Usages of the Hyperreals in Basic Analysis

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January 24, 2013

There are good reasons to believe that non-standard Analysis, in some version or other, will be the Analysis of the future.
– Kurt Gödel

Introduction

This work should give an introduction to the so called non-standard Analysis, which is based upon the notion of adding infinitesimal numbers - numbers which are smaller than any real number - and infinite numbers - numbers which are greater than any real numbers - to the standard set of real numbers.
This mathematical field has first been developed by Abraham Robinson in his 1966 work [Rob66].

In the first parts, we will show how to construct such a system of hyperreal numbers and show that they do indeed fulfill the desired properties. To achieve this, we will use the so called ultrapower construction. This means we create the hyperreal numbers by use of a/n (non-principal) Ultrafilter, the existence of which has to be proven first. Another key property of hyperreal numbers $^*\mathbb{R}$ is that as a superset of $\mathbb{R}$ they fulfill many of the proven properties of $\mathbb{R}$ and vice-versa. This is formalized by the so called *Transfer Principle*, a theorem from the field of logics. Yet due to its non-analytical nature we won't investigate it further. It should be merely stated that understanding its proof is not relevant for understanding its statement.

In the latter parts, we will use the constructed number system to give non-standard proofs for fundamental real-Analysis Theorems on convergence and continuity. Though this is nothing new - and non-standard Analysis hasn't achieved fundamentally new results in real-Analysis yet. But non-standard proofs have their advantages in terms of comprehensibility and length.
We show that definitions of terms like convergence or continuity often resemble the original, illustrative ideas behind them in non-standard Analysis.

Even more, touching a topic which is not directly part of this work, as its requisites would blow up the length of this work, non-standard concepts can be used in much more general setups of Analysis and functional Analysis and can give extremely short proofs for theorems like *Tychonoff's Theorem.*
For a deeper insight into non-standard Analysis, [Gold98] and [HuLo85] are to be recommended. This work is mostly based upon the first chapters of those two books.

Contents

1 Filters and Ultrafilters

1.1 Fundamentals

Definition 1.1.1. Let S be a non-empty set and 2^S its power set. We call a non-empty $\mathcal{F} \subseteq 2^S$ a *filter* if it satisfies the following conditions:

(F1) if $A_1 \in \mathcal{F}$ and $A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$

(F2) if $A_1 \in \mathcal{F}$ and $A_1 \subseteq A_2$ then $A_2 \in \mathcal{F}$

(F3) $\emptyset \notin \mathcal{F}$

(F3) makes sure that $\mathcal{F} \neq 2^S$. Often a filter is defined without this requirement and the above definition is called a *proper filter*.

Example 1.1.2. Let S be an infinite set. Then

$$\mathcal{F}^{co} := \{A \subseteq S : A^c \text{ is finite}\}$$

is a filter on S, called the *cofinite filter*. This is readily shown, as for $A, B \in \mathcal{F}^{co}$ the set $(A \cap B)^c = A^c \cup B^c$ is finite, hence $A \cap B \in \mathcal{F}^{co}$. Similarly, the superset property follows by the observation that for $A \subseteq B$, $B^c \subseteq A^c$ and, therefore, B^c must be finite if $A \in \mathcal{F}^{co}$. Eventually, $\emptyset$ is finite and $\emptyset^c = S$ is infinite, hence we really have a (proper) filter.

Definition 1.1.3. An *ultrafilter* $\mathcal{U}$ is a filter that additionally satisfies

(UF) for any $A \subseteq S$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

Lemma 1.1.4. *The* generated filter *by a non-empty collection $\mathcal{C} \subseteq \mathcal{P}(S)$, which fulfills the finite intersection property (i.e., the intersection of every non-empty finite subcollection of $\mathcal{C}$ is non-empty), is the smallest filter on S including $\mathcal{C}$ and defined by*

$$\mathcal{F}^{\mathcal{C}} := \{A \subseteq S : A \supseteq C_1 \cap \cdots \cap C_n \text{ for some } n \in \mathbb{N} \text{ and some } C_i \in \mathcal{C}, i = 1 \ldots n\}$$

Proof. It is obvious that $\mathcal{C} \subseteq \mathcal{F}^{\mathcal{C}}$, as for any $A \in \mathcal{C}$ we can simply define $n := 1$ and $C_1 := A$.
To proof that $\mathcal{F}^{\mathcal{C}}$ is a filter we first note that because of $C_1 \cap \cdots \cap C_n \neq \emptyset$ for any $n \in \mathbb{N}$ $\mathcal{F}^{\mathcal{C}}$ cannot contain $\emptyset$.
Furthermore for $A_1, A_2 \in \mathcal{F}^{\mathcal{C}}$ there exists $C_1 \ldots C_m, D_1 \ldots D_n \in \mathcal{C}$ for some $m, n \in \mathbb{N}$ which fulfill $C_1 \cap \cdots \cap C_m \subseteq A_1$ and $D_1 \cap \cdots \cap D_n \subseteq A_2$.
Hence, $A_1 \cap A_2 \supseteq C_1 \cap \cdots \cap C_m \cap D_1 \cap \cdots \cap D_n$ belongs to $\mathcal{F}^{\mathcal{C}}$.
Finally, for any $A_2 \supseteq A_1$ with $A_1 \in \mathcal{F}^{\mathcal{C}}$ the same intersection $C_1 \cap \cdots \cap C_n$ that is part of A_1 is also part of its superset A_2, so A_2 belongs to $\mathcal{F}^{\mathcal{C}}$, which thereby is proven to be a filter.
To show that $\mathcal{F}^{\mathcal{C}}$ is the smallest filter including $\mathcal{C}$ we take any filter $\mathcal{G}$ on S with $\mathcal{C} \subseteq \mathcal{G}$ and note that, because $\mathcal{G}$ is a filter, any finite intersection $C_1 \cap \cdots \cap C_n$ for some $n \in \mathbb{N}$ of elements $C_1 \ldots C_n \in \mathcal{C}$ must be in $\mathcal{G}$ as well. But then, again due to $\mathcal{G}$ being a filter, any superset of such an intersection $B \in S, B \supseteq C_1 \cap \cdots \cap C_n$ has to be part of $\mathcal{G}$, too. Hence, any set which is a part of $\mathcal{F}^{\mathcal{C}}$ is also a part of $\mathcal{G}$. $\square$

Example 1.1.5. $\mathcal{F}^a = \{A \subseteq S : a \in A\}$ is an ultrafilter and is called the *principle ultrafilter* generated by a on S. If S is finite, every ultrafilter is of the form $\mathcal{F}^a$ for some $a \in S$.
It is easy to see that this is just a special case of a generated filter with the collection $\mathcal{C} = \{a\}$.
Due to the definition of $\mathcal{F}^a$ any set that does not include a is not part of the filter. But each of those is the complement of a set containing a. Hence, $\mathcal{F}^a$ is an ultrafilter.

Corollary 1.1.6. *$\mathcal{F}$ is an ultrafilter on S, if and only if it is a maximal filter on S, i.e. a filter that cannot be properly extended to a larger filter on S.*

Proof.
" $\Rightarrow$ " : If $\mathcal{F}$ is an ultrafilter and we extend it to any collection $\mathcal{F}_A$ including some $A \subseteq S, A \notin \mathcal{F}$, then both, A and A^c are part of $\mathcal{F}_A$. As $A \cap A^c = \emptyset$, $\mathcal{F}_A$ cannot be a filter. Hence, $\mathcal{F}$ is already maximal.
" $\Leftarrow$ " : Suppose $\mathcal{F}$ is a filter but not an ultrafilter. This means that there exists an $A \subseteq S$ that fulfills $A, A^c \notin \mathcal{F}$.
Now $A \cap B \neq \emptyset$ for all $B \in \mathcal{F}$ or $A^c \cap B \neq \emptyset$ for all $B \in \mathcal{F}$. To see this, we suppose that there are $B_1, B_2 \in \mathcal{F}$ that fulfill $B_1 \cap A = \emptyset$ and $B_2 \cap A^c = \emptyset$. This implies that $B_2 \subseteq A$ and $B_1 \subseteq A^c$ and because $\mathcal{F}$ is a filter, both A and A^c would be elements of $\mathcal{F}$. Hence, such sets B_1, B_2 do not exist.
This implies that the collection $A \cup \mathcal{F}$ or the collection $A^c \cup \mathcal{F}$ fulfill/s the finite intersection property. Therefore, it generates the filter $\mathcal{F}^{\mathcal{F} \cup A}$ or $\mathcal{F}^{\mathcal{F} \cup A^c}$ that properly includs $\mathcal{F}$. This means $\mathcal{F}$ cannot be maximal. □

Definition 1.1.7. An ultrafilter $\mathcal{U}$ is *free* if it does not contain any finite subsets of S, i.e.

(FREE) $|A| \geq \aleph_0$, for all $A \in \mathcal{U}$.

Lemma 1.1.8. *If an ultrafilter $\mathcal{U}$ is not free, it contains a one-element set and it is, therefore, principal.*

Proof. Let us assume that $\mathcal{U}$ is not free and contains no one-element set. Then $\mathcal{U}$ must contain all sets of the form $S \backslash \{a\}$ for all $a \in S$, because it is an ultrafilter.
Furthermore, because $\mathcal{U}$ is not free, there is an $A \in \mathcal{U}$ with $A = \{a_1 \ldots a_n\}$ for some $n \in \mathbb{N}$.
Since $\mathcal{U}$ contains all sets of the form $S \backslash \{a_k\}$ $k = 1 \ldots n$, $\mathcal{U}$ being a filter implies

$$\bigcap_{k=1}^{n} S \backslash \{a_k\} = A^c \in \mathcal{U},$$

which contradicts $A \in \mathcal{U}$. □

As a consequence, we see that any non-principal ultrafilter must contain all *cofinite sets* (cf. Example 1.1.5).

Lemma 1.1.9. *Let $(\mathcal{F}_i)_{i \in I}$ be a collection of filters $\mathcal{F}_i$ on S, which is totally ordered by $\subseteq$. Then $\bigcup_{i \in I} \mathcal{F}_i$ is a filter on S.*

Proof. Since $(\mathcal{F}_i)_{i\in I}$ is totally ordered by set inclusion, for any $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}$ we have $A_1, A_2 \in \mathcal{F}_{i_1}$ or $A_1, A_2 \in \mathcal{F}_{i_2}$. Hence, $A_1 \cap A_2 \in \bigcup_{i\in I} \mathcal{F}_i$ as $\mathcal{F}_{i_1}$ and $\mathcal{F}_{i_2}$ are filters.
Furthermore, because any $A \in \bigcup_{i\in I} \mathcal{F}_i$ is an element of a filter $\mathcal{F}_j$ for some $j \in I$, all its supersets are part of $\mathcal{F}_j$ and therefore part of $\bigcup_{i\in I} \mathcal{F}_i$.
Finally, because $\emptyset$ is in no filter of our collection, it is not part of $\bigcup_{i\in I} \mathcal{F}_i$ either. □

Lemma 1.1.10. *Let $\mathcal{F}$ be an ultrafilter and $A_1, \ldots, A_n$, $n \in \mathbb{N}$ a finite collection of pairwise disjoint sets such that $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$.*
Then $A_i \in \mathcal{F}$ for exactly one $i \in \{1, \ldots n\}$.

Proof. To prove that at least one $A_i \in \mathcal{F}$ assume the contrary. Then, due to (UF), $A_i^c \in \mathcal{F}$ for $i = 1, \ldots, n$ and, therefore, due to (F1), $\bigcap_{i=1}^{n} A_i^c \in \mathcal{F}$. As $\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i\right)^c$, this contradicts $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$.
To prove that not more than one $A_i \in \mathcal{F}$, assume $A_{i_1}, A_{i_2} \in \mathcal{F}$ for distinct $i_1, i_2 \in \{1, \ldots, n\}$.
Because those two are disjoint, $A_{i_1} \subseteq A_{i_2}^c$ and due to (F2) $A_{i_2}^c \in \mathcal{F}$, which contradicts (UF). □

1.2 Existence of Ultrafilters

It is not obvious that a non-principal ultrafilter even exists. However, using *Zorn's Lemma* we will prove that such an object indeed exists on any infinite set.

Lemma 1.2.1 (Zorn's Lemma)**.** *Suppose $\mathcal{P}$ is a set partially ordered by the relation $\preceq$, in which every totally ordered subset has an upper bound in $\mathcal{P}$. Then $\mathcal{P}$ contains at least one maximal element.*

A proof for this assertion can be found for example in [BuDa98].

Theorem 1.2.2 (Ultrafilter Theorem)**.** *Any collection of subsets of S that has the finite intersection property is contained in an ultrafilter on S.*

Proof. Suppose $\mathcal{C}$ is a collection with the finite intersection property and define

$$\mathcal{P} := \left\{\mathcal{F} \subseteq 2^S : \mathcal{F} \text{ is a filter on } S,\, \mathcal{F}^{\mathcal{C}} \subseteq \mathcal{F}\right\}.$$

$\mathcal{P}$ is partially ordered by set inclusion '$\subseteq$' and, therefore, $\bigcup_{\mathcal{F}\in\mathcal{O}} \mathcal{F}$ is a filter and part of $\mathcal{P}$ for any totally ordered subset $\mathcal{O} \subseteq \mathcal{P}$ (cf. 1.1.9). This means that all of those chains have an upper bound in $\mathcal{P}$. Due to Zorn's Lemma, $\mathcal{P}$ has a maximal element $\mathcal{M}$. Thereby $\mathcal{M}$ is a maximal filter on S, hence, it is an ultrafilter on S (cf. Corollary 1.1.6.).
To see this, suppose $\mathcal{M}$ can be properly extended, i.e. for an $A \in S$, $A \notin \mathcal{M}$ there exists the filter $\mathcal{F}^{\mathcal{M}\cup A}$. As $\mathcal{F}^{\mathcal{M}\cup A} \supseteq \mathcal{M} \supseteq \mathcal{F}^{\mathcal{C}}$, $\mathcal{F}^{\mathcal{M}\cup A}$ is an element of $\mathcal{P}$. Hence, $\mathcal{M}$ cannot be maximal, which is obviously a contradiction to $\mathcal{M}$ being maximal. □

For this proof, Zorn's Lemma was employed. The question arises, whether the Ultrafilter Theorem is equivalent to the Axiom of Choice. But it can be shown that it is a weaker statement.

Corollary 1.2.3. *Any infinite set has a non-principal ultrafilter on it.*

Proof. If S is infinite, Example 1.1.2 shows that $\mathcal{F}^{co}$ is a filter. Clearly, it fulfills the finite intersection property. Hence, it is included in an ultrafilter $\mathcal{F}$ (cf. Theorem 1.2.2.). But for any $s \in S$ we have $S \backslash \{s\} \in \mathcal{F}^{co} \subseteq \mathcal{F}$. Hence, $\mathcal{F}$ cannot be of the form $\mathcal{F}^s$. □

2 Hyperreals

2.1 The Construction of the Hyperreals

We will start our construction with a basic result about sequences. $\left(\mathbb{R}^{\mathbb{N}}, +, \cdot\right)$ forms a ring. In this context, $+$ and $\cdot$ stand for the component-wise multiplication and addition of members of sequences. However, $\left(\mathbb{R}^{\mathbb{N}}, +, \cdot\right)$ is not a field as can be seen in the next example.

Example 2.1.1. For the sequences $\langle a_n \rangle = \langle 1, 0, 1, 0, 1, 0, \ldots \rangle$ and $\langle b_n \rangle = \langle 0, 1, 0, 1, 0, 1, \ldots \rangle$ we observe that

$$\langle a_n \rangle \cdot \langle b_n \rangle = \langle 0, 0, 0, 0, 0, 0, ... \rangle = \langle 0 \rangle.$$

Hence, those two sequences cannot have a multiplicative inverse.

Definition 2.1.2. Let $\mathcal{F}$ be a non-principal ultrafilter on the set $\mathbb{N}$.
Then we can define a relation $\equiv$ on $\mathbb{R}^{\mathbb{N}}$ by putting

$$\langle a_n \rangle \equiv \langle b_n \rangle \quad \text{iff} \quad \{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{F}$$

Lemma 2.1.3. $\equiv$ *is an equivalence relation on* $\mathbb{R}^{\mathbb{N}}$.

Proof. The Reflexivity and Symmetry are obvious. The Transitivity holds because for $\{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{F}$ and $\{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$ it follows ($\mathcal{F}$ is a Filter) that $\{n \in \mathbb{N} : a_n = b_n = c_n\} \in \mathcal{F}$. But this is a subset of $\{n \in \mathbb{N} : a_n = c_n\}$, which, therefore, belongs to $\mathcal{F}$. □

If $\langle a_n \rangle \equiv \langle b_n \rangle$, we say that $\langle a_n \rangle$ and $\langle b_n \rangle$ are *equivalent modulo* $\mathcal{F}$.
It is important to note that two convergent sequences may have the same limit for $n \to +\infty$, but, are not equivalent.

Example 2.1.4. $a_n := \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle \not\equiv \langle 0, 0, 0, \ldots \rangle := b_n$ as $\{n \in \mathbb{N} : b_n = a_n\} = \emptyset \notin \mathcal{F}$.

$\equiv$ eliminates the problem that non-zero sequences can have zero-products as we will show below.

Example 2.1.5. Consider the sequences $\langle a_n \rangle := \langle 1, 0, 1, 0, 1, 0, ... \rangle$ and $\langle b_n \rangle := \langle 0, 1, 0, 1, 0, 1, ... \rangle$. In case that $\langle a_n \rangle \not\equiv \langle \mathbf{0} \rangle$, i.e $\{n \in \mathbb{N} : a_n = 0\} \notin \mathcal{F}$, for the complement we have $\{n \in \mathbb{N} : a_n = 0\}^c = \{n \in \mathbb{N} : a_n \neq 0\} = \{n \in \mathbb{N} : b_n = 0\} \in \mathcal{F}$, i.e. $\langle b_n \rangle \equiv \langle \mathbf{0} \rangle$. So we see that exactly one of these sequences is equivalent to zero.

Definition 2.1.6. We denote the equivalence class of a sequence $a = \langle a_n \rangle$ under $\equiv$ by $[a]$. Thus

$$[a] = \left\{b \in \mathbb{R}^{\mathbb{N}} : a \equiv b\right\}$$

Now we define the Hyperreals as the quotient set of $\mathbb{R}^{\mathbb{N}}$ by $\equiv$ and write

$$^*\mathbb{R} := \mathbb{R}^{\mathbb{N}}/\equiv = \left\{[a] : a \in \mathbb{R}^{\mathbb{N}}\right\}.$$

Definition 2.1.7. For $a = \langle a_n \rangle, b = \langle b_n \rangle$ we set

$$\begin{aligned}
&[a] + [b] = [\langle a_n + b_n \rangle] \\
&[a] \cdot [b] = [\langle a_n \cdot b_n \rangle] \\
&[a] < [b] \text{ iff } \{n \in \mathbb{N} : a_n < a_n\} \in \mathcal{F}
\end{aligned}$$

As this is a definition on equivalence classes, we have to show that they are well-defined. Pick $\langle a_n \rangle, \langle a'_n \rangle, \langle b_n \rangle, \langle b'_n \rangle$ such that $\{n \in \mathbb{N} : a_n = a'_n\} \in \mathcal{F}$ and $\{n \in \mathbb{N} : b_n = b'_n\} \in \mathcal{F}$. Then $\{n \in \mathbb{N} : a_n + b_n = a'_n + b'_n\} \supseteq \{n \in \mathbb{N} : a_n = a'_n\} \cap \{n \in \mathbb{N} : b_n = b'_n\}$ is also a member of $\mathcal{F}$.
We achieve the same result for $\cdot$ in the same fashion. Hence, $+$ and $\cdot$ are well-defined on $^*\mathbb{R}$.
For '$<$' we observe that

$$\{n \in \mathbb{N} : a_n = a'_n\} \cap \{n \in \mathbb{N} : b_n = b'_n\} \cap \{n \in \mathbb{N} : a_n < b_n\} \subseteq \{n \in \mathbb{N} : a'_n < b'_n\}$$

and

$$\{n \in \mathbb{N} : a_n = a'_n\} \cap \{n \in \mathbb{N} : b_n = b'_n\} \cap \{n \in \mathbb{N} : a'_n < b'_n\} \subseteq \{n \in \mathbb{N} : a_n < b_n\}.$$

Hence, $\{n \in \mathbb{N} : a_n < b_n\} \in \mathcal{F}$, if and only if $\{n \in \mathbb{N} : a'_n < b'_n\} \in \mathcal{F}$.

Corollary 2.1.8. *The structure* $(^*\mathbb{R}, +, \cdot, <)$ *is an ordered field with zero* $[\mathbf{0}]$ *and unity* $[\mathbf{1}]$.

Proof. It is readily shown that $^*\mathbb{R}$ is a commutative ring with zero $[\mathbf{0}]$ and unity $[\mathbf{1}]$ and the additive inverse $-[\langle a_n : n \in \mathbb{N} \rangle] = [\langle -a_n : n \in \mathbb{N} \rangle]$. Suppose that $[a] \neq 0$, i.e. $a \not\equiv 0$. Because $\mathcal{F}$ is an ultrafilter, $J := \{n \in \mathbb{N} : a_n \neq 0\} \in \mathcal{F}$.
Now define a sequence b by putting

$$b_n = \begin{cases} \frac{1}{a_n} & \text{if } n \in J \\ 0 & \text{otherwise} \end{cases}.$$

From $\{n \in \mathbb{N} : a_n \cdot b_n = 1\} = J \in \mathcal{F}$ we conclude $a \cdot b \equiv \mathbf{1}$. Hence, $[a] \cdot [b] = [a \cdot b] = [\mathbf{1}]$ in $^*\mathbb{R}$; i.e. $[b] = [a]^{-1}$.
$<$ on $^*\mathbb{R}$ is a total ordering, because $\mathbb{N}$ is the disjoint union of

$$\{n \in \mathbb{N} : a_n < b_n\}, \{n \in \mathbb{N} : a_n = b_n\}, \{n \in \mathbb{N} : a_n < b_n\}.$$

So exactly one of them belongs to $\mathcal{F}$. Therefore, exactly one of the relations

$$[a] < [b], [a] = [b], [a] > [b]$$

holds true.
Furthermore, the sum of two positive elements $a, b \in {}^*\mathbb{R}$ is positive, because of

$$\{n \in \mathbb{N} : a_n > 0\} \cap \{n \in \mathbb{N} : a_n > 0\} \subseteq \{n \in \mathbb{N} : a_n + b_n > 0\}.$$

The multiplication of two positive elements $a, b \in {}^*\mathbb{R}$ is positive, because of

$$\{n \in \mathbb{N} : a_n > 0\} \cap \{n \in \mathbb{N} : a_n > 0\} \subseteq \{n \in \mathbb{N} : a_n \cdot b_n > 0\}.$$

This shows that $(^*\mathbb{R}, +, \cdot, <)$ is indeed an ordered field. □

Definition 2.1.9. To include the reals in the hyperreals we define the mapping

$$\begin{aligned} * : \ & \mathbb{R} \to {}^*\mathbb{R} \\ & a \mapsto {}^*a := [\langle a, a, a, \dots \rangle] \end{aligned}$$

Lemma 2.1.10. *The mapping $*$ is an injective, order-preserving homomorphism from $\mathbb{R}$ into ${}^*\mathbb{R}$.*

Proof. It is clear that $*$ is a homomorphism. Moreover ${}^*a = {}^*b$ is equivalent to $\langle a, a, a, \dots \rangle \equiv \langle b, b, b, \dots \rangle$ and, hence, $a = b$. So $*$ is injective. The field and order properties are trivial to prove. □

We name members of $\mathbb{R}$ *standard* and members of ${}^*\mathbb{R}\backslash\mathbb{R}$ *non-standard*.

Example 2.1.11. Define $\varepsilon = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle = \langle \frac{1}{n} : n \in \mathbb{N} \rangle$ and compare it to $\langle 0, 0, 0, \dots \rangle = \langle \frac{0}{n} : n \in \mathbb{N} \rangle$. Then $\left\{ n \in \mathbb{N} : \frac{0}{n} < \frac{1}{n} \right\} = \mathbb{N} \in \mathcal{F}$, i.e. $[\mathbf{0}] < [\varepsilon]$ in ${}^*\mathbb{R}$.
Furthermore, for any positive $a \in \mathbb{R}$, the set $\left\{ n \in \mathbb{N} : \frac{1}{n} < a \right\}$ is cofinite. As $\mathcal{F}$ is a non-principal ultrafilter and hence free, it contains all cofinite sets. Consequently, $[\varepsilon] < {}^*a$ in ${}^*\mathbb{R}$.

Example 2.1.12. Define $\omega = \langle 1, 2, 3, \dots \rangle = \langle \frac{1}{n} : n \in \mathbb{N} \rangle$ and pick any positive $a \in \mathbb{R}$. Then the set $\{n \in \mathbb{N} : a < n\}$ is cofinite and, therefore, belongs to $\mathcal{F}$. Hence, $[\omega] < {}^*a$.

As the above examples show, the choice of constructing ${}^*\mathbb{R}$ with a non-principal ultrafilter leads to elements of ${}^*\mathbb{R}$ that are not equivalent to elements of $\mathbb{R}$. Even more, using the notation from above, it is easy to see that $\varepsilon \cdot \omega = \mathbf{1}$, so $[\varepsilon] = [\omega]^{-1}$ and $[\omega] = [\varepsilon]^{-1}$. We call such elements of ${}^*\mathbb{R}$ *infinitesimal* and *unlimited*.

2.2 Enlargement and Extensions

Definition 2.2.1. For $A \subseteq \mathbb{R}$ define its *enlargement*

$${}^*A = \left\{ [\langle a_n \rangle] \in {}^*\mathbb{R} : \{n \in \mathbb{N} : a_n \in A\} \in \mathcal{F} \right\}$$

This is a proper definition, as for $a \equiv a'$ and $\{n \in \mathbb{N} : a_n \in A\} \in \mathcal{F}$ it follows that

$$\{n \in \mathbb{N} : a_n = a'_n\} \cap \{n \in \mathbb{N} : a_n \in A\} \in \mathcal{F}$$

and, hence,

$$\{n \in \mathbb{N} : a_n = a'_n\} \cap \{n \in \mathbb{N} : a_n \in A\} \subseteq \{n \in \mathbb{N} : a'_n \in A\}.$$

So $\{n \in \mathbb{N} : a'_n \in A\} \in \mathcal{F}$ as well. The converse is also true.

For $b \in A$ observe that ${}^*b = [\langle b, b, b, \dots \rangle] \in {}^*A$, because $\{n \in \mathbb{N} : b \in A\} = \mathbb{N} \in \mathcal{F}$. Identifying b with ${}^*b$ we can say that $A \subseteq {}^*A$.

To get a better idea of enlarged sets, consider the following theorem.

Theorem 2.2.2. *For $A \in \mathbb{R}$,*

(a) if A is infinite, ${}^*A$ contains non-standard members, hence ${}^*A \supsetneq A$.

(b) if A is finite, ${}^*A$ does not contain non-standard members, hence ${}^*A = A$.

Proof.

(a) Because A is infinite, there exists a sequence $\langle a_n \rangle$, $a_n \in A$ with $a_i \neq a_j$ for all $i \neq j, i, j \in \mathbb{N}$. Therefore, $\{n \in \mathbb{N} : a_n \in A\} = \mathbb{N} \in \mathcal{F}$, i.e. $[a] \in {}^*A$. But for any $b \in A$, $\{n \in \mathbb{N} : a_n = b\}$ is either $\emptyset$ or a singleton, neither of which belongs to $\mathcal{F}$, because $\mathcal{F}$ is non-principal. Hence, $[a] \in {}^*A \backslash A$.

(b) If A is finite, then for any sequence $\langle a_n \rangle$, $a_n \in A$ the set $\{a_n : n \in \mathbb{N}\}$ is finite. Therefore, the set $\{n \in \mathbb{N} : a_n \in \mathbb{N}\}$ equals $\{n \in \mathbb{N} : a_n = b_1\} \uplus \cdots \uplus \{n \in \mathbb{N} : a_n = b_m\}$ for some $b_1, \ldots, b_m \in A$.
As this is a disjoint, finite union, $\{n \in \mathbb{N} : a_n \in \mathbb{N}\} \in \mathcal{F}$ if and only if $\{n \in \mathbb{N} : a_n = b_j\} \in \mathcal{F}$ for some $j \in \{1, \cdots, m\}$. Hence, for any $[a] \in {}^*A$, $[a] \equiv {}^*b$, for some $b \in A$.

□

Definition 2.2.3. For $f : \mathbb{R}^m \to \mathbb{R}$ define its *hyperreal extension* ${}^*f : {}^*\mathbb{R}^m \to {}^*\mathbb{R}$ as ${}^*f\left([\langle a_n^1 \rangle], \ldots, [\langle a_n^m \rangle]\right) = [\langle f(a_n^1, \ldots, a_n^m) \rangle]$.

This is a proper definition, because

$$\left\{n \in \mathbb{N} : a_n^1 = a_n^{1\prime}, \ldots, a_n^m = a_n^{m\prime}\right\} \subseteq \left\{n \in \mathbb{N} : f(a_n^1, \ldots, a_n^m) = f(a_n^1, \ldots, a_n^{m\prime})\right\}$$

and, therefore, $a^1 \equiv a^{1\prime}, \ldots, a^m \equiv a^{m\prime}$ implies $f \circ (a^1, \ldots, a^m) \equiv f \circ (a^1, \ldots, a^{m\prime})$.
However, the given definition does not apply to functions of the form $f : A^1 \times \cdots \times A^m \to \mathbb{R}$, $A^1, \ldots, A^m \subseteq \mathbb{R}$ as for $l \notin \{n \in \mathbb{N} : a_n^j \in A^j\}$ for some $j \in \{1, \ldots, m\}$, $f(a_l^1, \ldots, a_l^m)$ is undefined.

Definition 2.2.4. For $f : A^1 \times \cdots \times A^m \to \mathbb{R}$, $A^1, \ldots, A^m \subseteq \mathbb{R}$ define its *hyperreal extension* ${}^*f : {}^*A^1 \times \cdots \times {}^*A^m \to {}^*\mathbb{R}$ as follows: for each $\langle a_n^1 \rangle, \ldots, \langle a_n^m \rangle \in \mathbb{R}^{\mathbb{N}}$ with $[\langle a_n^1 \rangle] \in {}^*A^1, \ldots, [\langle a_n^m \rangle] \in {}^*A^m$, so that $\{n \in \mathbb{N} : a_n^1 \in A^1\} \in \mathcal{F}, \ldots, \{n \in \mathbb{N} : a_n^m \in A^m\} \in \mathcal{F}$ define the sequence

$$b_n := \begin{cases} f(a_n^1, \ldots, a_n^m) & \text{if } n \in \{n \in \mathbb{N} : a_n^1 \in A^1\} \cap \cdots \cap \{n \in \mathbb{N} : a_n^m \in A^m\} \\ 0 & \text{if } n \notin \{n \in \mathbb{N} : a_n^1 \in A^1\} \cap \cdots \cap \{n \in \mathbb{N} : a_n^m \in A^m\} \end{cases}$$

and set

$${}^*f([\langle a_n^1 \rangle], \ldots, [\langle a_n^m \rangle]) = [\langle b_n \rangle]$$

An important example for this concept is the extension of sequences to *hypersequences*. For $s : \mathbb{N} \to \mathbb{R}$ we have ${}^*s : {}^*\mathbb{N} \to {}^*\mathbb{R}$. In particular, ${}^*s$ is defined for non-standard $n \in {}^*\mathbb{N}$.

A k-ary relation P on $\mathbb{R}$ is a set of k-tuples, i.e. a subset of $\mathbb{R}^k$.

Definition 2.2.5. Let P be a k-ary relation on $\mathbb{R}$. Then P can be enlarged to a k-ary relation ${}^*P$ on ${}^*\mathbb{R}$ by defining for given sequences $a^1, \dots, a^k \in \mathbb{R}^{\mathbb{N}}$

$$ {}^*P([a^1], \dots, [a^k]) \quad \text{iff} \quad \{n \in \mathbb{N} : P(a_n^1, \dots, a_n^k)\} \in \mathcal{F} $$

Again, we have to prove that this notion is well-defined. For sequences $\langle a_n^1 \rangle, \dots, \langle a_n^k \rangle$, $\langle b_n^1 \rangle, \dots, \langle b_n^k \rangle$ we have that

$$ \{n \in \mathbb{N} : a_n^1 = b_n^1\} \cap \dots \cap \{n \in \mathbb{N} : a_n^k = b_n^k\} \cap \{n \in \mathbb{N} : P(a_n^1, \dots, a_n^k)\} \subseteq \{n \in \mathbb{N} : P(b_n^1, \dots, b_n^k)\}, $$

which gives us the desired result. Furthermore, for real numbers $a^1, \dots, a^k$ we have that

$$ P(a^1, \dots, a^2) \quad \text{iff} \quad {}^*P({}^*a^1, \dots, {}^*a^k). $$

Hence, ${}^*P$ is really an enlargement of P.

It is only natural to denote the same functions and relations with the same symbols, even if they operate on different fields. Therefore, we will drop the $*$-notation for enlarged relations and extended functions in the following chapters, unless further clarification is necessary.

2.3 Uniqueness of the Hyperreals

Up to now, we have constructed the hyperreals as a quotient set of $\mathbb{R}^{\mathbb{N}}$ depending on the particular non-principal ultrafilter $\mathcal{F}$. The question arises, whether this system is unique or whether we achieve many different such hyperreal number systems, when we choose different ultrafilters.
Due to the theoretical nature of ultrafilters this question cannot be answered using the ZFC-system of mathematical axioms. It has, however, been proven that, if we assume that the *Continuum Hypothesis* holds, all of the constructed hyperreal fields are isomorphic to each other. The proof for this can be found in [ErGiHe55].

2.4 The Transfer Principle

After extending functions and relations from $\mathbb{R}$ to ${}^*\mathbb{R}$ we still have to prove that those extensions follow the same rules as the original functions. Therefore we need some basic knowledge about symbolic logic for relational systems (cf. [Raut08]), in order to understand that the *Transfer Principle* does exactly that for us.

Definition 2.4.1. A *relational system* is a structure $\mathfrak{S} = (S, \{P_i : i \in I\}, \{f_j : j \in J\})$ consisting of a set S, a collection of finitary relations $P_i (i \in I)$ on S and a collection of finitary functions $f_j (j \in J)$ on S.

The particular relational structures that are interesting for us are

$$\begin{aligned} \mathfrak{R} &= (\mathbb{R}, Rel_{\mathbb{R}}, Fun_{\mathbb{R}}) \text{ and} \\ {}^*\mathfrak{R} &= ({}^*\mathbb{R}, \{{}^*P : P \in Rel_{\mathbb{R}}\}, \{{}^*f : f \in Fun_{\mathbb{R}}\}) \end{aligned}$$

with

$$\begin{aligned} Rel_{\mathbb{R}} &= \{P : P \text{ is a finitary relation on } \mathbb{R}\} \text{ and} \\ Fun_{\mathbb{R}} &= \{f : f \text{ is a finitary function on } \mathbb{R}\}. \end{aligned}$$

Remark. A relational structure consisting of a set S and all finitary relations and functions defined on S is called a *full structure*. By its definition $\mathfrak{R}$ is such, but ${}^*\mathfrak{R}$ is not, because there exist relations on ${}^*\mathbb{R}$ that are not of the form ${}^*P$.

Definition 2.4.2. For a relational structure $\mathfrak{S}$ we define its *symbolic language* $\mathfrak{L}_{\mathfrak{S}}$ by a basic set of symbols and combinations of those basic symbols:

- Logical Connectives: $\wedge$ and $\neg$, to be interpreted as "and" and "not".
- Quantifier Symbols: $\forall$, interpreted as "for all".
- Parentheses: [,], (and) to be used for bracketing.
- Variable Symbols: a countable collection of symbols such as letters, to be used as "variables".
- Equality Symbol: =, to be interpreted as "equals"
- Constant Symbols: A symbol s for each $s \in S$.
- Relation Symbols: A symbol P for each $P \in \mathfrak{S}$.
- Function Symbols: A symbol f for each $f \in \mathfrak{S}$.

Furthermore, we want to combine such symbols, which leads to the next definition.

Definition 2.4.3. We define *terms* of $\mathfrak{L}_{\mathfrak{S}}$ as follows:

(a) Constant and variable symbols are terms.

(b) For f being function of n variables and $\tau^1, \ldots, \tau^n$ are terms, then $f(\tau^1, \ldots, \tau^n)$ is a term.

A term containing no variables is called a *constant term*.

Example 2.4.4. The expression $xy + xz$ is a term in $\mathfrak{L}_{\mathfrak{R}}$, which can be seen by defining functions

$$\begin{aligned} A(a, b) &:= a + b \\ M(a, b) &:= ab \end{aligned}$$

for $a, b \in \mathbb{R}$. Hence, $xy + xz$ denotes $A(M(x, y), M(x, z))$.

Definition 2.4.5. A *formula* in $\mathfrak{L}_{\mathfrak{S}}$ is defined by:

(a) *Atomic Formula*:

- for terms a, b, $a = b$ is a formula
- for an n-ary relation P, $P(\tau^1, \ldots, \tau^n)$ is a formula

(b) *Formula*

- for a formula Φ, $\neg\Phi$ is a formula
- for formulae Φ, Ψ, $\Phi \wedge \Psi$ is a formula
- for a variable x and a formula Φ, $\forall x \Phi$ is a formula

Example 2.4.6. Let I name the inequality relation $<$. Then $I(x, 2)$, usually written as $x < 2$ is an atomic formula.

Remark. It is convenient to use more symbols than those which are defined in Definition 2.4.2.
For formulas Φ and Ψ we will denote

- $\neg(\neg\Phi \wedge \neg\Psi)$ as $\Phi \vee \Psi$
- $(\neg\Phi) \vee \Psi$ as $\Phi \Rightarrow \Psi$.
- $\Phi \Rightarrow \Psi$ as $\Psi \Leftarrow \Phi$.
- $(\Phi \Rightarrow \Psi) \wedge (\Psi \Leftarrow \Phi)$ as $\Phi \Leftrightarrow \Psi$.
- $\neg(\forall x \Phi)$ as $\exists x \neg\Phi$.

Furthermore, for any $A \subseteq S$ we define a relation E by putting $E(a)$, iff a is an element of A. In this case we write $a \in A$.

Definition 2.4.7. An occurence of a variable x in a formula Φ can be *free* or *bound.*

- If Φ is an atomic formula, x is free in Φ, if and only if x occurs in Φ. In an atomic formula, no variable is bound.
- The occurence of x is free in $\neg\Phi$, if and only if it is free in Φ. The occurence of x is bound in $\neg\Phi$, if and only if is bound in Φ.
- The occurence of x is free in $(\Phi \wedge \Psi)$, if it is free in Φ or Ψ. The occurence of x is bound in $(\Phi \wedge \Psi)$, if it is bound in Φ and Ψ.
- The occurence of x is free in $\forall y \Phi$, if and only if x is free in Φ and x is a different Symbol from y. The occurence of x is bound, if and only if x is y or x is bound in Φ.

Example 2.4.8. Let e=1. In the formula

$$(x < e) \wedge (\forall x)(x > y)$$

the first occurence of x is free, the others are bound and the only occurence of y is free.
The variable z has no occurence in the formula and, thus, is neither bound nor free.
e is a constant and, therefore, not free or bound either.
To summarize, the only variables in the above formula are x and y and they are both free, because both occur free for at least once.

Definition 2.4.9. A formula in which all variables are bound is called a sentence.

Example 2.4.10. The fact that the distributive law holds in $\mathbb{R}$ is expressed by

$$(\forall x)(\forall y)(\forall z)[x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge z \in \mathbb{R} \Rightarrow x \cdot (y+z) = x \cdot y + x \cdot z].$$

It is an example for a sentence in $\mathfrak{L}_{\mathfrak{R}}$.

Example 2.4.11. For variables x and y, the expression

$$(\forall x)[x \in \mathbb{R} \wedge x < 1 \Rightarrow x < y]$$

is not a sentence, since the variable y is not bound.

Definition 2.4.12. The $*$-transform ${}^*\Phi$ of a formula Φ in $\mathfrak{L}_{\mathfrak{R}}$ is defined by

- replacing each constant term τ occuring in Φ with ${}^*\tau$
- replacing each function term $f(\tau^1, \ldots, \tau^n)$ occuring in Φ with ${}^*f(\tau^1, \ldots, \tau^n)$
- replacing each relation P occuring in Φ with ${}^*P$

The definitions given in this section allow us to phrase the Transfer Principle for the real and hyperreal numbers.

Theorem 2.4.13 (Transfer Principle)**.** *A sentence Φ in $\mathfrak{L}_{\mathfrak{R}}$ is true if and only if ${}^*\Phi$ is true in $\mathfrak{L}_{{}^*\mathfrak{R}}$*

A proof for this theorem is given in [HuLo85]. When using it later on, we will often times simply use phrases like "transferring" or "by transfer".

Example 2.4.14. The distributive law given in Example 2.4.9. holds on $\mathbb{R}$. Hence, by transfer we obtain

$$(\forall x)(\forall y)(\forall z)[x \in {}^*\mathbb{R} \wedge y \in {}^*\mathbb{R} \wedge z \in {}^*\mathbb{R} \Rightarrow x \cdot (y+z) = x \cdot y + x \cdot z],$$

which describes the fact that the distributive law holds on ${}^*\mathbb{R}$. In the above expression, $x \in {}^*\mathbb{R}$ denotes the hyperreal extension of the relation $x \in \mathbb{R}$. (cf. Definition 2.2.5)

2.5 Terms and Arithmetics

Definition 2.5.1. We call a number $b \in {}^*\mathbb{R}$

- *limited*, if $x < b < y$ for some $x, y \in \mathbb{R}$.
- *positive unlimited*, if $x < b$ for all $x \in \mathbb{R}$.
- *negative unlimited*, if $b < x$ for all $x \in \mathbb{R}$.
- *unlimited*, if x is positive unlimited or negative unlimited.
- *positive infinitesimal*, if $0 < b < x$ for all $x \in \mathbb{R}$.
- *negative infinitesimal*, if $x < b < 0$ for all $x \in \mathbb{R}$.
- *infinitesimal*, if x is positive infinitesimal, negative infinitesimal or 0.
- *appreciable*, if x is limited, but not infinitesimal.

Example 2.5.2. If $k \in {}^*\mathbb{N}$ is limited, then $k \leq n$ for some $n \in \mathbb{N}$. But then by transferring the sentence

$$\forall m \in \mathbb{N} : m \leq n \Rightarrow m = 1 \vee m = 2 \vee \cdots \vee m = n$$

we get $k \in \{1, 2, \ldots, n\}$. Hence, any limited hypernatural number is already a natural number, i.e ${}^*\mathbb{N}/\mathbb{N}$ contains only unlimited numbers.

Theorem 2.5.3. *Let ε, δ be infinitesimal, a, b appreciable, U, V unlimited and $n \in {}^*\mathbb{N}$.*

⋆ **Sums**
$\varepsilon + \delta$ is infinitesimal
$a + \delta$ is appreciable
$a + b$ is limited
$U + \varepsilon$ and $U + a$ are unlimited

⋆ **Opposites**
$-\varepsilon$ is infinitesimal
$-a$ is appreciable
$-U$ is unlimited

⋆ **Products**
$\varepsilon \cdot \delta$ and $\varepsilon \cdot b$ are infinitesimal.
$a \cdot b$ is appreciable
$a \cdot U$ and $U \cdot V$ are unlimited

⋆ **Reciprocals**
$\frac{1}{\varepsilon}$ is unlimited if $\varepsilon \neq 0$
$\frac{1}{a}$ is appreciable
$\frac{1}{U}$ is infinitesimal

⋆ **Roots**
if $\varepsilon > 0$, $\sqrt[n]{\varepsilon}$ is infinitesimal
if $a > 0$, $\sqrt[n]{a}$ is appreciable
if $H > 0$, $\sqrt[n]{H}$ is unlimited

Proof. All of those statements can be proven via straight forward verification of Definition 2.5.1. □

The root-function in Theorem 2.5.3. denotes the hyperreal extension of the root-function on $\mathbb{R}$.
Using the notation from Theorem 2.5.3., the expressions $\frac{\varepsilon}{\delta}$,$\frac{U}{V}$, $\varepsilon \cdot U$ and $U + V$ cannot be categorized in similar fashion. They are called *Indeterminate Forms*.

Theorem 2.5.4. *The limited and infinitesimal numbers in* $^*\mathbb{R}$ *both form subrings of* $^*\mathbb{R}$.

Proof. Let ε, δ be infinitesimal. For any standard number $a > 0$, we have $|\varepsilon| < \frac{a}{2}$ and $|\delta| < \frac{a}{2}$, (by extending the absolute value function to $^*\mathbb{R}$). $|\varepsilon + \delta| < a$ and $|\varepsilon - \delta| < a$. Moreover, $\varepsilon, \delta < \sqrt{a}$ implies $|\varepsilon \cdot \delta| < a$. Consequently the infinitesimals form a subring of $^*\mathbb{R}$.
For limited numbers $a, b \in {}^*\mathbb{R}$, taken such that $\alpha_1 < a < \alpha_2$, $\beta_1 < b < \beta_2$, for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ we get $\alpha_1 + \beta_1 < a + b < \alpha_2 + \beta_2$ and $\alpha_1 - \beta_2 < a - b < \alpha_2 - \beta_1$.
Furthermore we have that $-|\alpha_2 \cdot \beta_2| < |a \cdot b| < |\alpha_2 \cdot \beta_2|$. So the limited numbers form a subring of $^*\mathbb{R}$, as well. □

Definition 2.5.5. $a, b \in {}^*\mathbb{R}$ are called

- *near* or *infinitesimally close* if $a - b$ is infinitesimal. In this case we write $a \simeq b$.
  The set $\mathrm{hal(a)} := \{\mathrm{c} \in {}^*\mathbb{R} : \mathrm{c} \simeq \mathrm{a}\}$ is called the *halo* or *monade* of a.

- a *limited distance apart* if $a - b$ is limited and write $a \sim b$ for this fact.
  The set $\mathrm{gal(a)} := \{\mathrm{c} \in {}^*\mathbb{R} : \mathrm{c} \sim \mathrm{a}\}$ is called the *galaxy* of a.

It can be readily checked that $\simeq$ and $\sim$ are equivalence relations on $^*\mathbb{R}$.

Theorem 2.5.6. *If* $a \in {}^*\mathbb{R}$ *is limited, there is a unique standard real number* $a' \in \mathbb{R}$ *with* $a \simeq a'$.

Proof. Define

$$A := \{b \in \mathbb{R} : a \leq b\} \text{ and } B := \{b \in \mathbb{R} : b < a\}$$

Since a is limited, there is a number $r \in \mathbb{R}$ such that $-r < a < r$. Thus A and B are non-empty and A is bounded below and B is bounded above.
As $\mathbb{R}$ is a Dedekind-complete, ordered field, there exists a least upper bound a' for B.
For each real $\varepsilon > 0$, it follows from $(a' + \varepsilon) \in A$ and $(a' - \varepsilon) \in B$ that $|a' - a| \leq \varepsilon$, i.e. $a \simeq a'$.
To prove its uniqueness let $b' \in \mathbb{R}$ be near a, as well. Then $|a' - b'| \leq |a' - a| + |a - b'| < 2\varepsilon$ for all standard $\varepsilon > 0$. Hence, $b' = a'$. □

We call this number a' the *shadow* or *standard part* of a, denoted by sh(a).

In the proof given above, we use the Dedekind completeness of $\mathbb{R}$ to attain the existence of shadows. Indeed, as the following remark shows, the existence of those numbers is an alternative formulation for completeness. As we will use the Cauchy completeness for this fact, it should be noted that Dedekind and Cauchy completeness happen to be equivalent on Archimedian, ordered fields, so in particular on $\mathbb{R}$.

Remark. The assertion "every limited hyperreal is infinitely close to a real number" implies the completeness of $\mathbb{R}$.
To see this let $\langle s_n \rangle$ be a Cauchy sequence. In particular, there exists a $k \in \mathbb{N}$ such that

$$\forall m, n \in \mathbb{N} : m, n \geq k \Rightarrow |s_m - s_n| < 1$$

By transfer, it follows that for an unlimited $N \in {}^*\mathbb{N}$ we have $k, N \geq k$ and, therefore,

$$|s_k - s_N| < 1$$

and thus s_N is limited. By our assertion we get that there is an $L \in \mathbb{R}$ such that $s_N \simeq L$. For a real number $\varepsilon > 0$ we have the Cauchy-property

$$\forall m, n \in \mathbb{N} : m, n \geq j \Rightarrow |s_m - s_n| < \varepsilon$$

for some $j \in \mathbb{N}$. Transferring this and choosing a natural $m > j$, we have $m, N \geq j$. Therefore, $|s_m - s_N| < \varepsilon$. It follows that

$$|s_m - L| < |s_m - s_N| + |s_N - L| < \varepsilon + d,$$

where $d \in {}^*\mathbb{R}$ denotes the infinitesimal distance between s_N and L. Because $|s_m - L|$ and ε are both real numbers, this implies that $|s_m - L| \leq \varepsilon$. Hence $\langle s_N \rangle$ converges to L and $\mathbb{R}$ is complete.

3 Convergence of Sequences

3.1 Hyperreal Characterization of Convergence

Theorem 3.1.1. *A real-valued sequence $\langle s_n \rangle$ converges to $L \in \mathbb{R}$ if and only if $s_n \simeq L$ for all unlimited $n \in {}^*\mathbb{N}$.*

Proof.
"$\Rightarrow$" : The standard definition of convergence for $\langle s_n \rangle$ is that for an arbitrary, given, real $\varepsilon > 0$ there exists an $m_\varepsilon \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} : n \geq m_\varepsilon \Rightarrow |s_n - L| < \varepsilon.$$

By transfer we get

$$\forall n \in {}^*\mathbb{N} : n \geq m_\varepsilon \Rightarrow |s_n - L| < \varepsilon.$$

As m_ε is natural, we get for any unlimited $N \in {}^*\mathbb{N}$ that $|s_N - L| < \varepsilon$ and because $\varepsilon > 0$ was arbitrarily chosen, $|s_N - L|$ must be infinitesimal. Hence, $s_N \simeq L$.
"$\Leftarrow$" : Fix an unlimited $N \in {}^*\mathbb{N}$. Then for any $n \in {}^*\mathbb{N}$ if $n > N$ it follows that n is unlimited. Hence, $s_n \simeq L$, i.e. $|s_n - L| < \varepsilon$ for all $\varepsilon > 0$. Therefore, the sentence

$$\exists m \in {}^*\mathbb{N} : \forall n \in {}^*\mathbb{N} \wedge n > m \Rightarrow |s_n - L| < \varepsilon$$

is true. Transferring we have

$$\exists m \in \mathbb{N} : \forall n \in \mathbb{N} \wedge n > m \Rightarrow |s_n - L| < \varepsilon$$

for all $\varepsilon > 0$. Hence, $\langle s_n \rangle$ converges to L. □

This theorem shows that in non-standard Analysis we can replace the role of the standard tail of a sequence by the *extended tail* consisting of the *extended terms*, i.e. members of the sequence with an unlimited index. The standard open neighborhoods $(L - \varepsilon, L + \varepsilon)$ are replaced by the *infinitesimal neighborhood* hal(L).

Theorem 3.1.2. *For a real-valued sequence $\langle s_n \rangle$ there can only be one limit*

Proof. For $\langle s_n \rangle$ converging to M and N, we take an unlimited $n \in {}^*\mathbb{N}$ and get $s_n \simeq M$ and $s_n \simeq N$, so $M \simeq N$, and because they are both real, $M = N$. □

3.2 Non-standard Proofs on Convergence

Theorem 3.2.1. *If a real-valued sequence $\langle s_n \rangle$ is either*

- *bounded above in $\mathbb{R}$ and non-decreasing, or*
- *bounded below in $\mathbb{R}$ and non-increasing,*

then it converges in $\mathbb{R}$.

Proof. We only prove the first case, as the second one can be shown in a similar fashion. Let s_N be an extended term, i.e. $N \in {}^*\mathbb{N}$ is unlimited. By hypothesis there exists a real number u that is an upper bound for $\langle s_n \rangle$. Therefore, $s_1 \leq s_n \leq u$ is true for all $n \in \mathbb{N}$. This can be transferred, meaning it holds for all $n \in {}^*\mathbb{N}$. Therefore, we have $s_1 \leq s_N \leq u$. Hence, s_N is limited and has a shadow L.
Transferring the property non-decreasing, we get that $n \leq m$ implies $s_n \leq s_m$ for all $n, m \in {}^*\mathbb{N}$. In particular, $s_n \leq s_N \simeq L$ for all $n \in \mathbb{N}$. Therefore, $s_n \leq L$, because they are both real numbers, meaning L is an upper bound for the sequence. It is the least upper bound, as for any $r \in \mathbb{R}$ that is an upper bound for $\langle s_n \rangle$, by transfer we obtain $s_n \leq r$ for all $n \in {}^*\mathbb{N}$. We conclude $L \simeq s_N \leq r$, obtaining $L \leq r$, as both are real.
Summarizing, we have shown that for any unlimited $n \in {}^*\mathbb{N}$ the real number $\mathrm{sh(s_n)}$ is a least upper bound for the real sequence $\langle s_n \rangle$. Since a real sequence can only have one least upper bound, all of those shadows must denote the same real number L. By definition of the shadow it has the property $s_n \simeq L$ for all unlimited $n \in {}^*\mathbb{N}$. Hence, by Theorem 3.1.1 $\langle s_n \rangle$ converges to L. □

Example 3.2.2. For a real $c \in [0, 1)$ we use the theorem above to prove that the sequence $s_n := c^n$ converges to zero.
$\langle s_n \rangle$ is non-increasing and bounded below, hence, it converges to some real number L. Thus, if $N \in {}^*\mathbb{N}$ is unlimited, we have $c^N \simeq L$ and $c^{N+1} \simeq L$. By transfer of

$$\forall n \in \mathbb{N} : c^{n+1} = c \cdot c^n$$

we have

$$L \simeq c^{N+1} = c \cdot c^N \simeq c \cdot L.$$

Hence, $L = c \cdot L$, as both are real numbers. Because of $c \neq 1$, L must be 0.

Theorem 3.2.3. *A real-valued sequence $\langle s_n \rangle$ is bounded in $\mathbb{R}$ if and only if its extended terms are all limited.*

Proof.
" $\Rightarrow$ " : Transferring boundedness we get

$$|s_n| < b \quad \text{for all } n \in {}^*\mathbb{N}$$

for some real $b > 0$. In particular, this holds for all unlimited n.
" $\Leftarrow$ " : If s_n is limited for all unlimited $n \in {}^*\mathbb{N}$, it is limited for all $n \in {}^*\mathbb{N}$. To see this, note that ${}^*\mathbb{N}$ only consists of $\mathbb{N}$ and unlimited numbers (c.f Example 2.5.2) and every s_m with a limited index $m \in \mathbb{N}$ is limited.
Therefore, for a hyperreal, positive, unlimited $r \in {}^*\mathbb{R}$ we get $|s_n| < r$ for all $n \in {}^*\mathbb{N}$, so that the sentence

$$\exists y \in {}^*\mathbb{R} : \forall n \in {}^*\mathbb{N} : \ |s_n| < y$$

is true. By transfer it follows that

$$\exists y \in \mathbb{R} : \forall n \in \mathbb{N} : \ |s_n| < y,$$

i.e. $\langle s_n \rangle$ is bounded. □

Theorem 3.2.4. *A real-valued sequence $\langle s_n \rangle$ is Cauchy if and only if $s_m \simeq s_n$ for all unlimited $n, m \in {}^*\mathbb{N}$.*

Proof.
"$\Rightarrow$": Choose a real $\varepsilon > 0$, then because $\langle s_n \rangle$ is Cauchy, there exists a $j \in \mathbb{N}$ such that

$$\forall m, n \in \mathbb{N} : m, n \geq j \Rightarrow |s_m - s_n| < \varepsilon$$

holds. Transferring we obtain

$$\forall m, n \in {}^*\mathbb{N} : m, n \geq j \Rightarrow |s_m - s_n| < \varepsilon$$

Because $j \in \mathbb{N}$, for unlimited $m, n \in {}^*\mathbb{N}$, this statement is true regardless of the chosen $\varepsilon > 0$, hence, $|s_m - s_n| \simeq 0$.
"$\Leftarrow$": Suppose that $\langle s_n \rangle$ is not Cauchy. Then there is an $\varepsilon > 0$ such that

$$\forall N \in \mathbb{N}\ \exists m, n \in \mathbb{N} : m, n \geq N\ \wedge\ |s_m - s_n| \geq \varepsilon$$

Transferring, we obtain in particular that there are unlimited $m, n \in {}^*\mathbb{N}$ such that $|s_m - s_n| \geq \varepsilon$, hence, $s_m \not\simeq s_n$. □

Theorem 3.2.5 (Cauchy's Convergence Criterion)**.** *A real-valued sequence $\langle s_n \rangle$ converges in $\mathbb{R}$ if and only if it is Cauchy.*

Proof.
"$\Rightarrow$": If $\langle s_n \rangle$ converges to L, then for any two unlimited $n, m \in {}^*\mathbb{N}$ we have (c.f Theorem 3.1.1.) $s_n \simeq L$ and $s_m \simeq L$, hence, $s_n \simeq s_m$.
"$\Leftarrow$": Using a standard result we know that every Cauchy sequence is bounded. Thus, for an unlimited $m \in {}^*\mathbb{N}$, the extended term s_m is limited and so is its shadow $L \in \mathbb{R}$. For any different extended term s_n we know that $s_n \simeq s_m$, because $\langle s_n \rangle$ is Cauchy (c.f Theorem 3.2.4.). Therefore, we also have $s_n \simeq L$. But $s_n \in$ hal(L) for all unlimited $n \in {}^*\mathbb{N}$ is exactly the non-standard criterion for convergence to L.
□

As we can see, instead of using the completeness of $\mathbb{R}$ explicitly, the existence of shadows for limited, hyperreal numbers can be employed.

4 Continuous Functions

4.1 Hyperreal Characterization

Theorem 4.1.1. *A function $f : A \to \mathbb{R}$ is continuous at $a \in A \subseteq \mathbb{R}$, if and only if*

$$f\,(\mathrm{hal(a)} \cap {}^*\mathrm{A}) \subseteq \mathrm{hal(f(a))},$$

or in other words, if and only if for all $x \in {}^*\mathbb{R}$ the property $x \simeq a$ implies $f(x) \simeq f(a)$.

Proof.
"$\Rightarrow$": The standard definition of continuity is:
For all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\forall x \in \mathbb{R} : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Now for a fixed $\varepsilon > 0$, there is a $\delta > 0$ such that by transfer

$$\forall x \in {}^*\mathbb{R} : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

is true. For any $x \in \mathrm{hal(a)}$, $|x - a| < \delta$ and, therefore, $|f(x) - f(a)| < \varepsilon$. As the real $\varepsilon > 0$ was arbitrary, it follows $f(x) \simeq f(a)$.
"$\Leftarrow$": Fix a real $\varepsilon > 0$ and let d denote an arbitrary, positive infinitesimal number. Then for all $x \in {}^*\mathbb{R}$ with $|x - a| < d$ we have $x \simeq a$ and by our assumption $f(x) \simeq f(a)$. Because $\varepsilon > 0$ is real, we also have $|f(x) - f(a)| < \varepsilon$. Thus,

$$\exists \delta \in {}^*\mathbb{R}\ \delta > 0 \forall x \in {}^*\mathbb{R} \ : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

Transferring this sentence we have proven the continuity of f on $\mathbb{R}$. □

Examining the second part of the proof from above in detail we obtain an even stronger result

Theorem 4.1.2. *Let $a \in A$.*

1. *f is continuous at $a \in A$.*
2. *for all $x \in {}^*A$, $x \simeq a$ implies $f(x) \simeq f(a)$*
3. *There exists a positive infinitesimal d such that for all $x \in {}^*A$, $|x - a| < d$ implies $f(x) \simeq f(a)$.*

To show that this only works for standard points $a \in \mathbb{R}$, we examine the following example.

Example 4.1.3. For $f(x) = \frac{1}{x}$ we know that this function is continuous on standard intervals $(0, 1)$. Now let $\delta \in {}^*(0, 1)$ be infinitesimal. Then $\delta \simeq 2\delta$, but $f(\delta) - f(2\delta) = -\frac{1}{2\delta}$ is not infinitesimal.

The stronger requirement $a \in {}^*\mathbb{R}$ turns out to be equivalent to *uniform continuity.*

Theorem 4.1.4. *A function $f : A \to \mathbb{R}$ is uniformly continuous on $A \subseteq \mathbb{R}$, if and only if*

$$f\,(\text{hal(a)} \cap {}^*\text{A}) \subseteq \text{hal(f(a))},$$

for all $a \in {}^*A$ or, in other words, if and only if for all $a, b \in {}^*A$ the property $a \simeq b$ implies $f(a) \simeq f(b)$.

Proof.
"$\Rightarrow$": The standard definition of uniform continuity is:
For all $\varepsilon > 0$ there exists a $\delta > 0$ for all $x, a \in A : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.
Now for a fixed $\varepsilon > 0$ and the corresponding $\delta > 0$, the sentence

$$\forall a, x \in A : \;\; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

is true. By transfer, for all $a, x \in {}^*A$, $|a - x| < \delta$ implies $|f(a) - f(b)| < \varepsilon$. For $a \simeq x$ we have $|a - x| < \delta$ for every real $\delta > 0$, and hence, $|f(a) - f(x)| < \varepsilon$ for every real $\varepsilon > 0$. Therefore, $a \simeq x$ implies $f(a) \simeq f(x)$.
"$\Leftarrow$": Fix a real $\varepsilon > 0$ and let d denote an arbitrary, positive infinitesimal number. Then for all $a, x \in {}^*A$ that fulfill $|x - a| < d$ we have $x \simeq a$ and by our assumption $f(x) \simeq f(a)$. But because $\varepsilon > 0$ is real, we also have $|f(x) - f(a)| < \varepsilon$ and thus,

$$\exists \delta \in {}^*\mathbb{R} : \delta > 0 \wedge \forall a, x \in {}^*A : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Transferring this sentence we have proven the uniform continuity of f on A. □

4.2 Non-standard Proofs on Continuity

Theorem 4.2.1 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then for every real d strictly between $f(a)$ and $f(b)$ there exists a real $c \in (a, b)$ such that $f(c) = d$.*

Proof. Without loss of generality we deal with the case $f(a) < d < f(b)$.
For each $n \in \mathbb{N}$, we partition $[a, b]$ into n equal subintervals of width $\frac{(b-a)}{n}$.
Let s_n denote the greatest partition point with $f(s_n) < d$. So we have

$$\forall n \in \mathbb{N} : a \leq s_n < b \;\; \text{and} \;\; f(s_n) < d \leq f\left(s_n + \frac{(b-a)}{n}\right),$$

which we can transfer. Then the same statement now applies to all $n \in {}^*\mathbb{N}$.
For an unlimited N we get s_N and the next point $s_N + \frac{(b-a)}{N}$ are infinitesimally close to $c = \text{sh}(\text{s}_\text{N})$, as $\frac{(b-a)}{N}$ is infinitesimal.
Since f is continuous at c and c is real, $c \simeq s_N$ and $c \simeq s_N + \frac{(b-a)}{N}$ imply $f(c) \simeq s_N$ and $f(c) \simeq f\left(s_N + \frac{(b-a)}{N}\right)$, hence, $f(c) \simeq d$. Since d and $f(c)$ are both real, they must be equal. □

Theorem 4.2.2 (Extreme Value Theorem)**.** *For a continuous $f : [a, b] \to \mathbb{R}$ there exist real $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.*

Proof. We only prove that f obtains the absolute maximum, the proof for the absolute minimum is similar.
For each $n \in \mathbb{N}$, we partition $[a, b]$ into n equal subintervals with end points $a + \frac{k(b-a)}{n}$ for integers $0 \leq k \leq n$. Let s_n be the partition point with the largest f-value, meaning

$$\forall n, k \in \mathbb{N} : 0 \leq n \leq k \;\Rightarrow\; a \leq s_n \leq b \;\text{ and }\; f\left(a + \frac{k(b-a)}{n}\right) \leq f(s_n). \tag{1}$$

By transfer (1) also holds for all $n, k \in {}^*\mathbb{N}$ such that $0 \leq k \leq n$. Now we choose any hypernatural N and put $d = \mathrm{sh(s_N)}$. By continuity $f(s_N) \simeq f(d)$.
Let x be an arbitrary real element of $[a, b]$, then for each $n \in \mathbb{N}$ there is an integer $k < n$ with

$$a + \frac{k(b-a)}{n} \leq x \leq a + \frac{(k+1)(b-a)}{n}.$$

Hence, by transfer there exists a hyperinteger $K < N$ such that x lies in the interval

$$\left[a + \frac{K(b-a)}{N}, a + \frac{(K+1)(b-a)}{N}\right]$$

of infinitesimal width $\frac{(b-a)}{N}$. Therefore, $x \simeq \left(a + \frac{K(b-a)}{N}\right)$ and by continuity of f

$$f(x) \simeq f\left(\left(a + \frac{K(b-a)}{N}\right)\right).$$

But, due to (1), the values of f are dominated by $f(s_N)$ and we obtain

$$f(x) \simeq f\left(\left(a + \frac{K(b-a)}{N}\right)\right) \leq f(s_N) \simeq f(d),$$

which leads to $f(x) \leq f(d)$, since both are real. $\square$

Theorem 4.2.3. *If the real function f is continuous on the closed interval $[a, b] \subseteq \mathbb{R}$, then f is uniformly continuous on $[a, b]$.*

Proof. Take $x, y \in {}^*[a, b]$ with $x \simeq y$ and denote $c = \mathrm{sh(x)}$. Since $a \leq x \leq b$ and $x \simeq c$, we have $c \in [a, b]$. Hence, f is continuous at c. It follows that $f(x) \simeq f(c)$ and $f(y) \simeq f(c)$. Therefore, $f(x) \simeq f(y)$ and we have proven uniform continuity (c.f Theorem 4.1.4.). $\square$

Definition 4.2.4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz-continuous* if it fulfills the *Lipschitz-condition*, meaning there is a positive real constant L, such that

$$\forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq L|x - y|.$$

A *contraction mapping* is a Lipschitz-continuous function with a constant $L < 1$. We can now prove a version of *Banach's Fix Point Theorem* for $(\mathbb{R}, |\cdot|)$ using only non-standard Analysis.

Theorem 4.2.5. *Any contraction mapping $f : \mathbb{R} \to \mathbb{R}$ has a unique fixed point.*

Proof. Let $L < 1$ be the the Lipschitz-constant for f and take any $x \in \mathbb{R}$. We define a sequence $\langle s_n \rangle$ by putting $s_0 = x$ and $s_{n+1} = f(s_n)$ for $n \geq 0$.
Using the Lipschitz-condition of f we get

$$|s_n - s_{n+1}| \leq L^n |s_0 - s_1|, \text{ for } n \in \mathbb{N}. \tag{1}$$

Using this we can estimate $|s_0 - s_n|$ by

$$\begin{aligned} |s_0 - s_n| &\leq |s_0 - s_1| + \cdots + |s_{n-1} - s_n| \leq \\ &\leq |s_0 - s_1| + \cdots + L^{n-1}|s_0 - s_1| = \\ &= |s_0 - s_1|(1 + \cdots + L^{n-1}) = \\ &= \frac{1 - L^n}{1 - L}|s_0 - s_1| \end{aligned}$$

and, therefore,

$$|s_0 - s_n| \leq \frac{1}{1 - L}|s_0 - s_1| \quad \text{for all } n \in \mathbb{N}.$$

Transferring this sentence and picking an unlimited $m \in {}^*\mathbb{N}$ we get

$$|s_0 - s_m| \leq \frac{1}{1 - L}|s_0 - s_1|.$$

Because the right side of this inequation is real, s_m is limited and has a shadow $\mathrm{sh(s_m)} = \mathrm{S}$. As f is continuous, $s_m \simeq S$ implies $f(s_m) \simeq f(S)$. By transfer of the definition of $\langle s_n \rangle$ we get $f(s_m) = s_{m+1}$ and by transfer of (1) we get $s_{m+1} \simeq s_m$. Altogether we have

$$f(S) \simeq f(s_m) = s_{m+1} \simeq s_m \simeq S.$$

Since $f(S)$ and S are both real they must be equal.
For the uniqueness we note that for $f(x) = x$ and $f(y) = y$ we get

$$|x - y| = |f(x) - f(y)| \leq L|x - y|,$$

Since $L < 1$, it follows that $|x - y| = 0$ and, therefore, $x = y$. □

Example 4.2.6. For any contraction mapping $f : \mathbb{R} \to \mathbb{R}$ we construct the sequence $s_0 = x$, $s_{n+1} = f(s_n)$ for $n \geq 0$. In Theorem 4.2.6 we have shown, that s_m has a shadow for all unlimited $m \in {}^*\mathbb{N}$ and that this shadow $\mathrm{sh(s_m)}$ is a fixed point. But as the same theorem shows, there can only be one fixed point. Hence, $c := \mathrm{sh(s_m)} = \mathrm{sh(s_n)}$ for all unlimited $m, n \in {}^*\mathbb{N}$. By Theorem 3.1.1 this implies the convergence of the sequence to c.

4.3 Sequences of Functions

Another important case of extensions that we have to investigate is the extension of sequences of functions. As of yet, for a sequence of real functions $f_n : A \to \mathbb{R}$, $n \in \mathbb{N}$, we can extend each one of them to a function ${}^*f_n : {}^*A \to {}^*\mathbb{R}$, $n \in \mathbb{N}$. But we would like this sequence to be extended to a hypersequence, i.e. for all $n \in {}^*\mathbb{N}$.
To achieve this, note that for such a sequence we define $F : \mathbb{N} \times A \to \mathbb{R}$ by putting $F(n, x) = f_n(x)$. For such a function, we know that there is an extension
${}^*F : {}^*\mathbb{N} \times {}^*A \to {}^*\mathbb{R}$.
For a fixed $n \in \mathbb{N}$ we transfer

$$\forall x \in \mathbb{R} : \quad F(n, x) = f_n(x)$$

and obtain that ${}^*F(n, x) = {}^*f_n(x)$ holds for all $x \in {}^*\mathbb{R}$. So this approach does not produce different extensions ${}^*f_n(x)$. Similarly, we can transfer for fixed $x \in \mathbb{R}$

$$\forall n \in \mathbb{N} : \quad F(n, x) = f_n(x)$$

and obtain that the extension of the real-number sequence $\langle f_n(x) \rangle$ matches with this approach. Hence, for sequences of functions, we obtain hypersequences of hyperreal-valued functions by the rules of extension, established in Section 2.2.

Theorem 4.3.1. *A sequence of real-valued functions $\langle f_n \rangle$ defined on $A \subseteq \mathbb{R}$ converges point-wise to the function $f : A \to \mathbb{R}$ if and only if for each $x \in A$ and each unlimited $n \in {}^*\mathbb{N}$, $f_n(x) \simeq f(x)$.*

Proof. This follows immediately from the hyperreal characterization of convergence, Theorem 3.1.1. □

Theorem 4.3.2. *A sequence of real-valued functions $\langle f_n \rangle$ defined on $A \subseteq \mathbb{R}$ converges uniformly to the function $f : A \to \mathbb{R}$ if and only if for each $x \in {}^*A$ and each unlimited $n \in {}^*\mathbb{N}$, $f_n(x) \simeq f(x)$.*

Proof.
"$\Rightarrow$": By the definition of uniform convergence

$$\forall \varepsilon > 0 \;\; \exists m \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq m \Rightarrow \forall x \in A : |f_n(x) - f(x)| < \varepsilon$$

Fixing an $\varepsilon > 0$, we have a certain $m \in \mathbb{N}$, such that by transfer

$$\forall n \in {}^*\mathbb{N} : n \geq m \Rightarrow \;\forall x \in {}^*A : \;\; |f_n(x) - f(x)| < \varepsilon.$$

For any infinite $N \in {}^*\mathbb{N}$ we have that $N \geq m$ and obtain that $f_n(x) \simeq f(x)$ for all $x \in {}^*A$, because $\varepsilon > 0$ was arbitrarily picked.
"$\Leftarrow$": Fix an unlimited $N \in {}^*\mathbb{N}$. For all $n \geq N$, $n \in {}^*\mathbb{N}$ we have $f_n(x) \simeq f(x)$ for all $x \in {}^*A$. Hence, for any real $\varepsilon > 0$

$$\exists m \in {}^*\mathbb{N} : n \geq m \Rightarrow \;\forall x \in {}^*\mathbb{R} : |f_n(x) - f(x)| < \varepsilon.$$

Transferring we get the desired result. □

Theorem 4.3.3. *If the functions $\langle f_n \rangle$ are all continuous on $A \subseteq \mathbb{R}$, then for any $n \in {}^*\mathbb{N}$ and any $y \in {}^*A$ there exists a positive infinitesimal d such that $f_n(x) \simeq f_n(y)$ for all $x \in {}^*A$ with $|x - y| < d$.*

Proof. f_n being continuous on A for all $n \in \mathbb{N}$ means that

$$\forall n \in \mathbb{N} \ , \ \forall y \in A \ , \ \forall \varepsilon > 0 \ , \ \exists \delta > 0 : \ \forall x \in A : |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$$

is true. By transfer for all $n \in {}^*\mathbb{N}$, $y \in {}^*A$ and all infinitesimal $\varepsilon > 0$, we get that there is some hyperreal $\delta > 0$ such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon$ for all $x \in {}^*A$. But this implies $f_n(x) \simeq f_n(y)$. Hence, there is a positive infinitesimal d such that the desired conclusion follows. □

This result allows us to proof a basic theorem on uniformly converging sequences of continuous functions.

Theorem 4.3.4. *If the functions $\langle f_n \rangle$ are all continuous on $A \subseteq \mathbb{R}$ and the sequence converges uniformly to the function $f : A \to \mathbb{R}$, then f is continuous on A.*

Proof. For $c \in A$ we want to prove that f is continuous at c. For an unlimited $n \in {}^*\mathbb{N}$, by the last result, there exists an infinitesimal $d > 0$ such that for any $x \in {}^*A$

$$|x - c| < d \text{ implies } f_n(x) \simeq f_n(c).$$

As $c, x \in {}^*A$ and $|x - c| < d$ implies $x \simeq c$, for an unlimited $n \in {}^*\mathbb{N}$, we have $f_n(x) \simeq f(x)$ and $f_n(c) \simeq f(c)$ (c.f Theorem 4.3.2.). Altogether we attain

$$f(x) \simeq f_n(x) \simeq f_n(c) \simeq f(c).$$

In conclusion, there is a positive $d \simeq 0$ such that for all $x \in {}^*A$

$$|x - c| < d \text{ implies } f(x) \simeq f(c),$$

so f is continuous at c. □

References

[Gold98] Robert Goldblatt. *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis.* Springer, 1998

[HuLo85] Albert E. Hurd, Peter A. Loeb. *An Introduction to Nonstandard Real Analysis.* Academic Press, Inc., 1985

[Rob66] Abraham Robinson. *Non-standard Analysis.* North-Holland Publishing Co., Amsterdam 1966

[ErGiHe55] P. Erdoes, L. Gillman and M. Henriksen. *An isomorphism theorem for real-closed fields.* Annals of Mathematics, 1955

[BuDa98] D.M. Burton, H. Dalkowsiki. *Elementary Number Theory.* McGraw-Hill, 1998

[Raut08] Wolfgang Rautenberg. *Einführung in die Mathematische Logik.* Vieweg+Teubner, 2008

[Kalt1] M. Kaltenbäck. *Analysis 1.* Lecture Notes, 2012

[Kalt2] M. Kaltenbäck. *Analysis 2.* Lecture Notes, 2012

[Kalt3] M. Kaltenbäck. *Analysis 3.* Lecture Notes, 2012