omega} \|\xi_i\|^2 < \infty$. With the inner product given by

$$((\xi_i)_{i\in\Omega}, (\eta_i)_{i\in\Omega}) = \sum_{i\in\Omega} (\xi_i, \eta_i)$$

this space becomes a Hilbert space.

Given operators $T_i \in B(H_i),\ i \in \Omega$ such that $\sup\limits_{i\in\Omega} \|T_i\| < \infty$, one can consider their *direct sum* $\underset{i\in\Omega}{\oplus} T_i$ defined by

$$(\underset{i\in\Omega}{\oplus} T_i)(\xi_i)_{i\in\Omega} = (T_i\xi_i)_{i\in\Omega}.$$

$\underset{i\in\Omega}{\oplus} T_i$ is bounded with $\left\|\underset{i\in\Omega}{\oplus} T_i\right\| = \sup\limits_{i\in\Omega} \|T_i\|$.

If all H_i are the same H, the direct sum is called the *amplification* of H by $card(\Omega)$. In that case, the amplification $T^\sim \in B(\underset{i\in\Omega}{\oplus} H)$ of a $T \in B(H)$ is defined by

$$T^\sim = \underset{i\in\Omega}{\oplus} T_i.$$

Recall some of the most important topologies on $B(H)$. For their definition we remember, that a topology is characterized by its convergent nets.

Definition 3. Let H be a Hilbert space

- The *weak topology* is defined by $T_i \to T \iff (T_i\xi, \eta) \to (T\xi, \eta), \quad \xi, \eta \in H$.
- The *strong topology* is given by $T_i \to T \iff \|(T - T_i)\xi\| \to 0, \quad \xi \in H$.

Remark 4. The strong topology is induced by the family of semi-norms $p_x : T \mapsto \|Tx\|$. Let X be a vector space with a topology induced by a family of semi-norms $(p_i)_{i\in I}$. We recall, that for every $x \in X$ a neighborhood basis is given by

$$\{\{y \in X : p_i(x - y) < \epsilon,\, i \in J\} : J \subseteq I \text{ finite}, \epsilon > 0\}.$$

We will also need the Spectral Theorem for self-adjoint operators. For the proof and the definition of spectral measures cf. Section 7 in [2].

Theorem 5. *Let H be a Hilbert space and $A \in B(H)$ be a self-adjoint operator. Let $\mathcal{A}$ be the σ-algebra of Borel sets on $\sigma(A)$. Then there exists a unique spectral measure E on $(\sigma(A), \mathcal{A}, H)$, such that*

$$A = \int t\, dE(t).$$

Moreover for $T \in B(H)$

$$TA = AT \iff TE(\Delta) = E(\Delta)T, \quad \Delta \in \mathcal{A}.$$

The following definition will be useful.

Definition 6. Let $B \subseteq B(H)$ and $X \subseteq H$. Set

$$BX := \{Tx \,:\, T \in B,\, x \in X\}$$

Finally, we will state a standard argument as a lemma.

Lemma 7. *Let H be a Hilbert space, $S, T \in B(H)$ and $D \subseteq H$ dense. If*

$$(S\xi, \eta) = (T\xi, \eta) \text{ for all } \xi, \eta \in D,$$

then $S = T$.

Proof. We have $((S - T)\xi, \eta) = 0$ for all $\xi, \eta \in D$. By the density of D and continuity of the inner product, this holds for $\eta \in H$. Thus $(S - T)\xi \in H^{\perp} = \{0\}$ for all $\xi \in D$. Again the density of D yields $S = T$. □

2.2. **Banach algebras.**

Definition 8. A *Banach algebra* is a Banach space $(A, \|.\|)$ together with an associative and distributive multiplication $\cdot : A \times A \to A$, such that

$$\|x \cdot y\| \leq \|x\|\|y\| \text{ for all } x, y \in A.$$

It is called *unital* if there exists a multiplicative unit. This unit is unique and will be denoted by 1. The set of invertible elements is denoted by

$$Inv(A) = \left\{x \in A : \exists x^{-1} \in A : xx^{-1} = x^{-1}x = 1\right\}.$$

Furthermore the *spectrum* of an element $x \in A$ is given by

$$\sigma(x) = \{\lambda \in \mathbb{C} : \quad x - \lambda 1 \notin Inv(A)\}.$$

The *spectral radius* is given by

$$r(x) = \max\{|\lambda| : \lambda \in \sigma(x)\}.$$

Remark 9. Note, that for $x \in A$, the multiplicative inverse x^{-1} is unique.

The following properties concerning the spectrum hold true.

Theorem 10. *Let A be a unital Banach algebra and $x, y \in A$. Then*

(1) $\sigma_A(x)$ is nonempty and compact.
(2) $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$. In particular $r(x) \leq \|x\|$.
(3) $\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}$
(4) If B is a Banach sub-algebra of A containing 0, then $\sigma_A(x) \subseteq \sigma_B(x)$ and $\delta(\sigma_B(x)) \subseteq \delta(\sigma_A(x))$ where δ is the topological boundary. Equivalently, $\rho_B(x)$ is open and closed as subset of $\rho_A(x)$.
(5) $\sigma_A(x^{-1}) = \left\{\lambda^{-1} : \lambda \in \sigma_A(x)\right\}$

Many proofs that technically assume the Banach algebra to be unital, can be extended to the non unital case by the following construction.

Lemma 11. *Let A be a non unital Banach algebra. Then $A_1 := A \oplus \mathbb{C}$ with coordinate wise addition, multiplication defined by $(x;\lambda)(y;\mu) = (xy + \mu x + \lambda y; \lambda\mu)$ and $\|(x;\lambda)\|_1 := \|x\| + |\lambda|$ is a unital Banach algebra.*

Elements of a Banach algebra A can also be viewed as elements of $B(A)$, if we identify them with their respective multiplication operators.

Definition 12. Let A be a Banach algebra. For $x \in A$, we define the *multiplication operator* $M_x \in B(A)$ by

$$y \mapsto xy.$$

Remark 13. The map $x \mapsto M_x$ from A to $B(A)$ is a homomorphism. Moreover, since $\|xy\| \leq \|x\| \, \|y\|$, we have $\|M_x\| \leq \|x\|$.

2.3. C*-Algebras.

Definition 14. A *Banach*-algebra* is a Banach algebra A with a conjugate-linear and isometric involution $* : A \to A$ such that

$$(xy)^* = y^* x^* \text{ for all } x, y \in A.$$

If additionally the so called C*-axiom

$$\|x^* x\| = \|x\|^2 \text{ for all } x \in A$$

holds, A is called a *C*-algebra*.

Let A, B be Banach*-algebras. An algebra-homomorphism $\phi : A \to B$ that satisfies $\phi(x^*) = \phi(x)^*$ for all $x \in A$ is called a **-homomorphism.*

A sub-algebra $S \subseteq A$, that is *-closed, i.e. $S^* \subseteq S$, is called a **-sub-algebra.*

Example 15. The following examples are fundamental in the sense that all C*-algebras, as we will see, can be represented as a *-sub-algebra of one of them.

(1) (Unital commutative C*-algebras) Let $C(X)$ be the set of all continuous complex-valued functions on a compact topological space X. Equipped with point-wise operations, the point-wise complex-conjugate and the norm given by $\|f\| = \sup \{f(x) : \quad x \in X\}$ it is a C*-algebra.
(2) (Non unital commutative C*-algebras) We consider the space

$$C_0(X) = \{f \in C(X) : \forall \epsilon > 0 \, \exists K \, compact : \, f(X \setminus K) \subseteq U_\epsilon(0)\}$$

for a locally compact space X. With the same structure as above one again obtains a C*-algebra.
(3) (Non commutative C*-algebras) Consider the space $B(H)$ of bounded linear operators on a Hilbert space H. With the operation which maps an operator to its adjoint, the usual algebraic operations - multiplication being the composition of operators - and the operator-norm, $B(H)$ is a C*-algebra.

In analogy to operators we can define normal, self-adjoint and unitary elements for C*-algebras.

Definition 16. For a *-algebra A, $x \in A$ is called

- *normal* if $x^* x = x x^*$
- *self-adjoint* if $x^* = x$
- *unitary* if $x^* = x^{-1}$for unital A.

Remark 17. The set of self-adjoint elements is denoted by A_{sa}.

We recall the following facts.

Theorem 18. *Let A be a unital C*-algebra and $x \in A$.*

(1) If x is unitary, then $\sigma_A(x) \subseteq \mathbb{T}$ where $\mathbb{T}$ denotes the unit circle.

(2) If x is self-adjoint then $\sigma_A(x) \subseteq \mathbb{R}$.

Lemma 19. *Let B be a unital C*-sub-algebra of a unital C*-algebra A (with the same unit), then $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$.*

A main characteristic of C*-algebras is, that algebraic (spectrum) properties are strongly linked to topological (norm) properties. This connection roots in the innocuous looking C*-axiom.

Lemma 20. *For a unital C*-algebra, let $x \in A$ be normal. Then $\|x\| = r(x)$ where $r(x)$ is the spectral radius. Furthermore, $\|x\|^2 = r(x^*x)$ for arbitrary $x \in A$.*

This also results in additional structure for *-homomorphisms.

Lemma 21. *Let $\varphi : A \to B$ be a *-homomorphism between unital C*-algebras A, B, such that $\varphi(1_A) = 1_B$. Then φ is automatically bounded with $\|\varphi\| \leq 1$. Moreover, $\sigma(\varphi(x)) \subseteq \sigma(x)$ $x \in A$.*

Remark 22. The previous lemma also holds true for the weaker assumption of A being a Banach *-algebra.

2.4. **Unital Commutative C*-algebras.** For the proofs in this section, the reader is referred to Section 1 in [3].

Definition 23. Let A be a C*-algebra. A *multiplicative functional* $\phi : A \to \mathbb{C}$ is a linear functional additionally satisfying $\phi(ab) = \phi(a)\phi(b)$. The *Gelfand space or spectrum* of A is defined by

$$\hat{A} := \{\phi : A \to \mathbb{C} |\, \phi \, is \, multiplicative; \, \phi \neq 0\}.$$

Lemma 24. *Let $\phi \neq 0$ be a multiplicative functional on a C*-algebra A. Then $\|\phi\| = 1$. If A is unital, then $\phi(1) = 1$.*

We recall that the w^*-topology on any subset $Y \subseteq A^*$ of the topological dual of a Banach space A, is the initial topology with respect to the maps

$$\iota(a) : Y \to \mathbb{C}; \; \phi \mapsto \phi(a),$$

where $a \in A$. We denote it by $\sigma(Y, \iota(A))$. The following Banach-Alaoglu type theorem for unital C*-algebras holds true.

Theorem 25. *Let A be a unital C*-algebra. Its Gelfand space $\hat{A}$ is a compact subspace of the topological dual space A^* with respect to the w^*-topology.*

Remark 26. From now on, if not otherwise stated explicitly, we assume $\hat{A}$ to be endowed with the w^*-topology.

The main result of this section is the following representation theorem.

Theorem 27. *Let A be a commutative unital C*-algebra. The map $\iota : A \to C(\hat{A})$ defined by $\iota(a) = (\phi \mapsto \phi(a))$ is an isometric isomorphism.*

Definition 28. Let A be a unital C*-algebra. For a normal $x \in A$ define

$$C^*(x) := c.l.s. \left\{ x^j (x^*)^k : j, k \in \mathbb{N}_0 \right\}$$

where $x^0 := 1$.

More generally for commuting $x_1, x_2, \ldots, x_n$, each of which is normal, we define $C^*(x_1, x_2, \ldots, x_n)$ to be the closed linear span of the products of the powers of the x_k and x_k^*.

Remark 29. It is easy to see, that $C^*(x)$ is a commutative C*-sub-algebra of A. Indeed, it is the smallest one containing x and 1.

As a corollary of Theorem (27) we obtain the continuous functional calculus for normal elements of unital C*-algebras.

Corollary 30. *Let A be a unital C*-algebra. For any normal element $x \in A$, $C^*(x)$ is isometrically isomorphic to $C(\sigma(x))$. The isomorphism Φ maps x to the identity $id : \sigma(x) \to \mathbb{C};\ t \mapsto t$. For $f \in C(\sigma(x))$ we define $f(x) := \Phi^{-1}(f)$. The following properties hold*

(1) $\sigma(f(x)) = f(\sigma(x))$.
(2) If $f \in C(\sigma(x))$, $g \in C(\sigma(f(x)))$, then $g \circ f \in C(\sigma(x))$ and $g \circ f(x) = g(f(x))$.

Proof. The proofs found in Section 1 in [3] may be applied to the present situation. □

Remark 31. This functional calculus may be used to prove the spectral theorem for normal operators.

Lemma 32. *Let A, B be unital C*-algebras, $\varphi : A \to B$ a *-homomorphism between them, such that $\varphi(1_A) = 1_B$ and $x \in A$ normal. Then $\varphi(f(x)) = f(\varphi(x))$ for any $f \in C(\sigma(x))$.*

Proof. We consider the unital commutative sub-algebras $C^*(x) \subseteq A$, $C^*(\varphi(x)) \subseteq B$ and the induced homomorphism between them. If $f(z) = \sum_{j,k=0}^{n} \lambda_{j,k} z^j \bar{z}^k$ is a polynomial, the claim is obvious. By Stone-Weierstrass these polynomials are dense in $C(\sigma(x))$. From Lemma (21) we know that $\sigma(\varphi(x) \subseteq \sigma(x)$. Denote by $|_{\sigma(\varphi(x))} : C(\sigma(x)) \to C(\sigma(\varphi(x))$ the restriction mapping and by Φ_x ($\Phi_{\varphi(x)}$) the mapping from Corollary (30) applied to x ($\varphi(x)$).We just showed, that

$$\varphi \circ \Phi_x^{-1}(f) = \Phi_{\varphi(x)}^{-1} \circ |_{\sigma(\varphi(x))}(f)$$

for any $f \in \mathbb{C}[z, \bar{z}]$ viewed as an element of $C(\sigma(x))$. According to the continuity of all involved mappings and due to the density of $\mathbb{C}[z, \bar{z}]$ we see that the above equation holds true for all $f \in C(\sigma(x))$, i.e. $\varphi(f(x)) = f(\varphi(x))$. □

As a first application of the functional calculus, we will strengthen the statement from Lemma (21).

Lemma 33. *Let $\phi : A \to B$ be a *-homomorphism such that $\phi(1_A) = 1_B$ between unital C*-algebras A, B. If ϕ is injective it is an isometry.*

Proof. We already know, that $\|\phi(x)\| \leq \|x\|$ for all $x \in A$, cf. Lemma (21). Let us suppose there is an $x \in A$, such that this inequality is strict. Then also

$$r := \|\phi(x^*x)\| < \|x^*x\| =: s.$$

We now consider the C*-algebra $C^*(x^*x)$ (note, that x^*x is normal) and the induced injective *-homomorphism $\phi : C^*(x^*x) \to B$. It is indeed injective since for $x \in A \cap C^*(x^*x)$ we have $\phi(x + \lambda 1) = \phi(x) + \lambda 1$ where $\phi(x)$ and 1 are obviously linearly independent. Consider the function $f := g|_{\sigma(x^*x)}$, where $g \in C([-s, s])$ vanishing on $[-s, r]$ with $g(s) = 1$. By the functional calculus we have

$$0 = f(\phi(x^*x)) = \phi(f(x^*x)).$$

Since f does not vanish on $\sigma(x^*x)$, the equality above contradicts the injectivity of ϕ. □

3. Further Structure

In the following $A, B, ...$ will always denote a C*-algebra unless otherwise stated. Moreover, the word functional always means linear functional.

3.1. **Unitization.**

A C*-algebra need not be unital. However, there is a canonical way to construct an enveloping unital C*-algebra.

Remark 34. If A is a C*-algebra, the map $x \mapsto M_x$ (cf. Definition (12)) is an isometric homomorphism. Only $\|M_x\| \geq \|x\|$ is left to proof. Indeed

$$\|M_x\| \geq \left\| M_x \frac{x^*}{\|x^*\|} \right\| = \frac{1}{\|x^*\|} \|xx^*\| = \|x\|.$$

Theorem 35. *Let A be a non unital C*-algebra. Then $A_1 := A \oplus \mathbb{C}$ with coordinate wise addition, $(x;\lambda)(y;\mu) = (xy + \mu x + \lambda y; \lambda\mu)$, $\|(x;\lambda)\|_1 = \sup_{\|y\|=1} \|xy + \lambda y\|$ and $(x;\lambda)^* = (x^*;\overline{\lambda})$ is a unital C*-algebra. The unit is given by $(0;1)$.*

Proof. $\|.\|_1$ is the operator norm on A_1 if its elements are viewed as sums of left multiplication operators and multiples of the identity on A, i.e. $(x, \lambda) \simeq M_x + \lambda I$. Thus $\|.\|_1$ actually is a norm and it also satisfies $\|xy\|_1 \leq \|x\|_1 \|y\|_1$.

To show completeness we first notice that equipped with the norm $\|(x;\lambda)\|_2 := \|x\| + |\lambda|$ and the algebraic rules given above, A_1 is a Banach algebra; (cf Lemma (11)).

The next step is to show, that $\|.\|_1$ and $\|.\|_2$ are equivalent. Since $\|xy + \lambda y\| \leq \|x\|\|y\| + |\lambda|\|y\| = \|(x;\lambda)\|_2$ for every $\|y\| = 1$,

$$\|(x;\lambda)\|_1 \leq \|(x;\lambda)\|_2.$$

Suppose there is no $n \in \mathbb{N}$ with $\|(x;\lambda)\|_2 \leq n\|(x;\lambda)\|_1$ for all $(x;\lambda) \in A_1$. Then there are sequences x_n, λ_n with $\|(x_n;\lambda_n)\|_2 \geq n\|x_n y + \lambda_n y\|$ for all $\|y\| = 1$. Take $y = x_n^*/\|x_n\|$ to see, that $\lambda_n \neq 0$ for $n > 1$. Setting $z_n := x_n/\lambda_n$ one gets

$$\|z_n\| + 1 \geq n\|z_n y + y\| \text{ for all } \|y\| = 1. \tag{3.1}$$

$(z_n)_{n\in\mathbb{N}}$ cannot be unbounded since for every $n \in \mathbb{N}$ we may take $y := z_n^*/\|z_n\|$, to obtain

$$\|z_n\| + 1 \geq n\|z_n y + y\| \geq n\|z_n\| - n.$$

Hence

$$\frac{n+1}{n-1} \geq \|z_n\|, \quad n > 1.$$

So by (3.1) for all $n, m \geq n(\epsilon)$ and $\|y\| = 1$ we get $\|z_n y + y\| \leq \varepsilon/2$ and therefore

$$\|(z_n - z_m)y\| \leq \|z_n y + y\| + \|z_m y + y\| \leq \epsilon.$$

By the previous remark (z_n) is a Cauchy sequence. Thus it converges to a limit $z \in A$ satisfying $zy = -y$ for all y. But this is impossible because A is non unital. So $\|...\|_1$ is a norm equivalent to $\|...\|_2$ showing that $(A_1; \|...\|_1)$ is complete.

Next we prove that the *-operation is an isometry and that the C*-axiom holds true. In fact

$$\|(x;\lambda)\|_1^2 = \sup_{\|y\|=1} \|xy + \lambda y\|^2 = \sup_{\|y\|=1} \|(xy + \lambda y)^*(xy + \lambda y)\|$$

$$= \sup_{\|y\|=1} \|y^*x^*xy + \lambda y^*x^*y + \bar{\lambda}y^*xy + |\lambda|^2 y^*y\| \leq$$

$$\leq \sup_{\|y\|=1} \|x^*xy + \lambda x^*y + \bar{\lambda}xy + |\lambda|^2 y\| = \|(xx^* + \lambda x^* + \bar{\lambda}x; |\lambda|^2)\|_1$$

$$= \|(x;\lambda)^*(x;\lambda)\|_1 \leq \|(x;\lambda)^*\|_1 \|(x;\lambda)\|_1.$$

We get $\|(x;\lambda)\|_1 \leq \|(x;\lambda)^*\|_1$ and by symmetry equality. In turn, we see, that the above inequality is an equality, proving the C*-axiom. The remaining properties are easily verified by a straight forward calculation. □

Remark 36. By Remark (34) we have that $\|.\|_1|_A = \|.\|$ for a non unital A. Therefore, A can be viewed as the subset $A \oplus \{0\}$ of A_1 by the isometric embedding $a \mapsto (a;0)$. As such it is a closed ideal with $A_1/A \cong \mathbb{C}$.

Definition 37. Let A be a unital C*-algebra. If A is unital, then we set. Otherwise we set $A_1 := A \oplus \mathbb{C}$ and provide this space with the C*-algebra structure from Theorem (35). Moreover, for a normal $x \in A$ we define

$$C^*(x) := c.l.s. \left\{ x^j (x^*)^k : j,k \in \mathbb{N}_0 \right\} \subseteq A_1$$

where $x^0 := 1$. More generally for commuting $x_1, x_2, \dots, x_n$, each of which is normal, we define $C^*(x_1, x_2, \dots, x_n)$ to be the closed linear span of the products of the powers of the x_k and x_k^*.

Lemma 38. *Let X be a non compact but locally compact Hausdorff space. If $ac(X)$ denotes the one-point (Alexandroff) compactification of X, then $C_0(X)_1 \cong C(ac(X))$ i.e. there exists an isometric *-isomorphism between those C*-algebras.*

Proof. We show, that the map $\Phi : C_0(X)_1 \to C(ac(X))$ defined by

$$\Phi((f;\lambda))(x) = \begin{cases} f(x) + \lambda & x \in X \\ \lambda & x = \infty \end{cases}$$

is an isometric *-isomorphism.

It is easy to see, that the map is well-defined and a *-isomorphism. By Lemma (33) it is also isometric .

Alternatively we can show isometry directly. In order to do so, note that $f \in C_0(X)$ takes its maximum at a point $y \in X$. By definition we get $\|(f;\lambda)\|_1 = \sup\{\|f + \lambda g\| : g \in C_0(X),\ \|g\| = 1\} = f(y) + \lambda$. Since, $\|\Phi((f;\lambda))\| = f(y) + \lambda$, Φ is an isometry. □

Remark 39. Let A, B be C*-algebras and $\varphi : A \to B$ a *-homomorphism. If A is unital, then by the continuity of the product $\varphi(1_A)$ is a unit of the *-subalgebra $\overline{\varphi(A)}$. Nevertheless it can happen, that B is not unital or that $1_B \neq \varphi(1_A)$.

An example for $1_B \neq \varphi(1_A)$ is easily constructed:

Let $\mathbb{M}(n)$ denote the C*-algebra of n-dimensional matrices (cf. Example (15.3)). Then the embedding

$$\iota : \mathbb{M}(m) \to \mathbb{M}(n), \quad m < n$$

given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

is a *-homomorphism between this C*-algebras. Obviously, we have $\iota(E_n) \neq E_m$.

Remark 40. Let A, B be C*-algebras and $\varphi : A \to B$ a *-homomorphism. If A is non unital, then it is easy to check, that $(x; \lambda) \mapsto \phi(x) + \lambda 1$ is a *-homomorphism from A_1 to B_1 extending φ. This extension will also be denoted by φ.

If $\varphi(A)$ is dense in B, then $\varphi : A_1 \to B_1$ is the unique extension of $\varphi : A \to B$ to a homomorphism. Assume, that $\psi : A_1 \to B_1$ is another extension of $\varphi : A \to B$. If $B = B_1$, i.e. B is unital, then it follows from the previous remark, that $1_B = \psi(1_{A_1})$ and by linearity $\psi = \varphi$ on A_1. If B is not unital, then $\psi(1_{A_1}) \notin B$, since otherwise $\psi(1_{A_1})$ would be a unit in B, which contradicts our assumption. Consequently, $\psi(A_1)$ is dense in B_1. As before, we derive $1_{B_1} = \psi(1_{A_1})$ and in turn $\psi = \varphi$ on A_1.

Lemma 41. *Any *-homomorphism $\varphi : A \to B$ between C*-algebras A, B is a contraction. If φ is injective, then it is isometric.*

Proof. Replacing B by the closure of $\varphi(A)$, we can assume φ to have dense range. We saw in the previous remark, that φ then admits a unique extension $\varphi : A_1 \to B_1$ which according to Lemma (21) is a contraction.

If φ is injective and A is not unital, then $1_{B_1} \notin \varphi(A)$, since otherwise injectivity would give the existence of a unit in A. Thus, also $\varphi : A_1 \to B_1$ is injective. According to Lemma (33) φ is isometric. □

Corollary 42. *If A is a non-unital C*-algebra and B a unital C*-Algebra containing A, then $A + c.l.s.\{1_B\}$ is isometrically isomorphic to A_1.*

Proof. According to Remark (40) the inclusion mapping $A \to B$ extends to an injective *-homomorphism. By the previous lemma this extension is an isometric isomorphism. □

Remark 43. For a C*-algebra A and $x \in A$, let $\sigma(x) = \sigma_A(x)$ be the spectrum of x, when x is considered as an element of A_1. By the previous corollary and Lemma (19) we have $\sigma_A(x) = \sigma_B(x)$ for any C*-algebra $B \subseteq A$.

3.2. **Commutative C*-algebras.** In this section we will expand the results of Section 2.3. to the non unital case. We start with examining the Gelfand space of a unitization of a non unital C*-algebra A.

Remark 44. By Lemma (24) we have $\phi(1) = 1$ for all $\phi \in \hat{A}_1$ (cf. Def (23)). Thus by Remark (40), we can uniquely extend any $\phi \in \hat{A}$ to a multiplicative functional on A_1 by setting $\phi(1) = 1$. By default we will identify ϕ with its unique extension.

Moreover, since it also holds, that $\phi|_A$ is a multiplicative functional on A for any $\phi \in \hat{A}_1$, we get $\hat{A}_1 = \hat{A} \cup \{\phi_1\}$ where $\phi_1|_A = 0$ and $\phi_1(1) = 1$.

Indeed an even stronger proposition is true.

Lemma 45. *Let A be a non unital C*-algebra. $\hat{A}_1$ is homeomorphic to the one point compactification $ac(\hat{A})$ of $\hat{A}$.*

Proof. We use the notation introduced previous to Theorem (25). By the remark above, $(\hat{A}, \sigma(\hat{A}, \iota(A)))$ is homeomorphic to $(\hat{A}_1 \setminus \{\phi_1\}, \sigma(\hat{A}_1 \setminus \{\phi_1\}, \iota(A_1)))$. To see this let $\Psi : \hat{A} \to \hat{A}_1 \setminus \{\phi_1\}$ map $\phi \in \hat{A}$ to its unique extension. Obviously this is a bijection. For all $a \in A$ one has $\iota_{(a;\lambda)} \circ \Psi = \iota_a + \lambda$ and $\iota_a \circ \Psi^{-1} = \iota_{(a;0)}$. The functions on the right side of the equation are continuous. Thus by the definition of the respective initial topologies, Ψ is bi-continuous.

By Theorem (25) $\hat{A}_1$ is compact. Furthermore $\hat{A}_1 \setminus \{\phi_1\}$ is open and dense in $(\hat{A}_1, \sigma(\hat{A}_1, \iota(A_1)))$. So using the uniqueness of the one point compactification we get, that $\hat{A}_1$ is homeomorphic to $ac(\hat{A})$. $\square$

Corollary 46. *$(\hat{A}$,$\sigma(\hat{A}, \iota(A)))$ is locally-compact (the existence of a unit is not required).*

Proof. We recall, that a space is locally compact if its one-point compactification is Hausdorff. Indeed, by the above lemma, the one point compactification is homeomorphic to the Hausdorff space $\hat{A}_1$. $\square$

We now have everything in order, to extend Theorem (27) to the non unital case.

Corollary 47. *Let A be a commutative C*-algebra. Then A is isometrically isomorphic to $C_0(\hat{A})$.*

Proof. We already know, that A_1 is isometrically isomorphic to $C(\hat{A}_1)$, so we can assume, that A is non-unital. By Lemma (45) $\hat{A}_1$ is homeomorphic to $ac(\hat{A})$. Let $\Phi : \hat{A}_1 \to ac(\hat{A})$ be the corresponding a homeomorphism. Then the map $\Upsilon(f) = f \circ \Phi$ from $C(\hat{A}_1)$ to $C(ac(\hat{A}))$ is an isometric isomorphism. We now use Lemma (45) to see, that $C(ac(\hat{A}))$ is also isometrically isomorphic to $C_0(\hat{A})_1$. Putting it all together on gets an isometric isomorphism $\Omega : A_1 \to C_0(\hat{A})_1$. Taking a closer look on this isomorphism one sees, that $\Omega(A) = C_0(\hat{A})$, where we consider $C_0(\hat{A})$ as a subset of $C_0(\hat{A})_1$ by Remark (36). $\square$

3.3. **positive Elements.** Let A be a C*-algebra for the remainder of this section.

Definition 48. A self-adjoint x is called *positive* if $\sigma(x) \subseteq [0, \infty)$. The set of all positive elements is denoted A_+. We also write $x \geq 0$ to indicate, that x is positive.

Remark 49. Since $\sigma(f) = f(X)$ for $f \in C(X)$ and a compact space X, the positive elements of $C(X)$ are exactly the non negative functions. By Lemma (45) the same is true for $C_0(X)$.

It is useful to know, that every element can be written as a linear combination of four positive elements.

Lemma 50. *Let $y \in A_{sa}$ and $x \in A$ then*

(1) $x = Re(x) + iIm(x)$ where $Re(x) = \frac{x+x^*}{2}$ and $Im(x) = \frac{x-x^*}{2i}$ are self-adjoint.
(2) For $y \in A_{sa}$ one has the decomposition $y = y_+ - y_-$ where $y_+, y_- \in A_+$are unique satisfying $y_+y_- = 0$. We also have $\sigma(y_-) \cap (-\infty, 0) = \sigma(y) \cap (-\infty, 0)$ and $\sigma(y_+) \cap (0, \infty) = \sigma(y) \cap (0, \infty)$.

Proof. The first assertion is immediately seen by a straightforward calculation.

For (2) consider the representation of the (commutative) C*-algebra $C^*(y)$. It is easy to see, that y_+, y_- represented by $f(t) := \max(t, 0)$, $g(t) := -\min(t, 0)$ respectively, satisfy the conditions. The claim about the spectra follows from the previous theorem. $\square$

From the functional calculus in the previous section one derives some basic facts about positive elements.

Lemma 51. *For $x \in A$, the following assertions hold true:*

(1) If x is normal then $x^*x \geq 0$.
(2) If $x \in A_{sa}$ and $\|x\| \leq 2$, then $x \geq 0 \Longleftrightarrow \|1-x\| \leq 1$ (in A_1).
(3) If $x \geq 0$ then there exists a square-root $x^{1/2} \geq 0$.
(4) $(x;\lambda) \geq 0$ in A_1 if and only if $x \in A_{sa}$ and $\lambda \geq \|x_-\|$.

Proof. In each case it is possible to consider the C*-algebra $C^*(x)$ instead of A. Therefore, x can be represented by some $f \in C(\sigma(x))$. The first three properties then follow immediately using that fact. Concerning (4), note that $\sigma((x_-;\lambda)) = \sigma(x_-) + \lambda$. Then

$$\sigma((x;\lambda)) \cap (-\infty,0) = \sigma((x_-;\lambda)) \cap (-\infty,0) = (\sigma(x_-)+\lambda) \cap (-\infty,0)$$

yields the fourth property. □

We continue with some further properties. Note, that (2) is a considerable improvement of (1) in the previous lemma.

Theorem 52. *Let $x \in A$.*

(1) If $x, y \geq 0$ then $x+y \geq 0$, i.e. A_+ is a closed cone.
(2) $x^*x \geq 0$.
(3) $x^*yx = (y^{1/2}x)^*(y^{1/2}x) \geq 0$ for $y \geq 0$.

Proof. (1) By Lemma (51) we have,

$$A_+ \cap K_1(0) = A_{sa} \cap K_1(0) \cap \{x : \|1-x\| \leq 1\}$$

with $K_1(0)$ being the closed unit ball in A. Since A_{sa} is closed by the continuity of * , this is a closed and convex set. For $x, y \in A_+$, there is a $C > 0$ such, that $\frac{x}{C}, \frac{y}{C} \in A_+ \cap K_1(0)$. By convexity

$$\frac{1}{2C}(x+y) \in A_+ \cap K_1(0).$$

Since positive multiples of positive elements are positive we have $x+y \in A_+$.

(2) To prove the second property, according to Lemma (50) we decompose $x^*x = c_+ - c_-$ and define $h := xc_-$. A calculation yields

$$-h^*h = c_-^3 \geq 0.$$

Using Lemma (51),(1) along with (1) we obtain

$$hh^* = (h^*h + hh^*) + (-h^*h) = 2Re(h)^2 + 2Im(h)^2 + (c_-^3) \geq 0.$$

Since $h^*h \geq 0 \Longleftrightarrow hh^* \geq 0$ by Theorem (10),(3), it follows that $c_- = 0$.

(3) immediately follows from (2). □

Remark 53. By property (2) and the existence of square roots we get that $x \in A$ is positive if and only if there exists $y \in A$ such that $x = y^*y$. This characterization does not hold in general Banach *-algebras. The reason for this is that one does not have the functional calculus which is used to obtain the decomposition $x^*x = c_+ - c_-$.

Definition 54. For $x, y \in A$ one writes $x \geq y$ if $x - y \geq 0$.

Remark 55. This relation defines a partial order on A satisfying $x \leq y \Rightarrow x + z \leq y + z$ for all $z \in A$. To assert transitivity let $x \leq y$ and $y \leq z$. Then

$$z - x = (z - y) + (y - x) \geq 0$$

because the sum of positive elements is positive by the previous theorem. We get $x \leq z$.

Lemma 56. *If $y \leq z$, then $x^*yx \leq x^*zx$ and if $y \in A_+$, then $0 \leq x^*yx \leq \|y\|x^*x$.*

Proof. By (3) of the previous theorem, we have $x^*(z - y)x \geq 0$. Hence we get $x^*yx \leq x^*zx$. From Theorem (10),(2) we obtain $y \leq \|y\|\,1$. The second inequality immediately follows from this fact and the first part of the present assertion. □

The next lemma and its corollary will be used in the next section, to proof the existence of approximate units.

Lemma 57. *If $0 \leq x \leq y$ and $x \in Inv(A)$ then $y \in Inv(A)$ and $0 \leq y^{-1} \leq x^{-1}$.*

Proof. If $x \in Inv(A) \cap A_+$ then $\varepsilon 1 \leq x$ for some $\epsilon > 0$ as one sees by identifying x with $f \in C(\sigma(x))$. Therefore, also $\epsilon 1 \leq y$. Identifying y with a function g, we see, that y is invertible. If x and y commute, we may restrict our considerations to $C^*(x, y)$ and use representation theory from Theorem (27) for commutative C*-algebras to get the inequaltiy. Indeed, x and y are both represented by functions f and g, on the Gelfand space of $C^*(x, y)$, respectively, and we have $0 \leq f \leq g$. Thus $0 \leq g^{-1} \leq f^{-1}$ and finally $0 \leq y^{-1} \leq x^{-1}$. For the general case note, that by Lemma (56)

$$y^{-1/2}xy^{-1/2} \leq y^{-1/2}yy^{-1/2} = 1.$$

From the special case we get

$$1 \leq (y^{-1/2}xy^{-1/2})^{-1} = y^{1/2}x^{-1}y^{1/2}.$$

Finally,

$$y^{-1} = y^{-1/2}1y^{-1/2} \leq y^{-1/2}y^{1/2}x^{-1}y^{1/2}y^{-1/2} = x^{-1}.$$

□

Corollary 58. *If $0 \leq x \leq y$, then $\|x\| \leq \|y\|$.*

Proof. Let $\lambda > r(y)$, then $0 \leq \lambda - y \leq \lambda - x$. By the previous lemma, since $\lambda - y \in Inv(A)$, $\lambda - x$ is also invertible. Therefore, $\|x\| = r(x) \leq r(y) = \|y\|$. □

3.4. **Approximate Units.**

Definition 59. An *approximate unit* for a C*-algebra A is a net $(h_\lambda)_{\lambda \in \Lambda}$ of positive elements in A with $\|h_\lambda\| \leq 1$ and $h_\lambda x \to x$ for all $x \in A$.

There always exists an approximate unit for a C*-algebra. Indeed for any dense two-sided ideal $I \subseteq A$ there is an approximate unit contained in I.

Theorem 60. *If $I \subseteq A$ is a dense two-sided Ideal one defines $\Lambda_I := \{x \in I_+ : \|x\| < 1\}$ with $I_+ := A_+ \cap I$. Then there exists a direction on Λ_I, given by the partial order $\leq$, such that it is an approximate unit.*

Proof. Step 1: We start showing, that $\leq$ is a direction on Λ_I. To do so, we will prove, that the map $\alpha : x \mapsto (1-x)^{-1} - 1$ defines an order-isomorphism from Λ_I onto I_+, where the order is given by $\leq$.

Indeed, for $x \in \Lambda_I$, $\alpha(x)$ is well defined and belongs to I_+, since

$$(1-x)^{-1} - 1 = \sum_{k=1}^{\infty} x^k = x(\sum_{k=0}^{\infty} x^k) \in I_+$$

by $\|x\| < 1$. By Lemma (57) α is order preserving.

If $y \in I_+$ then, $(1-(1+y)^{-1})(1+y) = y$.

The invertibility of $(1+y)$ (cf. Lemma (57)) yields

$$1-(1+y)^{-1} = y(1+y)^{-1} \in I_+.$$

Let y be represented by the continuous function g on the Gelfand space of $C^*(y)$. Then $y(1+y)^{-1}$ is represented by $\frac{g}{1+g}$. The compactness of the Gelfand space yields $\|y(1+y)^{-1}\| < 1$. Therefore,

$$1-(1+y)^{-1} \in \Lambda_I.$$

Thus the mapping $\beta : x \mapsto 1-(1+x)^{-1}$ maps I_+ onto Λ_I and obviously satisfies $\alpha \circ \beta = id_{I_+}$, $\beta \circ \alpha = id_{\Lambda_I}$. Hence, α is a bijection with the inverse mapping β.

As I_+ is a directed set $(x, y \in I_+ \Rightarrow x, y \leq x+y)$, the same is true for Λ_I.

Step 2: In order to prove, that Λ_I is an approximate unit, we have to show, that the net $(x^*(1-y)x)_{y \in \Lambda_I}$ converges to 0 for every $x \in \Lambda_A$. Indeed from $(1-y)^2 = (1-y)^{\frac{1}{2}}(1-y)(1-y)^{\frac{1}{2}} \leq (1-y)$ we get

$$\|(1-y)x\|^2 = \left\|x^*(1-y)^2 x\right\| \leq \|x^*(1-y)x\|.$$

Hence, from the fact, that by Lemma (50) every $x \in A$ can be written as a linear combination of elements from $\Lambda_A = \{x \in A_+ : \|x\| < 1\}$this suffices.

We first notice that Λ_I is dense in Λ_A. Indeed, let $y \in I$ be a sufficently good approximation of $x^{\frac{1}{2}}$ for $x \in \Lambda_A$, then by

$$\|x - yy^*\| \leq \|x^{\frac{1}{2}}(x^{\frac{1}{2}} - y^*)\| + \|(x^{\frac{1}{2}} - y)y^*\|$$

As a consequence without loss of generality we may assume that $I = A$. We now fix $x \in \Lambda_A$ and view x as a function on the Gelfandspace of $C^*(x)$. For every $\epsilon > 0$, that is sufficiently small, there exists a point $\mu \in \left\{\lambda \in \hat{C^*(x)} : f(\lambda) \geq \epsilon\right\}$ independent of ϵ, where f obtains its maximum. We see, that for those $\epsilon > 0$ one finds an $n \in \mathbb{N}$ such that

$$\|x(1-x^{\frac{1}{n}})x\| \leq \left\|f(1-f^{\frac{1}{n}})f\right\|_{\infty} \leq \max\left\{f(\mu)(1-\epsilon^{\frac{1}{n}})f(\mu), \epsilon^2\right\} \leq \epsilon.$$

By Lemma (56), we see that $x^*(1-y)x$ is decreasing. Therefore, $(x^*(1-y)x)_{y \in \Lambda_I}$ converges to 0 and the proof is completed.

we get that $yy^* \in \Lambda_I$ approximates x. □

The next lemma establishes some properties of approximate units.

Lemma 61. *If* $(h_\lambda)_{\lambda \in \Lambda}$*is an approximate unit in* A *then*

(1) $xh_\lambda \to x$ and $h_\lambda x h_\lambda \to x$ for all $x \in A$.
(2) $(h_\lambda^\alpha)_{\lambda \in \Lambda}$ is an approximate unit for any $\alpha > 0$.

Proof. 1) Since h_λ is positive $xh_\lambda \to x$ follows from $h_\lambda x^* \to x^*$. By decomposition one can assume that $x \geq 0$. Then

$$h_\lambda x h_\lambda = (h_\lambda x^{\frac{1}{2}})(x^{\frac{1}{2}} h_\lambda) \to x^{\frac{1}{2}} x^{\frac{1}{2}} = x.$$

2) One has

$$\|h_\lambda^2 x - x\| \leq \|h_\lambda^2 x - h_\lambda x h_\lambda\| + \|h_\lambda x h_\lambda - x\| = \|h_\lambda\| \, \|h_\lambda x - x h_\lambda\| + \|h_\lambda x h_\lambda - x\| \to 0.$$

By induction

$$\|h_\lambda^{2^n} x - x\| \to 0$$

for any $n \in \mathbb{N}$. Hence, there exits a λ_0, such that for all $\lambda \geq \lambda_0$, we have

$$\|x^* h_\lambda^{2^n} x - x^* x\| \leq \varepsilon.$$

If $\alpha > 0$, choose n with $2^{n-1} \geq \alpha$. Then

$$x^* h_\lambda^{2^n} x \leq x^* h_\lambda^{2\alpha} x \leq x^* h_\lambda^{\alpha} x \leq x^* x.$$

Combining these inequalities, by Corollary (58), we obtain

$$\|x^* h_\lambda^\alpha x - x^* h_\lambda^{2\alpha} x\| \leq \varepsilon, \quad \|x^* x - x^* h_\lambda^\alpha x\| \leq \varepsilon.$$

Therefore,

$$\begin{aligned} \|x - h_\lambda^\alpha x\|^2 = \|(x^* - x^* h_\lambda^\alpha)(x - h_\lambda^\alpha x)\| &= \|x^* x - 2x^* h_\lambda^\alpha x + x^* h_\lambda^{2\alpha} x\| \\ &\leq \|x^* x - x^* h_\lambda^\alpha x\| + \|x^* h_\lambda^\alpha x - x^* h_\lambda^{2\alpha} x\| \leq 2\varepsilon. \end{aligned}$$

□

Remark 62. For the approximate unit constructed in Theorem (60) there is an easier way to see (2). Indeed, if we examine the last part of the proof of Theorem (60), we see, that there also exists an $n \in \mathbb{N}$, such that $\|x(1 - x^{\alpha/n})x\| \leq \epsilon$.

4. Representation Theory of C*-Algebras

4.1. Representations.

Definition 63. A pair (H, π) is called a *representation* of a Banach *-algebra A if $\pi : A \to B(H)$ is a *-homomorphism into the space of bounded operators on the Hilbert space H.

Remark 64. Sometimes we will call just π a representation if it is clear what Hilbert space is meant.

We introduce some notions for representations.

Definition 65. Let (H, π) be a representation of a Banach *-algebra A.

- It is called *faithful* if π is injective.
- It is called *irreducible* if there are no non trivial closed invariant subspaces U, i.e. $\pi(A)(U) \subseteq U \Rightarrow U = \{0\} \vee U = H$.
- Let $B \subseteq B(H)$ be a *-sub-algebra (i.e. a sub-algebra satisfying $B^* \subseteq B$). Consider the largest subspace $N \subseteq H$ such that $B|_N = 0$; i.e. $N = \bigcap_{A \in B} \ker A$. Its orthogonal complement $X := N^\perp$ is called the *essential subspace* of B. If $X = H$, we say that B acts nondegenerately on H. The representation π is called *nondegenerate* if $\pi(A) \subseteq B(H)$ acts nondegenerately on H.
- $\xi \in H$ is called a *cyclic vector* if $\overline{\pi(A)\xi} = H$. If there exists a cyclic vector ξ, we call (H, π, ξ) a cyclic representation.

Remark 66. If a representation π is irreducible, then it is also nondegenerate.

Irreducibility is formulated in terms of closed invariant subspaces. Since $S\overline{U} \subseteq \overline{SU}$ for all bounded operators S, we have $\pi(A)U \subseteq U \Rightarrow \pi(A)\overline{U} \subseteq \overline{U}$. Hence, the only invariant non trivial subspaces of an irreducible representation must be dense in H.

Lemma 67. *$B \subseteq B(H)$ acts nondegenerately if and only if $\overline{BH} = H$.*

Proof. If $\overline{BH} =: X \neq H$, then $N := X^\perp \neq \{0\}$. Since X is invariant and B is a *-sub-algebra, N also is. To see this take $x \in N$, then

$$(Bx, X) = (x, B^*X) = (x, BX)$$

where $BX \subseteq X$, because X is invariant. It follows that $(Bx, X) = \{0\}$. Hence $BN \subseteq N$. as $BN \subseteq \overline{BH} = X \perp N$, B acts degenerately. Now suppose $B_{|N} = 0$ on some $N \neq \{0\}$. Again, because N is invariant $X := N^\perp \neq H$ also is. For $x \in H$ we have $(Bx, N) = (x, B^*N) = \{0\}$. Hence, $BH \subseteq X$ and further $\overline{BH} \subseteq \overline{X} = X$. □

Before we give some characterizations of irreducibility we have to prove the von-Neumann bicommutant theorem.

Definition 68. For a Hilbert space H and $A \subseteq B(H)$ we define the *commutant* $A' := \{T \in B(H) : TS = ST \text{ for all } S \in A\}$.

Remark 69. Obviously $A' \subseteq B(H)$ is a sub-algebra. If A is * closed, than A' also is. Indeed

$$a^*b = (b^*a)^* = (ab^*)^* = ba^*$$

for $a \in A'$ and all $b \in A$.

Moreover, we have $A \subseteq A''$. Therefore, $A' \subseteq A'''$. Since A''' commutes with A'', and hence also with A, we get $A''' \subseteq A'$ and finally $A' = A'''$.

Lemma 70. *Let $A \subseteq B(H)$. Then the commutant $A^{'}$ is closed in the weak topology. In particular, it is strongly closed.*

Proof. Let $(T_i)_{i\in I} \in A^{'}$ be a net, with $(T_i\xi, \eta) \to (T\xi, \eta)$ for all $\xi, \eta \in H$. We need to prove, that $T \in A^{'}$. For an arbitrary $S \in A$, we get

$$(TS\xi, \eta) = \lim_{i\in I}(T_iS\xi, \eta) = \lim_{i\in I}(ST_i\xi, \eta) = \lim_{i\in I}(T_i\xi, S^*\eta) = (T\xi, S^*\eta) = (ST\xi, \eta).$$

By Lemma (7) $TS = ST$. □

Theorem 71. *(Bicommutant) Let H be a Hilbert space. If $B \subseteq B(H)$ is a *-sub-algebra that acts nondegenerately, then B is strongly dense in $B^{''}$.*

Proof. Let $T^{''} \in A^{''}$. We have to show, that every strong neighborhood (cf. Remark (4))

$$\mathcal{N}_{\epsilon,x_1,\ldots,x_n}(T^{''}) = \left\{T \in B(H) : \left\|(T^{''} - T)\xi_j\right\| < \epsilon,\, j = 1, \ldots, n\right\}$$

contains an element of B. Fixing $\xi_1, \xi_2, \ldots, \xi_n \in H$ and $\epsilon > 0$ a $T \in B$ with

$$\max\left\{\left\|(T^{''} - T)\xi_j\right\| : j = 1, \ldots, n\right\} < \epsilon$$

has to be constructed. Therefore, we consider the amplifications (cf. Definition (2))

$$H^{\sim} := \bigoplus_{j=1,\ldots,n} H,\ S^{\sim} := \bigoplus_{j=1,\ldots,n} S$$

for $S \in B(H)$. Moreover, we define $B^{\sim} = \{S^{\sim} : S \in B\}$. Obviously $B^{\sim}$ is a *-sub-algebra of $B(H^{\sim})$ that acts nondegeneratly. Now it suffices to show, that there is a $T^{\sim} \in B^{\sim}$ such that

$$\left\|(T^{\sim} - (T^{''})^{\sim})\xi\right\| < \epsilon$$

where $\xi = (\xi_1, \ldots, \xi_n)$.

For this consider $X = \overline{B^{\sim}\xi}$. Since both X and $X^{\perp}$ are invariant under $B^{\sim}$, we have $P_X \in (B^{\sim})^{'}$ for the orthogonal projection P_X onto X. Since $(T^{''})^{\sim}$ commutes with P_X, we get $(T^{''})^{\sim}X \subseteq X$. If we can show that $\xi \in X$ we are done, for then $(T^{''})^{\sim}\xi \in X$. For all $S \in B^{\sim}$ we have

$$S[(1 - P_X)\xi] = (1 - P_X)[S\xi] = 0$$

since $S\xi \in X$. Because B acts nondegeneratly we may conclude $(1 - P_X)\xi = 0$, i.e. $\xi \in X$. □

We will see, that we can reduce our attention to projections if we want to know whether some $T \in B(H)$ belongs to a certain commutant.

Lemma 72. *Let $B \subseteq B(H)$ be a *-sub-algebra and $T \in B(H)$. Then $T \in B^{'}$ if and only if the ranges of all spectral projections of $Re(T)$ and $Im(T)$ are B-invariant subspaces.*

Proof. For $T \in B^{'}$ we have $Re(T), Im(T) \in B^{'}$ as they are linear combinations of T, T^*. It follows from Theorem (5) , that their spectral projections $E(\Delta)$ are in $B^{'}$. Therefore, their ranges are invariant subspaces. Conversely, if the ranges of all spectral projections are invariant subspaces, then all projections are in $B^{'}$. Again by Theorem (5) we get $Re(T), Im(T) \in B^{'}$. Hence, $T \in B^{'}$. □

Now we have all that we need to prove the following characterizations of irreducible representations.

Theorem 73. *Let (π, H) be a representation of a C*-algebra A. The following assertions are equivalent:*

(1) π is irreducible.
(2) $\pi(A)^{'} = \mathbb{C}1$.
(3) $\pi(A)^{''} = B(H)$.
(4) $\pi(A)$ is strongly dense in $B(H)$.
(5) If $dim(H) \neq 1$, then every $0 \neq \xi \in H$ is a cyclic vector for π.

Proof. Lets start with 1) $\Leftrightarrow$ 2). Suppose π is irreducible. If $T \in \pi(A)^{'}$, then by the previous lemma and the definition of irreducibility $Re(T)$ and $Im(T)$ are multiples of the identity. Thus T is a multiple of the identity. On the other hand if $\pi(A)^{'} = \mathbb{C}1$, then again by the previous lemma the only invariant subspaces are trivial.

2) $\Rightarrow$ 3) is trivial. For 3) $\Rightarrow$ 2) note, that $B(H)$ has no nontrivial invariant subspaces. By the previous lemma $\pi^{'}(A) = \pi(A)^{'''} = B(H)^{'} = \mathbb{C}1$.

4) $\Rightarrow$ 3) is immediate by Lemma (70).

3)$\Rightarrow$4) Since 3) implies 1), this follows by the commutant theorem.

We will now prove 1) $\Leftrightarrow$ 5). Let π be irreducible. Then for $\xi \in H$, the closed invariant subspace $\overline{\pi(A)\xi}$ must be trivial. If $\overline{\pi(A)\xi} = \{0\}$, then $\mathbb{C}\xi$ is a closed invariant subspace and thus the whole space, so $dim(H) = 1$.

For the other direction, consider a nonzero closed invariant subspace $U \subseteq H$. If $dim(H) = 1$, then trivially $U = H$. If on the other hand every ξ is a cyclic vector, choose $\xi \in U$. Then $H = \overline{\pi(A)\xi} \subseteq U$. □

Definition 74. Let A be a C*-algebra and $(\pi_1, H_1), (\pi_2, H_2)$ representations. We call $T \in B(H_1, H_2)$ an *intertwining operator* if

$$T\pi_1(x) = \pi_2(x)T$$

for every $x \in A$. The set of all intertwining operators is denoted $\mathcal{R}(\pi_1, \pi_2)$.

If $U \in \mathcal{R}(\pi_1, \pi_2)$ is unitary $(\pi_1, H_1), (\pi_2, H_2)$ are called *unitarily equivalent*. If in addition, there exist cyclic vectors ξ_1, ξ_2 with $U\xi_1 = \xi_2$, then $(\pi_1, H_1, \xi_1), (\pi_2, H_2, \xi_2)$ are called unitarily equivalent as cyclic representations.

4.2. **Positive Functionals, States.** In order to describe commutative C*-algebras we used multiplicative functionals. This approach fails in general, since a multiplicative functional cannot distinguish between xy and yx for non commuting elements $x, y \in A$. It turns out, that in order to describe a non-commutative C*-algebra one has to introduce the more general notion of states.

Definition 75. Let A be a C*-algebra. A functional ω is called *positive* if $\omega(x) \geq 0$ for all $x \in A_+$. If it is normalized, i.e. $\|\omega\| = 1$, then it is called a *state*. The set of states on A is denoted by $S(A)$.

Example 76. By Lemma (24) any multiplicative functional on a commutative C*-algebra is a state.

Remark 77. For general Banach *-algebras a functional is called positive if $\omega(x^*x) \geq 0$ for all $x \in A$. Indeed this defining property is the one that is needed in what follows (see Lemma (81)). However by Remark (53) we see that this definition is equivalent to the one given above in the case of C*-algebras. Since we do not have this equivalence for general Banach *-algebras, a lot of the following proofs will not work in that case. Nevertheless the majority of these results hold in more general settings as well, though the proofs are more elaborate.

Let us start by proving that every positive functional is automatically bounded.

Lemma 78. *Let ω be a positive functional, then there exists $C > 0$ such that $\|\omega\| \leq C$.*

It is an important fact, that any positive linear functional gives rise to a pre-inner product.

Proof. After decomposition (cf. Lemma(50)) it suffices to show that $|\omega(x)| \leq C\|x\|$ for all $x \in A_+$ and a $C > 0$. Suppose there is no such C. Then for every $n \in \mathbb{N}$ there is $x_n \in A_+$ with $\|x_n\| = 1$ and $\omega(x_n) \geq 4^n$. We now consider $x := \sum_{n=1}^{\infty} 2^{-n} x_n$. This is a well defined element of A satisfying $\omega(x) \geq 2^{-n}\omega(x_n) \geq 2^n$ for any $n \in \mathbb{N}$, which cannot be true. $\square$

Definition 79. $(x, y)_\omega := \omega(y^* x)$ for any positive functional ω on A.

Lemma 80. *Let ω be a positive functional. Then $\omega(x^*) = \overline{\omega(x)}$.*

Proof. If $x \in A_{sa}$, then $x = x_+ - x_-$ with $x_+, x_- \in A_+$. We get

$$\omega(x) = \omega(x_+ - x_-) = \omega(x_+) - \omega(x_-) \in \mathbb{R}.$$

For $x \in A$, $x = a + ib$ with $a, b \in A_{sa}$, we have

$$\omega(x^*) = \omega(a - ib) = \omega(a) - i\omega(b) = \overline{\omega(x)}$$

since $\omega(a), \omega(b) \in \mathbb{R}$. $\square$

Lemma 81. $(.,.)_\omega$ *is a pre-inner product on A.*

Proof. $(x, y)_\omega = \omega(y^* x) = \overline{\omega(x^* y)} = \overline{(y, x)_\omega}$ by Lemma (80), and $(x, x)_\omega = \omega(x^* x) \geq 0$ by Theorem (52). The sesquiliniarity is obvious. $\square$

Since we have shown that $(.,.)_\omega$ is a pre-inner product, one gets the Cauchy-Schwarz inequality.

Fact 82. $|\omega(y^* x)|^2 \leq \omega(x^* x)\omega(y^* y)$ *for any $x, y \in A$.*

The next theorem gives a characterization of positive functionals on unital algebras.

Theorem 83. *Let ω be a linear functional on A.*

(1) If $\omega \geq 0$, then $\|\omega\| = \sup\{\omega(x) : x \geq 0, \|x\| \leq 1\} = \lim \omega(h_\lambda)$ for an approximate unit $(h_\lambda)_{\lambda \in \Lambda}$. In particular, if A is unital $\|\omega\| = \omega(1) = |\omega(1)|$.
(2) If $\omega_1, \omega_2 \geq 0$, then $\|\omega_1 + \omega_2\| = \|\omega_1\| + \|\omega_2\|$.
(3) If A is unital, then $\|\omega\| = \omega(1) \Longleftrightarrow \omega \geq 0$.

Proof. 1) We have

$$\|\omega\| \geq \sup\{\omega(x) : x \geq 0, \|x\| \leq 1\} \geq \limsup \omega(h_\lambda).$$

Now on the other hand, for every $\varepsilon > 0$ there is a $\|y\| = 1$ such that,

$$\|\omega\|^2 - \varepsilon \leq |\omega(y)|^2 = \lim |\omega(h_\lambda^{1/2} y)|^2 \leq \liminf \omega(h_\lambda)\omega(y^* y) \leq \|\omega\| \liminf \omega(h_\lambda).$$

2) From (1) it follows, that

$$\|\omega_1 + \omega_2\| = \lim(\omega_1 + \omega_2)(h_\lambda) = \lim \omega_1(h_\lambda) + \lim \omega_2(h_\lambda) = \|\omega_1\| + \|\omega_2\|.$$

3) It suffices to show, that ω is positive on the C*-sub-algebras $C^*(x)$ where $x \in A^+$ if $\omega(1) = \|\omega\|$. We can use continuous functional calculus for normal elements, to identify $C^*(x) \cong C(\sigma(x))$. Therefore we may use the Riesz representation theorem, to identify ω with a complex measure μ. $\omega(1) = \|\omega\|$ yields $\|\mu\| = \mu(\sigma(x))$, which is only possible for $\mu \geq 0$. Hence, $\omega \geq 0$. □

An important consequence is, that any state on A uniquely extends to a state on A_1.

Fact 84. *Let ω be a positive linear functional on the non unital C*-algebra A. Then by $\omega(1) = \|\omega\|$, it extends to a positive functional on A_1. In particular, the extension of $\omega \in S(A)$ belongs to $S(A_1)$.*

The following estimate, will be useful later on.

Corollary 85. *If $x \in A$, then $|\omega(x)|^2 \leq \omega(x^*x)$ for a state ω.*

Proof. ω extends to A_1. So one can use Lemma (82) with $y = 1$. □

4.3. **The GNS Construction.** In this section we use the fact, that postive functionals induce pre-inner products to obtain representations. For every positive functional ω, the so called Gelfand-Naimark-Segal construction, yields a representation π_ω of the C*-algebra.

Theorem 86. *Let ω be a positive functional on A.*

(1) $N_\omega := \{x \in A : \omega(x^*x) = 0\}$ is a left ideal in A. The pre-inner product $(.,.)_\omega$ induces a well-defined inner product on A/N_ω by $(x+N_\omega, y+N_\omega)_\omega := (x,y)_\omega$. Its completion $(H_\omega, (.,.)_\omega)$ is a Hilbert space.
(2) Let $\pi_\omega(x)(y + N_\omega) := xy + N_\omega$ for $x, y \in A$. Then $\pi_\omega(x)$ is a bounded operator on A/N_ω with $\|\pi_\omega(x)\| \leq \|x\|$. It follows that it extends to a bounded operator on H_ω which we will also call $\pi_\omega(x)$.
(3) The map $\pi_\omega : A \to B(H_\omega)$ is a representation, called the GNS representation of A associated to ω.

Proof. 1) Let us start by showing that N_ω is a closed left ideal in A. It is obviously closed since it is the zero-set of a continuous map. For $y, z \in A$ and $x \in N_\omega$ using Lemma (82) we obtain

$$|\omega((yx)^*yx))|^2 = |\omega(x^*y^*yx)|^2 \leq \omega(x^*y^*y(x^*y^*y)^*)\omega(x^*x) = 0.$$

From $|\omega(x^*z)|^2 \leq \omega(z^*z)\omega(x^*x) = 0$ we infer

$$\omega((x+z)^*(x+z)) = \omega(x^*x+x^*z+z^*x+z^*z) = \omega(x^*z)+\omega(z^*x)+\omega(z^*z) = \omega(z^*z).$$

It follows that N_ω is a left ideal (for $N_\omega + N_\omega = N_\omega$ one chooses $z \in N_\omega$ in the above equation) and by employing the polar formula that the pre-inner product is well defined by acting on the representatives. From

$$0 = \omega((x + N_\omega)^*(x + N_\omega)) = \omega((x^* + N_\omega)(x + N_\omega)) = \omega(x^*x) \Rightarrow x + N_\omega = 0 + N_\omega$$

one concludes that $(.,.)_\omega$ is definit on A/N_ω.

2) Since N_ω is a left ideal $\pi_\omega(x)$ is a well defined operator on N/N_ω. Applying ω to

$$y^*x^*xy \leq \|x\|^2 y^*y$$

yields

$$\|\pi_\omega(x)\| \leq \|x\|.$$

3) π_ω is obviously a homomorphism. Furthermore

$$(\pi_\omega(x^*)(y+N_\omega), z+N_\omega)_\omega = \omega(z^*x^*y) = (y+N_\omega, \pi_\omega(x)(z+N_\omega))_\omega$$

$$= (\pi_\omega(x)^*(y+N_\omega), z+N_\omega)_\omega.$$

Using Lemma (7), we get $\pi_\omega(x^*) = \pi_\omega(x)^*$. □

We are going to see that representations constructed in this way are always cyclic and unique in a certain sense.

Theorem 87. *Let (H_ω, π_ω) be the GNS-representation of A associated to ω.*

(1) There is a vector $\xi_\omega \in H_\omega$ satisfying $\pi_\omega(x)\xi_\omega = x + N_\omega$ and $\omega(x) = (\pi_\omega(x)\xi_\omega, \xi_\omega)$. ξ_ω is cyclic for π_ω with $\|\xi_\omega\|^2 = \|\omega\|$.
(2) Let π be a representation of A on $B(H)$ with a ξ cyclic for π and $\omega(x) = (\pi(x)\xi, \xi)$. Then (H, π, ξ) is unitarily equivalent to $(H_\omega, \pi_\omega, \xi_\omega)$.

Proof. 1) In the unital case ξ_ω is simply $1 + N_\omega$. In the general case one considers the extension ω_1 of ω to A_1 with $\|\omega_1\| = \|\omega\|$. By Theorem (83),

$$\omega_1(1) = \|\omega\| = \lim \omega(h_\lambda) = \lim \omega(h_\lambda^2).$$

Hence,

$$\lim \|1 - h_\lambda\|^2_{H_{\omega_1}} = \lim \omega_1((1-h_\lambda)^2) = 0.$$

It follows that A/N_ω is dense in A_1/N_{ω_1} and H_ω can be identified with H_{ω_1}. Now $\xi_\omega = \xi_{\omega_1}$ satisfies $\omega(x) = (\pi_\omega(x)\xi_\omega, \xi_\omega)_H$.

2) If there was a unitary operator establishing a unitary equivalence, it must certainly send $\pi(x)\xi$ to $\pi_\omega(x)\xi_\omega$. So we define $U : \pi(A)H \to \pi_\omega(A)H_\omega$ by $U\pi(x)\xi = \pi_\omega(x)\xi_\omega$. One calculates

$$(U\pi(x)\xi, U\pi(y)\xi)_{H_\omega} = (\pi_\omega(x)\xi_\omega, \pi_\omega(y)\xi_\omega)_{H_\omega} = (\pi_\omega(y^*x)\xi_\omega, \xi_\omega)_{H_\omega} =$$

$$\omega(y^*x) = (\pi(y^*x)\xi, \xi)_H = (\pi(x)\xi, \pi(y)\xi)_H.$$

Thus U is an isometry. In particular, it is well defined. Since ξ, ξ_ω are cyclic, it uniquely extends to a unitary operator also denoted by U from H to H_ω. For all $x, y \in A$ we have

$$\pi_\omega(x)U\pi(y)\xi = \pi_\omega(xy)\xi_\omega = U\pi(xy)\xi = U\pi(x)\pi(y)\xi.$$

Hence, by a density argument $\pi_\omega(x)U = U\pi(x)$ for all $x \in A$ and U establishes a unitary equivalence. □

Remark 88. As mentioned in Remarks (53, 77) a lot of the things we have done do not work for Banach *-algebras. However, with some effort the results of this section may be obtained for Banach *-algebras with a bounded approximate identity. Central in this considerations is the Cohen-Hewitt Factorisation Theorem. For more details see [4] .

4.4. **The Space of States, Pure States.** As mentioned above, so far we did not need all the structure that comes with a C*-algebra. We have seen that for every state there exists a representation but this representation is not necessarily faithful. In order to get a faithful representation we will consider direct sums of GNS representations. If there are sufficiently many states this approach will work. Indeed, this is the point where we need most of the structure of C*-algebras.

We will start with a Banach-Alaoglou like theorem.

Theorem 89. *If A is a unital C*-algebra, then $(S(A), \sigma(S(A), \iota(A))$ is a compact convex subset of the dual space A^*.*

Proof. The proof works similar to Theorem (25) or Banach-Alaoglou respectively. For $x \in A_+$ consider the maps

$$\iota_x : A^{'} \to \mathbb{C};\ \phi \mapsto \phi(x).$$

Using these maps, we get

$$S(A) = K_1^{A^*}(0) \cap \{\phi \in A^* : \phi(1) = 1\} \cap \bigcap_{x \in A_+} \iota_x^{-1}([0, \infty)).$$

Because the ι_x are continuous, this is an intersection of w*-closed sets. Hence, $S(A)$ is also w*-closed. By Banach-Alaoglou, $K_1^{A^*}(0)$ is compact. Hence $S(A)$ is compact.

To see, that $S(A)$ is convex, note, that the sum of positive functionals is always a positive functional. Moreover, by Theorem (83)

$$\|\lambda\omega_1 + (1-\lambda)\omega_2\| = \lambda \|\omega_1\| + (1-\lambda) \|\omega_2\| = 1$$

if $\omega_1, \omega_2 \in S(A)$. Therefore, a convex combination of states is always a state. □

The next theorem one could understand as a version of the algebraic Hahn-Banach theorem for states.

Theorem 90. *If $B \subseteq A$ is C*-sub-algebra and ω is a state on B, then ω extends to a state ω' on A.*

Proof. By the Hahn-Banach theorem there is an extension ω' with norm 1. If B is unital, then by Theorem (83) ω' is a state since $\omega'(1) = \omega(1) = 1$. If B is not unital, one extends ω to $B_1 \subseteq A_1$ and then there is an extension ω' on A_1. By restricting to A one gets the wanted extension. □

If the set of positive functionals would fully describe a C*-algebra, obviously the subset of states also would. As it will eventually turn out, there is an even smaller subset sufficient for that task.

Definition 91. A state ω is called pure if it is an extrem point of $S(A)$, i.e. $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ with $0 < \lambda < 1$ and $\omega_1, \omega_2 \in S(A)$, yields $\omega_1 = \omega_2 = \omega$. The set of pure states is denoted $P(A)$.

Lemma 92. *A state ω is pure, if and only if all positive $\omega^\sim$ with $\omega^\sim \leq \omega$ are multiples of ω. Here $\omega^\sim \leq \omega$ mean, that $\omega^\sim(x) \leq \omega(x)$ for all $x \in A_+$.*

Proof. Let ω be pure and $\omega^\sim \leq \omega$. Since $\|\omega - \omega^\sim\| = \|\omega\| - \|\omega^\sim\|$ (cf. Theorem (83),(2)), we may assume, that $1 = \|\omega\| > \|\omega^\sim\|$. We get

$$\omega = \|\omega^\sim\| \frac{\omega^\sim}{\|\omega^\sim\|} + (1 - \|\omega^\sim\|) \frac{(\omega - \omega^\sim)}{(1 - \|\omega^\sim\|)}$$

where by Theorem (83),(2), $\frac{(\omega-\omega^{\sim})}{(1-\|\omega^{\sim}\|)}$ is a state. Thus $\frac{\omega^{\sim}}{\|\omega^{\sim}\|}=\omega$.

Now let all $\omega^{\sim}\leq\omega$ be multiples of ω. Consider a convex combination $\omega=\lambda\omega_1+(1-\lambda)\omega_2$ with $0<\lambda<1$. It suffices to show, that $\omega_1=\omega$. Obviously $\lambda\omega_1=\omega-(1-\lambda)\omega_2\leq\omega$. Thus ω_1 is a multiple of ω and since it is a state, $\omega_1=\omega$. □

Theorem 93. *Let A be a C*-algebra and $B\subseteq A$ a non trivial C*-sub-algebra. If ω is a pure state on B, then the set $ext(\omega)$ of extensions of ω to a state on A is a compact convex subset of $S(A)$ and any extrem point of this set is a pure state on A.*

Proof. Since $S(A)$ is convex, $ext(\omega)$ obviously also is. For compactness, note, that

$$ext(\omega)=K_1^{A^*}(0)\cap\{\phi\in A^*:\ \phi|_B=\omega\}\cap\bigcap_{x\in A_+}\iota_x^{-1}([0,\infty)).$$

Indeed, every functional on the right hand side ϕ satisfies $\|\phi\|=1$. This follows from (cf. Theorem (83))

$$\|\phi\|=\sup\{\omega(x):x\geq 0,\|x\|\leq 1,\,x\in A\}\geq$$

$$\geq\sup\{\omega(x):x\geq 0,\|x\|\leq 1,\,x\in B\}=\|\omega\|=1.$$

It is easy to see, that $\{\phi\in A^*:\ \phi|_B=\omega\}$ is weakly closed. Hence, $ext(\omega)$ is compact.

Let ω' be an extrem point of $ext(\omega)$. If $\omega'=\lambda\omega_1+(1-\lambda)\omega_2$ for $\omega_1,\omega_2\in S(A)$, then $\omega_1|_B=\omega_2|_B=\omega'$ because ω is a pure state, i.e. $\omega_1,\omega_2\in ext(\omega)$. Since ω' is an extrem point of $ext(\omega)$, we get $\omega_1=\omega_2=\omega'$ on A. Thus ω' is a pure state. □

Remark 94. Since $ext(\omega)$ is compact, by the Krein-Milman theorem there always exist extrem points (cf. [2], Section 5).

Corollary 95. *For a C*-algebra A and $x\in A$ there exists a pure state ω and a vector $\xi\in H_\omega$ with $\|\xi\|=1$, such that $\|\pi_\omega(x)\xi\|_{H_\omega}=\|x\|$. In particular, $\|\pi_\omega(x)\|=\|x\|$.*

Proof. Step 1: If $0\neq x\in A_{sa}$, then there is a pure state ω with $|\omega(x)|=\|x\|$.

One considers the C*-algebra $C^*(x)$ which is isometrically isomorphic to $C(\sigma(x))$. Let x be represented by a real valued f. Then f takes its maximum for some $\lambda\in\sigma(x)$. $\iota_\lambda:g\to g(\lambda)$ is a pure state on $C(\sigma(x))$ satisfying $|\iota_\lambda(f)|=\|f\|$. So there is a pure state on $C^*(x)$ satisfying the conditions. By Theorem (93) and its subsequent remark, it can be extended to a pure state ω on A_1. $x\in A$ implies, that $\omega|_A$ is a state. $\omega|_A$ is even a pure state on A. Indeed, if $\omega|_A=\lambda\omega_1+(1-\lambda)\omega_2$, then the unique extensions of ω_1,ω_2 to A_1 would give the same convex combination for ω. Thus $\omega_1=\omega_2=\omega|_A$.

Step 2: By Step 1 there exists a pure state ω with $\omega(xx^*)=\|xx^*\|=\|x\|^2$. We get

$$\|\pi_\omega(x)\xi_\omega\|^2_{H_\omega}=(\pi_\omega(xx^*)\xi_\omega,\xi_\omega)_{H_\omega}=\omega(xx^*)=\|x\|^2.$$

□

4.5. **The Theorem of Gelfand-Naimark.** As announced above considering direct sums of GNS-representations, we get a faithful representation.

Theorem 96. *For every C*-algebra A there is an isometric representation π.*

Proof. By Corollary (95) and the fact that $\left\| \bigoplus_{\omega \in P(A)} \pi_\omega(x) \right\| = \sup_{\omega \in P(A)} \|\pi_\omega(x)\|$,

$$\Big(\bigoplus_{\omega \in P(A)} H_\omega, \bigoplus_{\omega \in P(A)} \pi_\omega \Big)$$

is an isometric representation (cf. Definition (2) for the notation). □

This result is quite remarkable since it states, that every C*-algebra is basically nothing else but an operator algebra. The next few results are concerned with showing, that this representation is a direct sum of irreducible representations. Therefore, we will prove, that a GNS representation is irreducible if and only if the associated state is pure.

At first we make a trivial observation.

Corollary 97. *If ω is a positive functional on A, then by $\omega(T) = (T\xi_\omega, \xi_\omega)_\omega$ it induces a positive functional with the same norm on $B(H_\omega)$. In particular, a state induces a state.*

Proof. We have $|(T\xi_\omega, \xi_\omega)| \leq \|T\| \cdot \|\xi_\omega\|^2 = \|T\| \cdot \|\omega\|$. Hence, $\|T \mapsto (T\xi_\omega, \xi_\omega)\| \leq \|\omega\|$. Equality follows, if we set $T = I$. □

The next result may be seen as an operator theoretic version of the Radon-Nikodym theorem.

Theorem 98. *Given a C*-algebra A let $\omega_1 \leq \omega_2$ (i.e. $\omega_1(x) \leq \omega_2(x)$ for all $x \in A_+$) be two positive functionals on A. Then there is an unique operator $T \in \pi_{\omega_2}(A)'$ with $0 \leq T \leq 1$, such that $\omega_1(x) = \omega_2(T\pi_{\omega_2}(x))$ for all $x \in A$.*

Proof. We start by defining a pre-inner product that relates ω_1 to ω_2. In fact, $[\pi_{\omega_2}(x)\xi_{\omega_2}, \pi_{\omega_2}(y)\xi_{\omega_2}] := \omega_1(y^*x)$ is a pre-inner product on $\pi_{\omega_2}(A)\xi_{\omega_2}$. To see, that it is well defined, take $x \in A$, such that $x + N_{\omega_2} = \pi_{\omega_2}(x)\xi_{\omega_2} = 0$. Hence, $x \in N_{\omega_2}$, *i.e.* $\omega_2(x^*x) = 0$. Using $\omega_1 \leq \omega_2$, we obtain

$$\omega_1(x^*x) = 0.$$

Hence, $|\omega_1(y^*x)|^2 \leq \omega_1(x^*x)\omega(y^*y)_1 = 0$. Let $z \in A$. Then,

$$\begin{aligned} |[\pi_{\omega_2}(z)\xi_{\omega_2}, \pi_{\omega_2}(z)\xi_{\omega_2}]|^2 &= |\omega_1(z^*z)|^2 \leq |\omega_2(z^*z)|^2 \leq \omega_2(z^*z)\omega_2(z^*z) = \\ &= (\pi_{\omega_2}(z)^*\pi_{\omega_2}(z)\xi_{\omega_2}, \xi_{\omega_2})_{H_{\omega_2}}(\pi_{\omega_2}(z)^*\pi_{\omega_2}(z)\xi_{\omega_2}, \xi_{\omega_2})_{H_{\omega_2}} = \\ &= \|\pi_{\omega_2}(z)\xi_{\omega_2}\|^2 \|\pi_{\omega_2}(z)\xi_{\omega_2}\|^2 . \end{aligned}$$

Since $[.,.]$ is hermitian, $\|[.,.]\| = \sup_{\|z\|=1} \|[z,z]\| \leq 1$. By continuity this pre-inner product extends to H_{ω_2}.

The Lax-Milgram Theorem (cf. Proposition 3.2.6 in [2]), asserts, that there exists a unique operator $T \in B(H_{\omega_2})$ with $[\eta, \xi] = (T\eta, \xi)_{H_{\omega_2}}$ for all $\eta, \xi \in H_{\omega_2}$. By the calculation above we get $0 \leq T \leq 1$. One has $\omega_1(x) = [\pi_{\omega_2}(x)\xi_{\omega_2}, \xi_{\omega_2}] = (T\pi_{\omega_2}(x)\xi_{\omega_2}, \xi_{\omega_2})_{H_{\omega_2}} = \omega_2(T\pi_{\omega_2}(x))$ for all $x \in A$ (cf. Corollary (97)). To show $T \in \pi_{\omega_2}(A)'$ we calculate

$$(T\pi_{\omega_2}(x)[\pi_{\omega_2}(z)\xi_{\omega_2}], \pi_{\omega_2}(y)\xi_{\omega_2})_{H_{\omega_2}} = \omega_1(y^*(xz)) = \omega_1((x^*y)^*z) =$$

$$(T\pi_{\omega_2}(z)\xi_{\omega_2}, \pi_{\omega_2}(x^*)[\pi_{\omega_2}(y)\xi_{\omega_2}])_{H_{\omega_2}} = (\pi_{\omega_2}(x)T[\pi_{\omega_2}(z)\xi_{\omega_2}], \pi_{\omega_2}(y)\xi_{\omega_2})_{H_{\omega_2}}.$$

By density and continuity of the scalar product we get $T\pi_{\omega_2}(x) = \pi_{\omega_2}(x)T$. □

We are now able to characterise the irreducible GNS-representations.

Theorem 99. *For a state ω on A, the representation π_ω is irreducible if and only if ω is a pure state.*

Proof. Let π_ω be irreducible. By the previous theorem every $\omega' \leq \omega$ can be written as $\omega'(x) = \omega(T\pi_\omega(x))$ with $T \in \pi_\omega'(A)$. Irreducibility gives $T = cI$ according to Lemma (73). Therefore, ω' is a multiple of ω and by Lemma (92) ω is pure.

Conversely, for a pure ω , suppose there is a non trivial projection $P \in \pi_\omega(A)'$. We obtain $P\xi_\omega \neq 0$, for otherwise

$$P[\pi_\omega(x)\xi_\omega] = \pi_\omega(x)P\xi_\omega = 0$$

for all $x \in A$. Since ξ_ω is a cyclic vector, we get $P = 0$. Analogously, $(1-P)\xi_\omega \neq 0$. Therefore, we can decompose ω into the nonzero functionals (cf. Corollary (97)

$$\omega_1(x) := (\pi_\omega(x)P\xi_\omega, P\xi_\omega)_{H_\omega} = \omega(P\pi_\omega(x))$$

$$\omega_2(x) := (\pi_\omega(x)(1-P)\xi_\omega, (1-P)\xi_\omega)_{H_\omega} = \omega((1-P)\pi_\omega(x)),\ \omega = \omega_1 + \omega_2$$

These are positive because $\pi_\omega(x)$ is a positive operator for $x \in A_+$. Since ω is pure, due to Lemma (92), there exists $0 \leq \lambda \leq 1$ with $\omega_1 = \lambda\omega$ and $\omega_2 = (1-\lambda)\omega$. As $\omega_1, \omega_2 \neq 0$ we have $\lambda \neq 0, 1$. Using again, that ξ_ω is a cyclic vector, for every $\epsilon > 0$, we find $x \in A$ such that $\|\pi_\omega(x)\xi_\omega - P\xi_\omega\|_{H_\omega} \leq \epsilon$ and $\|\pi_\omega(x)\xi_\omega\|^2_{H_\omega} = \|P\xi_\omega\|^2_{H_\omega}$. It follows that

$$\|(1-P)\pi_\omega(x)\xi_\omega\|_{H_\omega} = \|(1-P)(\pi_\omega(x)\xi_\omega - P\xi_\omega)\|_{H_\omega} \leq \epsilon.$$

Thus we have

$$(1-\lambda)\|P\xi_\omega\|^2_{H_\omega} = (1-\lambda)\omega(x^*x) = \omega_2(x^*x) = \|(1-P)\pi_\omega(x)\xi_\omega\|^2_{H_\omega} \leq \epsilon^2.$$

This is a contradiction to $\lambda \neq 1$ and by Lemma (72) we conclude $\pi_\omega(A)' = \mathbb{C}1$. Using Theorem (73) we see that π_ω is irreducible. □

We now get, that the representation in Corollary (96) is a direct sum of irreducible ones.

Theorem 100. *The representation $(\bigoplus_{\omega\in P(A)} H_\omega, \bigoplus_{\omega\in P(A)} \pi_\omega)$ called the universal representation is isometric and a direct sum of irreducible representations.*

To end this section we will look once again at commutative C*-algebras. Using the characterization of irreducible GNS representations we can show that for a commutative A the Gelfand space $\hat{A}$ coincides with $P(A)$.

Lemma 101. *Let A be commutative. A functional ω is a pure state if and only if it is multiplicative.*

Proof. As seen in Example (76) any multiplicative functional ω is a state. Since ω is multiplicative we have $H_\omega = A \setminus J_\omega = A \setminus \ker\omega \cong \mathbb{C}$, as the co-dimension of $\ker\omega$ is equal to 1. By Theorem (73) π_ω is irreducible and in turn ω is pure.

Now suppose on the other hand, that ω is pure. We have $\pi_\omega(A) \subseteq \pi_\omega(A)^{'}$, since A is commutative. By Theorem (73), one gets $\pi_\omega(A)^{'} = \mathbb{C}I$ and therefore $\pi_\omega(A) = \mathbb{C}I$. Furthermore,

$$\omega(x) = (\pi_\omega(x)\xi_\omega, \xi_\omega)_{H_\omega} = (c\xi_\omega, \xi_\omega)_{H_\omega} = c$$

yields $\pi_\omega(x) = \omega(x)I$. Hence,

$$\omega(xy) = (\pi_\omega(x)\pi_\omega(y)\xi_\omega, \xi_\omega)_{H_\omega} = \omega(x)\omega(y)(\xi_\omega, \xi_\omega)_{H_\omega} = \omega(x)\omega(y).$$

□

Example 102. It is instructive to derive the representation of commutative C*-algebras by the Gelfand-transform from non commutative representation theory. We now know, that the set of pure states coincides with the Gelfand space. We have $H_\omega \simeq \mathbb{C}$ and $\pi_\omega(x)(z) = \omega(x)z$. Therefore, the universal representation acts on $H = \prod_{\omega \in \hat{A}} \mathbb{C}$ as point wise multiplication by $\omega(x)$. Thus it is given by $f : \hat{A} \to \mathbb{C}$, $\omega \mapsto \omega(x)$.

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