Invariant Subspace Problem

Bachelor Thesis

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February 3, 2016

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1 Introduction

In this paper we want to present a few results related to the invariant subspace problem. That is, the question whether an operator on a certain space, usually a Banach or Hilbert space, has a nontrivial invariant subspace. This problem seems to have been stated by Beurling and von Neumann and there are quite a lot of different theorems about it. Some of them give answers under conditions on the Banach or Hilbert space, others give their results under assumptions on the operator.

We divide the paper in two parts. In sections 2 and 3 some of the most popular positive answers to our question are shown and in section 4 we discuss one counterexample in detail, and present a few others in addition.

The chapter about similarity and quasisimilarity closely follows Chapter 4 of [1], while the chapter about polynomial boundedness follows Chapter 10 of [2]. The proof of Lomonosov's theorem is extracted from Chapter 4.4 of [3]. The Bernstein-Robinson and the Aronszajn-Smith theorem are taken from [5] and [4].

2 Classical Results about the Existence of Invariant Subspaces

In the first section we want to present a few important theorems, which give the existence of an invariant subspace under certain conditions. To start this section we present a few results, which should already be known, or have quite simple proofs. We continue with the classic Aronszajn-Smith theorem given in [4], and the Bernstein-Robinson theorem, extracted from [5], whose proves share some common features. Then we introduce hyperinvariant subspaces and Lomonosov's theorem, whose proof is different to the proofs of the first two.

2.1 Preliminary Notes

Before we start with the first theorem we recall and clarify the use of a few definitions

Definition 2.1. Generally $\mathscr{H}$ and $\mathscr{K}$ denote Hilbert spaces, while $\mathscr{B}$ denote a Banach space. $\mathcal{B}(\mathscr{H})$ denote all bounded operators of $\mathscr{H}$ into itself. Further:

(i) We call an invariant subspace S of $\mathscr{H}$ under an operator T, a proper or nontrivial invariant subspace, if $\{0\} \neq S \neq \mathscr{H}$.

(ii) A closed linear subspace M of $\mathscr{H}$ is said to be reducing for T if M and $M^{\perp}$ are invariant subspaces for T.

(iii) An operator with $\|T\| \leq 1$ is called a contraction.

Now we want to present a few conclusions already known from linear algebra and fundamental functional analysis, and one with a quite simple proof.

Theorem 2.2. *On finite dimensional vector spaces with dimension greater than one every nonzero operator has an eigenvector and hence a nontrivial invariant subspace.*

Theorem 2.3. *On a Hilbert space $\mathscr{H}$ that is not separable every operator has a proper invariant subspace.*

Proof. It is easily seen that the sequence $\{T^n : n \in \mathbb{N}\}$ spans a nontrivial subspace of $\mathscr{H}$, and is invariant. $\square$

We obtain another conclusion through the spectral theorem for normal operators:

Theorem 2.4. *Due to the spectral theorem every normal operator has an invariant subspace.*

Theorem 2.5. *Let T and L be nonzero operators on $\mathscr{H}$. If $LT = 0$, then* $\ker(L)$ *and* $\overline{\operatorname{ran}(T)}$ *are nontrivial invariant subspaces for T and L.*

Proof. If $LT = 0$, then $\operatorname{ran}(T) \subseteq \ker(L)$. Hence $T(\ker(L)) \subseteq T(\mathscr{H}) = \operatorname{ran}(T) \subseteq \ker(L)$. Since $T \neq 0$, we get that $\ker(L) \neq 0$ and on the other hand we have that $L \neq 0$ so that $\ker(L) \neq \mathscr{H}$. Therefore $\ker(L)$ is a nontrivial invariant subspace for T. Dually since $T^*L^* = 0$, $L^* \neq 0$ and $T^* \neq 0$, it follows that $\ker(T^*)$ is a nontrivial invariant subspace for L^*, and hence $\overline{\operatorname{ran}(T)} = \ker(T^*)^{\perp}$ is a nontrivial invariant subspace for L. Finally recall that $\ker(L)$ and $\overline{\operatorname{ran}(T)}$ are trivially invariant subspaces for L and T respectively. □

Corollary 2.6. *Every nilpotent operator has a nontrivial invariant subspace.*

We want to finish this section with another theorem, which gives us an example of a Banach space on which every continuous operator has a proper nontrivial invariant subspace. The proof can be found in [6], but it is quite long so it is omitted.

Theorem 2.7. *(Argyros,Haydon) There is an indecomposable Banach space with its dual space being isomorphic to ℓ_1. Every bounded linear operator on this space is expressible as $\lambda I + K$ with λ a scalar and K compact. In particular every continuous operator has an invariant subspace.*

2.2 The Aronszajn-Smith and Bernstein-Robinson Theorems

We now want to present two of the most classical results in invariant subspace theory. We start with the older Aronszajn-Smith Theorem and the proof we present follows the original one, presented in [4].

Theorem 2.8 (Aronszajn, Smith)**.** *Let T be a compact operator in a Banach Space $\mathscr{B}$. Then there exist proper invariant subspaces of T.*

In the proof of this theorem we use a map P on an arbitrary finite dimensional subspace $C \subseteq \mathscr{B}$, defined through

$$\|x - Px\| = \rho(x, C) = \min_{y \in C} \|x - y\|.$$

We are able to limit ourselves to a separable Banach space, because of Theorem 2.3, therefore we can define an equivalent strictly convex norm on our Banach space, according to [7], Theorem 9. Because of that, we shall suppose that our norm on $\mathscr{B}$ is strictly convex. Therefore, and because of the separability, there exists a unique point $Px \in C$ which realizes the minimal distance.

We refer to the upper map P as "metric projection", because it is quite similar to a projection, but not necessarily linear. Before we begin with the proof, we want to sum up a few general properties, which apply to the metric projection:

Lemma 2.9. *If P is a metric projection on a finite dimensional subspace C, following properties are fulfilled:*

1. *P is idempotent: $P^2 = P$*
2. *P is homogeneous: $P(\alpha x) = \alpha Px$*
3. *P is quasiadditve: $P(y + x) = y + Px$ for every $y \in C$*
4. $\|Px - x\| \leq \|x\|$, $\|Px\| \leq 2\|x\|$

5. $|\|x - Px\| - \|y - Py\|| \leq \|x - y\|$

6. *If $C' \subseteq C$ and P' is the metric projection on C', then $\|x - Px\| \leq \|x - P'x\|$.*

Proof. The properties *1.*, *4.* and *6.* are obvious from the definition and *5.* is the general property of the shortest distance from x to a fixed set C. To prove *2.* consider $P(x) = y$ and observe that $\|\alpha x - \alpha P(x)\| = \min_{y \in C} \|\alpha x - \alpha y\| = \|\alpha x - P(\alpha x)\|$.

Property *3.* follows with $\|(x+y) - P(x+y)\| = \min_{z \in C} \|x+y-z\| = \|(x+y) - (y + P(x))\|$, because $y \in C$. □

Now we can begin with a few constructions for the proof of the Aronszajn-Smith-Theorem: In finite dimensional spaces our theorem holds true, because of Theorem 2.2. Because of Theorem 2.3 we can restrict ourselves to infinite dimensional separable spaces and we have

(A) $\mathscr{B}$ is separable.

Now we are only interested in the case that

$$\operatorname{span}\{T^n f : n \in \mathbb{N}\} = \mathscr{B}, f \in \mathscr{B}. \tag{1}$$

because otherwise we would have already found our invariant subspace in the span of $\{T^n : n \in \mathbb{N}\}$. This formula implies the following property:

(B) $T^n f \neq 0$ and all elements $\{T^n f : n \in \mathbb{N}\}$ are linearly independent.

To prove **(B)** suppose that $\alpha_1 T^{n_1} f + \alpha_2 T^{n_2} f + ... + \alpha_k T^{n_k} f = 0$, with $0 \leq n_1 < n_2 < ... < n_k$ and $\alpha_i \neq 0$, $i \in \{1, ..., k\}$. We obtain that

$$T^{n_k} f = -\left(\frac{1}{\alpha_k}\right)(\alpha_1 T^{n_1} f + ... + \alpha_{k-1} T^{n_{k-1}} f +),$$

and hence that all $T^n f$ would lie in a subspace generated by $T^n f$ with $n < n_k$ which is a contradiction to (1) and the infinite dimension of $\mathscr{B}$.
Now we consider a sequence of closed subspaces $C_k \in \mathscr{B}$. The limes inferior of the sequence is defined as

$$\liminf_k C_k := \{x \in \mathscr{B} : \exists x_k \in C_k, x_k \to x\}.$$

The following two properties are easily verified:

(C) $\liminf_k C_k$ is a closed subspace.

(D) If every C_k is finite dimensional, then $x \in \liminf_k C_k$ if and only if $P_k x \to x$, where P_k denotes the projection on C_k.

With f satisfying (1) we construct the k-dimensional subspace

$$C_{(k)} = [T^n f]_0^{k-1}.$$

We denote by P_k the metric projection onto $C_{(k)}$. By (1) we obtain that $\liminf C_k = \mathscr{B}$ and hence that

$$P_k x \to x, x \in \mathscr{B}. \tag{2}$$

Now we consider the operator T_k on $C_{(k)}$ defined by

$$T_k x = P_k T x,\ x \in C_{(k)}.$$

Now we prove that T_k is linear. Through homogenity and quasiadditivity of the metric projection and by setting $x := \sum_{i=0}^{k-1} \xi_i T^i f$, we get that

$$T_k x = P_k T x = P_k \sum_{i=0}^{k-1} \xi_i T^{i+1} f = \sum_{i=0}^{k-2} \xi_i T^{i+1} f + \xi_{k-1} P_k T^k f,$$

and hence that T_k is linear.

Because T_k is a linear operator on a k-dimensional space we can find a triangular matrix to represent T_k. Therefore there exists an increasing sequence of subspaces

$$0 = C_k^0 \subset C_k^1 \subset ... \subset C_k^k = C_k, \tag{3}$$

where C_k^i is an i-dimensional invariant subspace of T_k.

Lemma 2.10. *Let $\{k_m\}$ and $\{i_m\}$ be sequences, such that $k_m \to \infty$ and $0 \le i_m \le k_m, \forall m \in \mathbb{N}$. Further let $x_m \in C_{k_m}^{i_m}$. If $Tx_m \to y$ then $y \in \liminf C_{k_m}^{i_m}$.*

Proof. In fact we have $P_{k_m} T x_m = T_{k_m} x_m \in C_{k_m}^{i_m}$. On the other hand, by Theorem 2.9,5, we have $\left| \|Tx_m - P^{(k_m)} T x_m\| - \|y - P^{(k_m)} T x_m\| \right| \le \|Tx_m - y\|$. With this and (2) we obtain

$$\|Tx_m - P_{(k_m)} T x_m\| \le \|y - P_{(k_m)} y\| + \|Tx_m - y\| \to 0$$

$$\|y - P_{(k_m)} T x_m\| \le \|y - Tx_m\| + \|Tx_m - P_{(k_m)} T x_m\| \to 0,$$

which proves the lemma, because $T_{k_m} x_m \to y$ and therefore $y \in \liminf_{k_m} C_{k_m}^{i_m}$. □

We continue the preparation for the proof of the Aronszajn-Smith Theorem with the following corollaries:

Corollary 2.11. *For any sequences $\{k_m\}$ and $\{i_m\}$ satisfying the conditions of Lemma 2.10, $\liminf_k C_{k_m}^{i_m}$ is an invariant subspace of T.*

Proof. If we have $x \in \liminf_k C_{k_m}^{i_m}$ we have by the definition of the $\liminf$ the existence of x_m such that $x_m \in C_{k_m}^{i_m}$, $x_m \to x$. By the continuity of T we obtain $Tx_m \to Tx$ and by Lemma 2.10 $Tx \in \liminf_k C_{k_m}^{i_m}$. □

Corollary 2.12. *If the $\liminf$ of every subsequence of $C_{k_m}^{i_m} = \{0\}$ then for any bounded sequence $x_m \in C_{k_m}^{i_m}$ we have $Tx_m \to \{0\}$.*

Proof. By compactness of T, the sequence x_m is transformed into a relatively compact subsequence Tx_m. Therefore it is enough to prove that if any subsequence Tx_{m_j} converges to some y, then $y = 0$. But this follows from our hypothesis, since by Lemma 2.10 $y \in \liminf_{k_m} C_{k_m}^{i_m}$. □

Proof. (Aronszajn-Smith) Now we choose an arbitrary $\alpha > 0$ with

$$0 < \alpha < 1, \|Tf\| > \alpha \|T\| \|f\|. \tag{4}$$

Since $f \in \mathscr{C}_k$ we have by Theorem 2.9-3 and 2.9-6

$$\|f\| = \|f - P_{k,0} f\| \ge \|f - P_{k,1} f\| \ge ... \ge \|f - P_{k,k} f\| = \{0\}.$$

Therefore there exists a unique index $i(k), 0 \leq i(k) < k$ for each $k \in \mathbb{N}$ such that

$$\|f - P_k^{i(k)} f\| \geq \alpha \|f\| > \|f - P_k^{i(k)+1} f\|. \tag{5}$$

Let u_k, $k = 1, 2, ...$ be an element of $C_k^{i(k)+1}$ such that

$$\|u_k\| = 1, \quad P_k^{i(k)} u(k) = 0. \tag{6}$$

An element with the given property can be obtained from an arbitrary $v \in C_k^{i(k)+1} \backslash C_k^{i(k)}$ by setting $u_k := \|v - P_k^{i(k)} v\|^{-1}(v - P_k^{i(k)} v)$. The property (6) is now proved by homogenity and quasiadditivity of P.

Since the dimensions of $C_k^{i(k)+1}$ and $C_k^{i(k)}$ differ by one, every element $y \in C_k^{i(k)+1}$ is representable in a unique way in the form $y = x + \beta u_k$ with $x = P_k^{i(k)} y$. Correspondingly we shall put

$$P_k^{i(k)+1} f = x_k + \beta_k u_k, \quad P_k^{i(k)+1} Tf = x'_k + \beta'_k u_k, \quad x_k, x'_k \in C_k^{i(k)}. \tag{7}$$

We have by Theorem 2.9-4,

$$\|x_k\| = \|P_k^{i(k)} P_k^{i(k)+1} f\| \leq 4\|f\|, \quad .\|x'_k\| \leq 4\|Tf\|. \tag{8}$$

Now we prove the following statements:

(E) For every sequence $k_m \to \infty, \liminf_{k_m} C_{k_m}^{i(k_m)} \neq \mathscr{B}$.

(F) For some sequence $k'_m \to \infty, \liminf_{k_m} C_{k'_m}^{i(k'_m)+1} \neq 0$.

(G) If for every sequence $k_m \to \infty, \liminf_{k_m} C_{k_m}^{i(k_m)} = \{0\}$ then for every sequence $k'_m \to \infty, \liminf_{k_m} C_{k'_m}^{i(k'_m)+1} \neq \mathscr{B}$.

If $\liminf_{k_m} C_{k_m}^{i(k_m)} = \mathscr{B}$, then by **(B)** $P_{k_m}^{i(k_m)} f \to f$ which contradicts (5), hence **(E)** holds true.

If **(F)** were not true we would have by Corollary 2.12 that the bounded subsequence $P_k^{i(k)+1} f$ (see Theorem 2.9-4) is transformed into a sequence $TP_k^{i(k)+1} f$ converging to 0. Since $Tf = T(f - P_k^{i(k)+1} f) + TP_k^{i(k)+1} f$ we get $\|Tf\| = \lim \|T(f - P_k^{i(k)+1} f)\| \leq \liminf_k \|T\| \|f - P_k^{i(k)+1} f\|$ which by (5) gives $\|Tf\| \leq 4\|T\| \|f\|$ in contradiction to (4).

Suppose that for some $k'_m \to \infty$, $\liminf_{k'_m} C_{k'_m}^{i(k'_m)+1} = \mathscr{B}$. By **(B)** we have $P_{k'_m}^{i(k'_m)+1} f \to f$ and $P_{k'_m}^{i(k'_m)+1} Tf \to Tf$. By (7) we have $f = \lim_{k_m}(x_{k'_m} + \beta_{k'_m} u_{k'_m})$ and $Tf = \lim_{k'_m}(x'_{k'_m} + \beta'_{k'_m} u_{k'_m})$. Further we obtain $Tf = \lim_{k'_m}(Tx_{k'_m} + \beta_{k'_m} Tu_{k'_m})$ and $T^2 f = \lim_{k'_m}(Tx'_{k'_m} + \beta'_{k'_m} Tu_{k'_m})$. By (8) and Corollary 2.12 it follows $Tf = \lim_{k'_m} \beta_{k'_m} Tu_{k'_m}$ and $T^2 f = \lim_{k'_m} \beta'_{k'_m} Tu_{k'_m}$. Therefore $\beta'_{k'_m} \backslash \beta_{k'_m}$ converges to some γ and $T^2 f = \gamma Tf$ in contradiction to **(B)**, hence **(G)** is proved.

Now we obtain the proof of our theorem as follows. If there is any sequence $k_m \to \infty$ such that $\Lambda = \liminf_{k_m} C_{k_m}^{i(k_m)} \neq \{0\}$ then in view of **(E)** and Corollary 2.11, Λ is a proper invariant subspace. If there is no such sequence, then by **(F)** we choose a sequence $k'_m \to \infty$ such that $\Lambda' = \liminf_{k_m} C_{k'_m}^{i(k'_m)+1} \neq \{0\}$. By **(G)** and Corollary 2.11 Λ' is a proper invariant subspace.

□

The Bernstein-Robinson theorem is an extension to the Aronszajn-Smith theorem, originally proved by using nonstandard analysis, in [5]. However we present the proof given by Halmos in [8], which still has a lot of features in common with Aronszajn-Smith's proof, but does not use nonstandard analysis.

Theorem 2.13 (Bernstein-Robinson)**.** *If A is an operator on a Hilbert space $\mathscr{H}$ of dimension greater than one and if p is a nonzero polynomial such that $p(A)$ is compact, there exists a nontrivial subspace of $\mathscr{H}$ under A.*

Before we can begin with the proof we need a short definition:

Definition 2.14. If f_n and g_n are sequences on $\mathscr{H}$ we shall write $f_n \sim g_n$ for $\|f_n - g_n\| \to 0$.

Proof. According to Theorem 2.2, we can assume the existence of a nonzero vector e such that e, Ae, A^2e,... are linearly independent and have $\mathscr{H}$ as their closed linear span. Otherwise the closed linear span would already be an invariant subspace.

Through Gram-Schmidt orthogonalization we can obtain an orthonormal basis $\{e_1, e_2, ...\}$ with the property, that $\{e_1, e_2, ..., e_m\}$ has the same linear span as $\{e, Ae, ..., A^{m-1}e\}$, for $m \in \mathbb{N}$. If $a_{m,n} := (Ae_n, e_m)$, it follows that $a_{m,n} = 0$, if $m > n+1$. The matrix entries of the kth power of A are given by $a^{(k)}_{m,n} = (A^k e_n, e_m)$ Through induction one can see that $a^{(k)}_{m,n} = 0$, if $m > n+k$ and

$$a^{(k)}_{n+k,n} = \prod_{1 \le j \le k} a_{n+j,n+j-1}.$$

Let $k \ge 1$ be the degree of our given polynomial p. If the matrix entries of p are given by $a^{(p)}_{m,n} = (p(A)e_n, e_m)$, then $a^{(p)}_{n+k,n}$ is a constant multiple of $a^{(k)}_{n+k,n}$. This is verified because the coefficient $a^{(l)}_{n+k,n} = 0$ if $l < k$. Since $\|p(A)e_n\| \to 0$, for $n \to \infty$, which we have because of the compactness of $p(A)$, there exists an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ of postive integers such that the corresponding subdiagonal terms $a_{k(n)+1,k(n)}$ converge to 0 for $n \to \infty$.

If H_n is the span of $\{e_1, ..., e_{k(n)}\}$, then $\{H_n\}_{n\in\mathbb{N}}$ is an increasing sequence of finite-dimensional subspaces of $\mathscr{H}$ with $\mathscr{H}$ as their span. If P_n is the projection with range H_n, then $P_n \xrightarrow{s} I$ (I being the identity operator). Since, for each operator, the identity P_nAP_n leaves H_n invariant, it follows that for each n there exists a chain of subspaces invariant under P_nAP_n

$$\{0\} = H_n^{(0)} \subset H_n^{(1)} \subset ... \subset H_n^{(k(n))} = H_n.$$

with $\dim H_n^{(i)} = i$, $i = 0, 1, ..., k(n)$, a construction similar to the proof of the Aronszajn-Smith Theorem.

If $f_n \in \mathscr{H}$ is a bounded sequence of vectors, we want to prove that

$$AP_nf_n \sim P_nAP_nf_n. \tag{9}$$

For the proof of (9), we have that $P_nf = \sum_{j=1}^{k(n)}(f, e_j)e_j$, if $f \in \mathscr{H}$ and that

$$AP_nf_n - P_nAP_nf_n = \sum_{j=1}^{k(n)}(f_n, e_j) \sum_{i=k(n)+1}^{\infty} a_{ij}e_i.$$

Since the largest j is $k(n)$ and the smallest i is $k(n)+1$ and since $a_{ij} = 0$ if $i > j+1$, it follows that $\|AP_nf_n - P_nAP_nf_n\| \le \|f_n\|a_{k(n)+1,k(n)} \to 0$, and therefore we have proved (9). (9) can be generalized to higher exponents:

$$A^kP_nf_n \sim (P_nAP_n)^kf_n \qquad k = 1, 2, ... \tag{10}$$

which can again be proved by induction. For $k = 0$, (10) says that $\|P_nf_n - f_n\| \to 0$, which is a stringent condition on the bounded sequence f_n. If that is satisfied, then (10) implies that

$$p(A)P_nf_n \sim p(P_nAP_n)f_n. \tag{11}$$

Now we return to our vector e. Since $P_n e = e$ for every n, it follows that $p(A)P_n e \sim p(P_nAP_n)e$. Since $p(A)e \neq 0$, which follows because the vectors $(e, Ae, ...)$ are linearly independent, we have

$$\epsilon = \|p(P_nAP_n)e\| = \|p(A)e\| > 0.$$

Consider for each n the numbers

$$\|p(P_nAP_n)e - p(P_nAP_n)P_n^{(0)}e\|,$$

$$\|p(P_nAP_n)e - p(P_nAP_n)P_n^{(1)}e\|,$$

$$\vdots$$

$$\|p(P_nAP_n)e - p(P_nAP_n)P_n^{(k(n))}e\|.$$

where $P_n^{(i)}$ is the projection with range $H_n^{(i)}$. Since $P_n^{(0)}$ is the zero projection the first of these numbers tends to ϵ. Since, on the other hand $P_n^{(k(n))} = P_n$, the last of these numbers is always 0. In view of these facts it is possible to choose for each n with a finite number of exceptions a positive integer $i(n)$, $1 \leq i(n) \leq k(n)$, such that

$$\|p(P_nAP_n)e - p(P_nAP_n)P_n^{(i(n)-1)}e\| \geq \frac{\epsilon}{2} \tag{12}$$

and

$$\|p(P_nAP_n)e - p(P_nAP_n)P_n^{(i(n))}e\| < \frac{\epsilon}{2}. \tag{13}$$

Further let $i(n)$ be the smallest integer for which these inequalities hold true.

Since both $P_n^{i(n)-1}$ and $P_n^{i(n)}$ are bounded sequences of operators, there exists an increasing sequence n_j of integers such that both $P_n^{i(n_j)-1}$ and $P_n^{i(n_j)}$ are weakly convergent. To simplify notation we set $Q_j^- := P_n^{i(n_j)-1}$ and $Q_j^+ := P_n^{i(n_j)}$. Let $M^- := \{f \in \mathscr{H} \mid Q_j^- f \xrightarrow{s} f\}$, and $M^+ := \{f \in \mathscr{H} | Q_j^+ f \xrightarrow{s} f\}$.

Now we are going to prove that M^- and M^+ are subspaces of $\mathscr{H}$ that are both invariant under A, and that at least one of them is nontrivial. To prove that M^- is closed, suppose that g is in the closure of M^-. We have to show that $g \in M^-$ and therefore that $Q_j^- g \to g$. Given a positive number δ, one has to find $f \in M^-$ so that $\|f - g\| < \frac{\delta}{3}$ and then find j_0 so that $\|Q_j f - f\| < \frac{\delta}{3}$ for $j \geq j_0$. It follows that, if $j \geq j_0$, then $\|Q_j^- g - g\| \leq \|Q_j^- g - Q_j^- f\| + \|Q_j^- f - f\| + \|f - g\| < \delta$. This proves that M^- is closed, the proof for M^+ is the same.

To prove that M^- is invariant under A, we suppose that $f \in M^-$, so that $Q_j^- f \to f$ and infer, first, that $AQ_j^- f \to Af$, because A is bounded and second, that $Q_j^- AQ_j^- f \sim Q_j^- Af$, because Q_j^- is uniformly bounded. Then we reason as follows:

$$Q_j^- Af \sim Q_j^- AQ_j^- f =^{(a)} Q_j^- P_{n_j} AP_{n_j} Q_j^- f =^{(b)} P_{n_j} AP_{n_j} Q_j^- f \sim^{(c)} AP_{n_j} Q_j^- f = AQ_j^- f \to Af.$$

(a) is valid because $Q_j^- \leq P_{n_j}$, (b) because the range of Q_j^- is invariant under $P_{n_j}AP_{n_j}$ and (c) because of (10). This proves that M^- is invariant under A, the prove that M^+ is invariant follows the same scheme.

The next step is to prove that $M^- \neq \mathscr{H}$. This is done by showing that $e \notin M^-$. For this purpose observe first that the operators $p(P_nAP_n)$ are uniformly bounded as one can see, by observing that

$$\|(P_nAP_n)^k\| \leq \|P_nAP_n\|^k \leq \|A\|^k.$$

and by using the polynomial whose coefficients are the absolut values of the coefficients of p. Now, because of (12) we have

$$\frac{\epsilon}{2} \leq \|P_{n_j} A P_{n_j}\| \|e - Q_j^- e\|.$$

Since $\|P_{n_j} A P_{n_j}\|$ is bounded from above, its reciprocal is bounded from zero, and consequently $\|e - Q_j^- e\|$ is bounded away from zero, which makes the convergence $Q_j^- e \to e$ impossible.

The corresponding step for M^+ says that $M^+ \neq \{0\}$, but with a quite different proof. The choice of the sequence $\{n_j\}$ implies that the sequence $\{Q_j^+ e\}$ is weakly convergent. The compactness of $p(A)$ implies therefore that the sequence $\{p(A) Q_j^+ e\}$ is strongly convergent to, say, f. The proof that follows, we have to show two parts:

1. $f \neq 0$
2. $f \in M^+$

To prove 1. we have $p(A) Q_j^+ e \sim p(P_{n_j} A P_{n_j}) Q_j^+ e$ by (11), which is within $\frac{\epsilon}{2}$ of $p(P_{n_j} A P_{n_j}) e$, by (13), whose norm tends to ϵ. It follows that $\|\{p(A) Q_j^+ e\}\|$ can not tend to zero, and hence that $f \neq 0$.

To prove 2. we have that $Q_j^+ f \sim Q_j^+ p(A) Q_j^+ e$, since Q_j^+ is uniformly bounded. Then we have $Q_j^+ p(A) Q_j^+ e \sim Q_j^+ p(P_{n_j} A P_{n_j}) Q_j^+ e$ by (11) and uniform boundedness. Because the range of Q_j^+ is invariant under $p(P_{n_j} A P_{n_j})$, we have $Q_j^+ p(P_{n_j} A P_{n_j}) Q_j^+ e = p(P_{n_j} A P_{n_j}) Q_j^+ e \sim p(A) Q_j^+ e$ with the last relation holding true because of (11). At last $p(A) Q_j^+ e \to f$ by definition.

If $M^+ \neq \mathscr{H}$ all is well. It remains to be proved that if $M^+ = \mathscr{H}$ then $M^- \neq \{0\}$. If $M^+ = \mathscr{H}$ then $Q_j^+ f \to f$ for all f, at least weakly. At the same time the sequence $\{Q_j^-\}$ is known to be weakly convergent to, say, Q^-. The operators Q_j^- and Q_j^+ are projections such that $Q_j^- \leq Q_j^+$ and such that $Q_j^+ - Q_j^-$ has rank 1. It follows that for each j there exists a unit vector f_j, such that $(Q_j^+ - Q_j^-) f = (f, f_j) f_j$ for all f.

Observe now that $Q_j^- e$ cannot weakly tend to e, for if it did, it would tend strongly to e, which is a property of projections, but was proved to not be the case. This implies that $Q^- e \neq e$, or, equivalently that $(1 - Q^-) e \neq 0$. Now the numbers $|(e, f_j)|$ can not be arbitrarily small. This follows because, since $|((Q_j^+ - Q_j^-) e, g)| \leq |(e, f_j)| \|g\|$ for all g, an affirmative answer would imply that $((1 - Q^-) e, g) = 0$ for all g, so that $(1 - Q^-) e = 0$ - a contradiction. So we have obtained that the numbers $|(e, f_j)|$ are bounded away from zero, which makes it possible to prove that $M^- \neq \{0\}$.

It turns out that if $g \perp (1 - Q^-) e$, then $g \in M^-$. Indeed since $(e, f_j)(f_j, g) \to ((1 - Q^-) e, g) = 0$, it follows that $(f_j, g) \to 0$ and hence that $(f, f_j)(f_j, g) \to 0$ for all f. This implies that $((1 - Q^-) f, g) = 0$ for all f, and hence that $(1 - Q^- g) = 0$. In other words $Q_j^- g \to g$ weakly, and therefore strongly. From this it follows that $g \in M^-$.

□

2.3 Lomonosov's Theorem

Lomonosov's Theorem is another classical result in invariant subspace theory, but it has a completely different proof to the Aronszajn-Smith Theorem and the Bernstein-Robinson Theorem. The proof we present follows the one in [3], Chapter 6.

Actually Lomonosov's Theorem is split in two parts, a lemma and the actual theorem. Before we get to these two we need to prove another theorem and provide a few definitions:

Definition 2.15. Let $\mathcal{B}(\mathscr{H}, \mathscr{K})$ be the space of all bounded operators from $\mathscr{H}$ to $\mathscr{K}$, with $\mathscr{H}$ and $\mathscr{K}$ being Hilbert spaces.

(i) With $\mathcal{K}(\mathscr{H}, \mathscr{K})$ we denote all compact operators from $\mathscr{H}$ to $\mathscr{K}$, and with $\mathcal{K}(\mathscr{H})$ all compact operators from $\mathscr{H}$ into itself.

(ii) Let $\mathrm{Lat}(T)$ denote the set of all invariant subspaces for T.

(iii) If $\mathcal{A} \subseteq \mathcal{B}(\mathscr{H})$ then let $\mathrm{Lat}(\mathcal{A}) := \bigcap\{\mathrm{Lat}(T) : T \in \mathcal{A}\}$.

Definition 2.16. If T is a linear operator, we shall call a subspace $C \subseteq \mathscr{H}$ hyperinvariant if $S(C) \subseteq C$ for all S commuting with T.

Definition 2.17. We denote the convex hull of a set S with $\mathrm{co}(S)$ and have $\mathrm{co}(S) := \{xt + (1-t)y : x, y \in S, t \in [0,1]\}$.

Theorem 2.18. *(Mazur's Theorem) If $\mathscr{H}$ is a Banach space and K is a compact subset of $\mathscr{H}$, then $\overline{\mathrm{co}(K)}$ is compact.*

Proof. Obviously, it suffices to show that $\overline{\mathrm{co}(K)}$ is totally bounded. Let $\epsilon > 0$ and choose $x_1, ..., x_n$ in K such that $K \subseteq \bigcup_{j=1}^{n} B(x_j, \epsilon/3)$. Put $C = \mathrm{co}\{x_1, ..., x_n\}$, then C is obviously compact. Hence there are vectors $y_1, ..., y_n$ in C such that $C \subseteq \bigcup_{j=1}^{n} B(y_j, \epsilon/3)$. If $w \in \overline{\mathrm{co}(K)}$, there is a z in $co(K)$ with $\|w - z\| < \epsilon/3$. Thus $z = \sum_{p=1}^{l} \alpha_p k_p$, where $k_p \in K$, $\alpha_k \geq 0$ and $\sum \alpha_k = 1$. Now for each k_p there is an $x_{j(k)}$ with $\|k_p - x_{j(p)}\| < \epsilon/3$. Therefore

$$\left\| z - \sum_{p=1}^{l} \alpha_p x_{j(p)} \right\| = \left\| \sum_{p=1}^{l} \alpha_k (k_p - x_{j(p)}) \right\| \leq \sum_{p=1}^{l} \alpha_k \|k_p - x_{j(p)}\| < \epsilon/3.$$

But $\sum_{p=1}^{l} \alpha_p x_{j(p)} \in C$, so there is an y_i with $\|\sum_{p=1}^{l} \alpha_p x_{j(p)} - y_i\| < \epsilon/3$. With the triangle inequality we get $\|w - y_j\| \leq \|w - z\| + \|z - \sum_{p=1}^{l} \alpha_p x_{j(p)}\| + \|\sum_{p=1}^{l} \alpha_p x_{j(p)} - y_i\| < \epsilon$, which shows us that $\overline{\mathrm{co}(K)} \subseteq \bigcup_{j=1}^{n} B(y_j, \epsilon)$ and so $\overline{\mathrm{co}(K)}$ is totally bounded.

□

Lemma 2.19. *If $\mathcal{A}$ is a subalgebra to $\mathcal{B}(\mathscr{H})$, such that $I \in \mathcal{A}$ and $Lat\mathcal{A} = \{\emptyset, \mathscr{H}\}$ and if K is a nonzero compact operator on $\mathscr{H}$, then there is an $A \in \mathcal{A}$ such that* $\ker(AK - I) \neq 0$.

Proof. It may be assumed that $\|K\| = 1$. Fix $x_0 \in \mathscr{H}$ such that $\|Kx_0\| > 1$ and put $S = \{x \in \mathscr{H} : \|x - x_0\| \leq 1\}$. If we have $0 \in S$, we get $1 < \|K(x_0)\| \leq \|K\|\|x_0 - 0\| \leq 1$ and hence a contradiction.

On the other hand, if $0 \in \overline{(K(S))}$ we have that there exists a sequence $y_n \in S$, $n \in \mathbb{N}$ such that $\lim_n K(y_n) = 0$ and hence there exists an $N \in \mathbb{N}$ such that $1 < \|Kx_0\| - \|K(y_N)\|$. Now we have $1 < \|Kx_0\| - \|K(y_N)\| \leq \|K\|\|x_0 - y_N\| \leq 1$ and therefore another contradiction.

In conclusion we have

$$0 \notin S \quad \textit{and} \quad 0 \notin \overline{K(S)}. \tag{14}$$

Now if $x \in \mathscr{H}$ and $x \neq 0$, $\overline{\{Tx : T \in \mathcal{A}\}}$ is an invariant subspace for $\mathcal{A}$, because $\mathcal{A}$ is an algebra, and it contains the nonzero vector x, because $I \in \mathcal{A}$. By hypothesis we get $\overline{\{Tx : T \in \mathcal{A}\}} = \mathscr{H}$. By (14) this says that for every $y \in \overline{K(S)}$, there is a T in $\mathcal{A}$ with $\|Ty - x_0\| < 1$. Equivalently

$$\overline{K(S)} \subseteq \bigcup_{T \in \mathcal{A}} \{y : \|Ty - x_0\| < 1\}.$$

Because $\overline{K(S)}$ is compact, there are $T_1, ..., T_n$ in $\mathcal{A}$ such that

$$\overline{K(S)} \subseteq \bigcup_{j=1}^{n} \{y : \|T_j y - x_0\| < 1\}. \tag{15}$$

For $y \in \overline{K(S)}$ and $1 \leq j \leq n$, let $a_j(y) = \max\{0, 1 - \|T_j y - x_0\|\}$. By (15), $\sum_{j=1}^{n} a_j(y) > 0$, for all $y \in \overline{K(S)}$. Define $b_j : \overline{K(S)} \to \mathbb{R}$, by

$$b_j(y) = \frac{a_j(y)}{\sum_{i=1}^{n} a_i(y)},$$

and define $\Psi : S \to \mathscr{H}$ by

$$\Psi(x) = \sum_{j=1}^{n} b_j(Kx) T_j Kx.$$

It is easy to see that $a_j : \overline{K(S)} \to [0, 1]$ is a continuous function and hence b_j and Ψ are continuous too.

If $x \in S$, then $Kx \in K(S)$. If $b_j(Kx) > 0$, then $a_j(Kx) > 0$ and so $\|T_j Kx - x_0\| < 1$. That is $T_j Kx \in S$, whenever $b_j(Kx) > 0$. Since S is a convex set and $\sum_{j=1}^{n} b_j(Kx) = 1$ for $x \in S$,

$$\Psi(S) \subseteq S.$$

Note that $T_j K \in \mathcal{K}(\mathscr{H})$ for each j so that $\bigcup_{j \leq n} T_j K(S)$ has compact closure. By Mazur's Theorem 2.18 $\overline{\text{co}(\bigcup T_j K(S))}$, is compact. But this convex set contains $\Psi(S)$ so that $\overline{\Psi(S)}$ is compact. This is, Ψ is a linear map. By the Schauder Fixed-Point Theorem, there is a vector $x_1 \in S$ such that $\Psi(x_1) = x_1$. Let $\beta_j = b_j(Kx_1)$ and put $A = \sum_{j=1}^{n} \beta_i T_j$. So $A \in \mathcal{A}$ and $AKx_1 = \Psi(x_1) = x_1$. Since $x_1 \neq 0$, because x_1 is in S, we obtain $\ker(AK - I) \neq 0$. □

Theorem 2.20. *(Lomonosov's Theorem) If $\mathscr{H}$ is a Banach space over $\mathbb{C}$, $T \in \mathcal{B}(\mathscr{H})$, T is not a multiple of the identity and $TK = KT$, for some nonzero compact operator K, then T has a nontrivial hyperinvariant subspace.*

Proof. Let $\mathcal{A} = \{T\}'$. We want to show that $\text{Lat}\mathcal{A} \neq \{0, \mathscr{H}\}$. If this is not the case then Lomonosov's Lemma implies that there is an operator A in $\mathcal{A}$ such that $\mathcal{N} = \ker(AK - I) \neq 0$. But $\mathcal{N} \in \text{Lat}(AK)$ and $AK|_{\mathcal{N}}$ is the identity operator. Since $AK \in \mathcal{K}(\mathscr{H})$, we get $\dim \mathcal{N} < \infty$. Since $AK \in \mathcal{A} = \{T\}'$ for any $x \in \mathcal{N}$. $AK(Tx) = TA(Kx) = Tx$, hence $T\mathcal{N} \subseteq \mathcal{N}$. But $\dim \mathcal{N} < \infty$, so that $T|_{\mathcal{N}}$ must have an eigenvalue λ. Thus $\ker(T - \lambda) = M \neq 0$. But $M \neq \mathscr{H}$, since T is not a multiple of the identity. It is easy to check that M is hyperinvariant for T:

$$x \in ker(T - \lambda) \Rightarrow 0 = (T - \lambda)x = K(T - \lambda)x = (T - \lambda)Kx \Rightarrow Kx \in ker(T - \lambda)$$

□

3 Similarity and Quasisimilarity

In this section we want to take a look from a different angle at the invariant subspace problem. So far we have proved theorems which give us the existence of an invariant subspace under certain conditions. Now we want to take a look at how far the existence of an invariant subspace for one operator, carries over to another operator with a certain relation to the first one. These relations are going to be similarity and quasisimilarity. We are going to take a closer look at hyperinvariant subspaces, which we have already introduced at Lomonosov's theorem. This section closely follows Chapter 4 of [1]. We can now start with a few definitions:

Definition 3.1. (i) $T \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ is called quasiinvertible, if it is an injective operator with dense range.

(ii) An operator $T \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ is called quasiaffine transform of $L \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ if there exists a quasiinvertible $X \in \mathcal{B}[\mathscr{K}, \mathscr{H}]$ such that $XT = LX$.

(iii) Two operators $T \in \mathcal{B}[\mathscr{H}]$ and $L \in \mathcal{B}[\mathscr{K}]$ are called quasisimilar, if there exist two operators $X \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ and $Y \in \mathcal{B}[\mathscr{K}, \mathscr{H}]$ such that $XT = LX$ and $YL = TY$.

(iv) An operator $X \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ intertwines $T \in \mathcal{B}[\mathscr{H}]$ to $L \in \mathcal{B}[\mathscr{K}]$ if $XT = LX$.

Lemma 3.2. *Let $T \in \mathcal{B}[\mathscr{H}]$, $L \in \mathcal{B}[\mathscr{K}]$ and $X \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ such that $XT = LX$. Suppose $C \subsetneq \mathscr{K}$ is a nontrivial invariant subspace for L. If*

$$\overline{\operatorname{ran}(X)} = \mathscr{K} \quad \textit{and} \quad \operatorname{ran}(X) \cap C \neq \{0\},$$

then $X^{-1}(C)$ is a nontrivial invariant subspace for T.

Proof. Let $C \subsetneq \mathscr{K}$ be an nontrivial invariant subspace for L. Since $X : \mathscr{H} \to \mathscr{K}$, is linear and continuous, $X^{-1}(C)$ is a subspace of $\mathscr{H}$. Moreover, since $X(X^{-1}(C)) \subseteq C$ it follows that $LXX^{-1}(C) \subseteq L(C)$. Hence, since $L(C) \subseteq C$, we have $LXX^{-1}(C) \subseteq C$. Because $LX = XT$, we get $XTX^{-1}(C) \subseteq C$ and so $X^{-1}XTX^{-1}(C) \subseteq X^{-1}(C)$. On the other hand, we get $TX^{-1}(C) \subseteq X^{-1}XTX^{-1}(C)$, because $A \subseteq X^{-1}(X(A))$ for all sets $A \subseteq \mathscr{H}$. Therefore we have

$$TX^{-1}(C) \subseteq X^{-1}(C),$$

in other words $X^{-1}(C)$ is an invariant subspace for T. Now we shall verify that the assumptions on $\operatorname{ran}(X)$ are enough to ensure that the invariant subspace $X^{-1}(C)$ is nontrivial. Take an arbitrary $y \in \operatorname{ran}(X) \cap C$ so that $y = Xu \in C$, with $u \in \mathscr{H}$. If $X^{-1}(C) = \{0\}$, then $u = 0$ because

$$X^{-1}(C) = \{0\} \Leftrightarrow \{x \in \mathscr{H} : Xx \in C\} = \{0\}.$$

Hence we obtain that $y = 0$, because X is linear. We conclude that, if $X^{-1}(C) = \{0\}$, it follows that $\operatorname{ran}(X) \cap C = \{0\}$. Equivalently, we have

$$\operatorname{ran}(X) \cap C \neq \{0\} \Rightarrow X^{-1}(C) \neq \{0\}.$$

If $X^{-1}(C) = \mathscr{H}$, then $\operatorname{ran}(X) = X(\mathscr{H}) = XX^{-1}(C)$. Thus since $X(X^{-1}(C)) \subseteq C$ we get $\overline{\operatorname{ran}(X)} \subseteq \overline{C} = C \neq \mathscr{K}$. We conclude that $X^{-1}(C) = \mathscr{H} \Rightarrow \overline{\operatorname{ran}(X)} \neq \mathscr{K}$, or equivalently $\overline{\operatorname{ran}(X)} = \mathscr{K} \Rightarrow X^{-1}(C) \neq \mathscr{H}$ □

If the intertwining operator X is surjective, then $X^{-1}(C)$ is a nontrivial invariant subspace for T, whenever C is a nontrivial invariant subspace for L. An even more particular case reads as follows:

Corollary 3.3. *If two operators are similar and one of them has an invariant subspace then so has the other.*

Proposition 3.4. *Let $\mathscr{K}$ be a Hilbert space, let M be a finite-dimensional subspace of $\mathscr{K}$ and let R be a linear manifold on $\mathscr{K}$. If $\overline{R} = \mathscr{K}$, then*

$$\overline{(R \cap M^{\perp})} = M^{\perp}.$$

Proof. The result holds trivially if $\mathscr{K}$ is a finite-dimensional Hilbert space, for in such a case $R = \mathscr{K}$ and therefore $R \cap M^\perp = M^\perp$. It also holds trivially if $\dim(M) = 0$, because then we have $M = \{0\}$ and therefore $M^\perp = \mathscr{K}$, which gives us $\overline{R \cap M^\perp} = \overline{R} = \mathscr{K} = M^\perp$. So from now on we can assume that $\mathscr{K}$ is infinite-dimensional and that $m := \dim(M) \geq 1$. First we shall verify that the above result holds true for $m = 1$. If $m = 1$ then $M = \operatorname{span}(e)$ for $\{0\} \neq e \in \mathscr{K}$. Since R is dense in $\mathscr{K}$ there exists $x \in \mathscr{K}$, such that

$$(e, x) \neq 0,$$

because otherwise, if $(e, x) = 0$ for all $x \in R$, we would have that $e \in R^\perp$, or $e = \{0\}$, because R is dense in $\mathscr{K}$. Now take an arbitrary $z \in M^\perp$. Since $\overline{R} = \mathscr{K}$ and $z \in \mathscr{K}$, there exists a sequence $\{z_j \in R : j \geq 1\}$ such that $z_j \to z$ as $j \to \infty$.

For each $j \geq 1$ set

$$y_j = z_j - \frac{(e, z_j)}{(e, x)} x,$$

and note that $y_j \in R$ for every $j \geq 1$, because $z_j, x \in R$ and R is a linear manifold. Further we have $y_j \in M^\perp$ for every $j \geq 1$, because $(e, y_j) = (e, z_j) - \frac{(e,z_j)}{(e,x)}(e, x) = 0$, so that $y_j \perp e$ and hence $y_j \in M^\perp$ and $y_j \to z$, $j \to \infty$, since $(e, z) = 0$, because $z \in M^\perp = \{e\}^\perp$. Therefore for every $z \in M^\perp$, there exists an $(R \cap M^\perp)$-valued sequence converging to z. Hence $(R \cap M^\perp)$ is dense in $M^\perp$ and we conclude that the result holds for $m = 1$.

Now suppose it holds for some $m \geq 1$, that is, suppose

$$\overline{R \cap M^\perp} = M^\perp,$$

for any m-dimensional subspace M of $\mathscr{K}$. Take an arbitrary $(m + 1)$-dimensional subspace of $\mathscr{K}$, say N. Let $\{e_l : 0 \leq l \leq m\}$, be an orthonormal basis for N, so that

$$N = \bigoplus_{l=0}^{m} [e_l]$$

Take an arbitrary integer $k \in \{0, ..., m\}$. Set

$$M_k = \bigoplus_{l=0, l \neq k}^{m} [e_l],$$

so that $\overline{R \cap M_k^\perp} = M_k^\perp$, once $\dim M_k = m$. Note that there exists $x_k \in R \cap M_k^\perp$ such that

$$(e_k, x_k) \neq 0,$$

because, if $(e_k, x_k) = 0$ for all $x_k \in R \cap M_k^\perp$, then $e_k \in (R \cap M_k^\perp)^\perp = \overline{(R \cap M_k^\perp)}^\perp = M_k^{\perp\perp} = \overline{M_k} = M_k$, which contradicts the fact that $0 \neq e_k \perp M_k$. Take an arbitrary $z \in N^\perp = (\bigoplus_{l=0}^m [e_k])^\perp$. Since $\overline{R} = \mathscr{K}$ and $z \in \mathscr{K}$, there exists a sequence $\{z_j \in R : j \geq 1\}$ such that $z_j \to z$ as $j \to \infty$.

For each $j \geq 1$ set

$$y_j = z_j - \sum_{k=0}^{m} \frac{(e_k, z_j)}{(e_k, x_k)} x_k$$

and note that $y_j \in R$ for every $j \geq 1$, because $z_j, x_k \in R$ and R is a linear manifold in $\mathscr{K}$. Further $y_j \in N^\perp$ for every $j \geq 1$, since $x_k \in M_k^\perp = (\bigoplus_{l=0, l\neq k}^m [e_l])^\perp \subseteq [e_n]^\perp$ for every $n \neq k$, $n \in \{0, ..., m\}$, it follows that $(e_l, x_k) = 0$, for every $l \neq k$, $l \in \{0, ..., m\}$. Hence

$$(e_l, y_j) = \left(e_l, z_j - \sum_{k=0}^{m} \frac{(e_k, z_j)}{(e_k, x_k)} x_k \right) = - \sum_{k=0, k\neq l}^{m} \frac{(e_k, z_j)}{(e_k, x_k)} (e_l, x_k) = 0$$

for every $j \geq 1$ and every $l \in \{,...,m\}$. Therefore $y_j \in \bigcap_{l=0}^m [e_l]^\perp = (\bigoplus_{l=0}^m [e_k])^\perp = N^\perp$ for every $j \geq 1$. Moreover $y_j \to z$ as $j \to \infty$, since $(e_k, z) = 0$, because $z \in N^\perp = \bigcap_{l=0}^m [e_l]^\perp$. Thus for every $z \in N^\perp$ there exists an $(R \cap N^\perp)$-valued sequence converging to z. Hence $(R \cap N^\perp)$ is dense in $N^\perp$ and we conclude that the result holds for $m+1$, whenever it holds for m, which ends the proof by induction. □

This proposition gives us the following two corollaries:

Corollary 3.5. *Take $T \in \mathcal{B}[\mathscr{H}]$, $L \in \mathcal{B}[\mathscr{K}]$ and $X \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ such that*

$$XT = LX.$$

Let $M \subset \mathscr{K}$ be a nontrivial finite-dimensional reducing subspace for L. If $\overline{\operatorname{ran}(X)} = \mathscr{K}$, then $X^{-1}(M^\perp)$ is a nontrivial invariant subspace for T.

Proof. This is an immediate conclusion from Proposition 3.4 and Lemma 3.2. □

Corollary 3.6. *If an operator T is a quasiaffine transform of another operator L, that has a nontrivial finite-dimensional reducing subspace, then T has a nontrivial invariant subspace.*

3.1 Hyperinvariant Subspaces

Definition 3.7. The commutant $\{T\}'$ of $T \in \mathcal{B}[\mathscr{H}]$ is the set of all operators in $\mathcal{B}[\mathscr{H}]$ that commute with T, or equally

$$\{T\}' := \{U \in \mathcal{B}[\mathscr{H}] : UT = TU\}.$$

Further let $T_x := \{y \in \mathscr{H} : y = Ux \text{ for some } U \in \{T\}'\}$.

Proposition 3.8. *For each $x \in \mathscr{H}$, $\overline{T_x}$ is a subspace of $\mathscr{H}$ which is hyperinvariant for T.*

Proof. Take any $x \in \mathscr{H}$ and consider the set $T_x \subseteq \mathscr{H}$. If $y_1, y_2 \in T_x$, then there exist $U_1, U_2 \in \{T\}'$, such that $y_1 = U_1 x$ and $y_2 = U_2 x$. Therefore we have $y_1 + y_2 = (U_1 + U_2)x \in T_x$, because obviously $U_1, U_2 \in \{T\}' \Rightarrow U_1 + U_2 \in \{T\}'$. Moreover $\alpha y \in T_x$ for every $\alpha \in \mathbb{C}$ and every $y \in T_x$, trivially. Therefore T_x is a linear manifold on $\mathscr{H}$. Now take $U \in \{T\}'$ arbitrary. If $y \in T_x$, then $y = U_0 x$ for some $U_0 \in \{T\}'$, so that $Uy = UU_0 x \in T_x$, for UU_0 is obviously in $\{T\}'$. Thus $U(T_x) \subseteq T_x$, and hence $\overline{U(T_x)} \subseteq \overline{T_x}$, because U is continuous.

Because the closure of a manifold is a linear subspace, we conclude that $\overline{T_x}$ is an invariant subspace for every $U \in \{T\}'$ and equivalently $\overline{T_x}$ is an hyperinvariant subspace for T. □

Lemma 3.9. *Let $T \in \mathcal{B}[\mathscr{H}]$, $L \in \mathcal{B}[\mathscr{K}]$, $X \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ and $Y \in \mathcal{B}[\mathscr{K}, \mathscr{H}]$ be such that*

$$XT = LX \quad \text{and} \quad YL = TY.$$

Suppose C is a nontrivial hyperinvariant subspace of L. If

$$\overline{\operatorname{ran}(X)} = \mathscr{K} \quad \text{and} \quad \ker(Y) \cap C = \{0\},$$

then $Y(C) \neq \{0\}$ and for each nonzero $x \in Y(C)$, $\overline{T_x}$ is a nontrivial hyperinvariant subspace for T.

Proof. According to Proposition 3.8 it is enough to verify, that under the above hypothesis, $\{0\} \neq \overline{T_x} \neq \mathscr{H}$, for every $0 \neq x \in Y(C) \neq \{0\}$. First note that $XUY \in \{L\}'$ for every $U \in \{T\}'$. Indeed, if $YL = TY$, $TU = UT$ and $XT = LX$, then

$$(XUY)L = XUTY = XTUY = L(XUY).$$

Since C is hyperinvariant for L, it follows that C is invariant for XUY, whenever $U \in \{T\}'$. Now take $x \in Y(C)$ arbitrary so that $x = Yu$ for some $u \in C \subset \mathscr{K}$. If $y \in T_x$, then $y = Ux = UYu$ for some $U \in \{T\}'$ and hence $Xy = XUYu$. But $u \in C$ and C is invariant for XUY. Thus $Xy \in C$. Therefore $X(T_x) \subseteq C$ so that, since X is continuous, we have

$$X(\overline{T_x}) \subseteq \overline{C} = C.$$

If $\overline{T_x} = \mathscr{H}$, then $\overline{\mathrm{ran}(X)} = \overline{X(\mathscr{H})} = \overline{X(\overline{T_x})} \subseteq C \neq \mathscr{K}$. We conclude that

$$\overline{\mathrm{ran}(X)} = \mathscr{K} \Rightarrow \overline{T_x} \neq \mathscr{H} \quad \forall x \in Y(C)$$

Finally, if $Y(C) = \{0\}$, then obviously $C \subseteq \ker(Y)$ and hence $C \cap \ker(Y) = \mathscr{C} \neq \{0\}$. Therefore

$$\ker(y) \cap C = \{0\} \Rightarrow Y(C) \neq \{0\}.$$

Hence $\overline{T_x} \neq \{0\}$, for every nonzero $x \in Y(C)$, because $\overline{T_x} = \{0\}$ if, and only if, $x = 0$. □

In particular, if $XT = LX$ and $YL = TY$, with $\overline{\mathrm{ran}(X)} = \mathscr{K}$ and $ker(Y) = \{0\}$, then there exists $x \in Y(C)$, such that $\overline{T_x}$ is a nontrivial hyperinvariant subspace for T, whenever C is a nontrivial hyperinvariant subspace for L. An even more particular case reads:

Corollary 3.10. *If two operators are quasisimilar, and one has a nontrivial hyperinvariant subspace then so has the other.*

3.2 Contractions Quasisimilar to Unitary Operators

Throgh this section T shall be a contraction on a Hilbert space $\mathscr{H}$. Now we classify these contractions, but first we need a short definition:

Definition 3.11. We call a contraction strongly stable if $\{T^n : n \in \mathbb{N}\}$ converges strongly to the zero operator or in signs $\{T^n : n \in \mathbb{N}\} \xrightarrow{s} 0$.

Let $C_{0.}$ be the class of all strongly stable contractions and let $C_{.0}$ be the class of all contractions, whose adjoint is strongly stable. Let $C_{1.}$ and $C_{.1}$ be the classes of contractions such that $T^n x \nrightarrow 0$ and $T^{*n} x \nrightarrow 0$, respectively for every nonzero $x \in \mathscr{H}$. All combinations are possible and lead to classes C_{00}, C_{10}, C_{01} and C_{11}. If T is a contraction, we have that the sequence $\{T^{*n}T^n : n \in \mathbb{N}\}$ is a bounded monotone sequence of self-adjoint operators, so that it converges strongly. Therefore it has a strong limit A, while $\{T^n T^{*n} : n \in \mathbb{N}\}$ has a strong limit A_*.

Proposition 3.12. *Let $T^{*n}T^n \xrightarrow{s} A$. Then A has the following properties:*

(i) *A is nonnegative and $\|A\| \leq 1$.*

(ii) $\|T^n\| \to \|A^{\frac{1}{2}}\|$ *as $n \to \infty$ for all $x \in \mathscr{H}$.*

(iii) $\ker(A) = \{x \in \mathscr{H} : T^n x \to 0\}$.

(iv) *$T^{*n}AT^n = A$ for every $n \geq 1$.*

Proof. A is nonnegative because it is the strong limit of a nonnegative sequence. A is a contraction, because it is the strong limit of a sequence of contractions, indeed $\|Ax\| = \lim_n \|T^{*n}T^n x\| \leq \|x\|$ for all $x \in \mathscr{H}$, and we have proven *(i)*.

We obtain *(ii)* by observing

$$\|T^n x\|^2 = (T^{*n}T^n x, x) \to (Ax, x) = \|A^{\frac{1}{2}}x\|^2.$$

Further we have that *(ii)* implies *(iii)*, because $\ker(A) = \ker(A^{\frac{1}{2}})$, for every nonnegative operator A.

Note that $T^{*k+n}T^{k+n} = T^{*n}T^{*k}T^kT^n$, for every k, $n \geq 1$, and also that $T^{*k+n}T^{k+n} \xrightarrow{s} A$ and $T^{*n}T^{*k}T^kT^n \to T^{*n}AT^n$ as $k \to \infty$, for every $n \geq 1$. Thus the identity in *(iv)* follows by uniqueness of the strong limit. □

With Proposition 3.12, it follows that $T \in C_{0\cdot}$ if and only if $A = 0$ and $T \in C_{1\cdot}$ if and only if $\ker(A) = \{0\}$. Therefore

$$T \in C_{00} \Leftrightarrow A = A_* = 0,$$

$$T \in C_{01} \Leftrightarrow A = 0 \quad \textit{and} \quad \ker(A_*) = \{0\},$$

$$T \in C_{10} \Leftrightarrow A_* = 0 \quad \textit{and} \quad \ker(A) = \{0\},$$

$$T \in C_{11} \Leftrightarrow \ker(A) = \ker(A_*) = \{0\}.$$

These properties enable us to show a few theorems, about which contractions possess nontrivial invariant subspaces. To accomplish this we start with the following proposition:

Proposition 3.13. *If a contraction is quasisimilar to a unitary operator, then it is of class* C_{11}.

Proof. If $T \in \mathcal{B}[\mathscr{H}]$ is a contraction and $U \in \mathcal{B}[\mathscr{K}]$ is a unitary operator such that $XT = UX$ and $YU = TY$, for a pair of quasiinvertible operators $X \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ and $Y \in \mathcal{B}[\mathscr{K}, \mathscr{H}]$, then $XT^n = U^nX$ and $Y^*T^{*n} = U^{*n}Y^*$ for every $n \geq 1$. Therefore if $x \in \mathscr{H}$ such that $\lim_n T^n x = 0$, then $\lim_n U^n Xx = 0$. Hence $Xx = 0$, so that $x = 0$. That is, by Proposition 3.12, $\ker(A) = \{0\}$. If one repeats this argument by putting U^* in the position of U and Y^* in the position of X, one obtains, that $\ker(A_*) = \{0\}$. Thus if a contraction T is quasismilar to a unitary operator, then $\ker(A) = \ker(A_*) = \{0\}$, or equivalently $T \in C_{11}$. □

In the following theorem we want to show that also the conversion to that proposition holds. Therefore we need to construct an isometry V associated to T such that the equality

$$VA^{\frac{1}{2}} = A^{\frac{1}{2}}T$$

holds. Recall that $\ker(A) = \ker(A_*)$, $\operatorname{ran}(A) \subseteq \operatorname{ran}(A^{\frac{1}{2}})$ and $\overline{\operatorname{ran}(A)} = \overline{\operatorname{ran}(A^{\frac{1}{2}})} = \ker(A^{\frac{1}{2}})^{\perp} = \ker(A)^{\perp}$.

Now we consider the decomposition $\mathscr{H} = \ker(A) \oplus \ker(A)^{\perp} = \ker(A) \oplus \overline{\operatorname{ran}(A)}$ and define a map

$$V : \overline{\operatorname{ran}(A)} \to \overline{\operatorname{ran}(A)},$$

as follows. Take an arbitrary $y \in \operatorname{ran}(A)$ so that there exists an $x_y \in \operatorname{ran}(A^{\frac{1}{2}})$ with $y = A^{\frac{1}{2}}x_y$.

Note that x_y is unique, for A is injective when acting on $\overline{\operatorname{ran}(A)}$ and so is $A^{\frac{1}{2}}$. Set $V_0 y = A^{\frac{1}{2}}Tx_y$, which defines a transformation $V_0 : \operatorname{ran}(A) \to \operatorname{ran}(A^{\frac{1}{2}}) = \operatorname{ran}(A)$ that is clearly linear, for $A^{\frac{1}{2}}$ and T are linear. Extend it to $\overline{\operatorname{ran}(A)}$ to get the linear transformation $V : \overline{\operatorname{ran}(A)} \to \overline{\operatorname{ran}(A)}$.

Further we have that $VA^{\frac{1}{2}}x = A^{\frac{1}{2}}Tx$ for all $x \in \operatorname{ran}(A^{\frac{1}{2}})$, so that $VA^{\frac{1}{2}}x = A^{\frac{1}{2}}Tx$ for all $x \in \overline{\operatorname{ran}(A^{\frac{1}{2}})} = \overline{\operatorname{ran}(A)}$. Also note that, if $x_0 \in \ker(A) = \ker(A^{\frac{1}{2}})$, then $A^{\frac{1}{2}}x_0 = 0$ and $A^{\frac{1}{2}}Tx_0 = 0$, for $\ker(A)$ is invariant under T, according to Proposition 3.12*(iii)*. Hence $VA^{\frac{1}{2}}x_0 = A^{\frac{1}{2}}Tx_0 = 0$ for all $x_0 \in \ker(A)$.

By putting T^* in the position of T one obtains V_* analogously.

Further, we need the following proposition. We present it without a proof, which can be found in [1].

Proposition 3.14. *If* $\ker(A) = \ker(A_*) = \{0\}$*, then*

$$VA^{\frac{1}{2}}A_*^{\frac{1}{2}} = A^{\frac{1}{2}}A_*^{\frac{1}{2}}V_*^*, \qquad A_*^{\frac{1}{2}}A^{\frac{1}{2}}V = V_*^*A_*^{\frac{1}{2}}A^{\frac{1}{2}},$$

and hence

$$A_*A^{\frac{1}{2}}V = TA_*A^{\frac{1}{2}}.$$

Theorem 3.15. *Every C_{11}-contraction is quasisimilar to a unitary operator*

Proof. Consider A and V as constructed above, so that for every contraction T on a Hilbert space $\mathscr{H}$

$$A^{\frac{1}{2}}T = VA^{\frac{1}{2}}.$$

Now recall that T is of class C_{11} if and only if $\ker(A) = \ker(A_*) = \{0\}$. Since A and A_* are self-adjoint this is equivalent to $\overline{\operatorname{ran}(A)} = \overline{\operatorname{ran}(A_*)} = \mathscr{H}$. Therefore $T \in C_{11}$ if, and only if A and A_* are quasiinvertible. In such a case

$$A_*A^{\frac{1}{2}}V = TA_*A^{\frac{1}{2}},$$

according to Proposition 3.14 and AA_* is quasiinvertible too. Indeed, if $\ker(A) = \ker(A_*) = \{0\}$, then $\ker(A_*A^{\frac{1}{2}}) = \ker(A^{\frac{1}{2}}A_*) = \{0\}$ and $\overline{\operatorname{ran}(A_*A^{\frac{1}{2}})} = \mathscr{H}$. We conclude that if $T \in C_{11}$ then it is quasisimilar to the isometry V.

Now we want to show that, if $T \in C_{11}$, V is unitary. Therefore we note that $\ker(T^*) \subseteq \ker(A_*)$, by Proposition 3.12*(iii)*. Since the latter is $\{0\}$ the same applies to $\ker(T^*)$. Thus $\ker(V^*) = \{0\}$, for $A^{\frac{1}{2}}V^* = T^*A^{\frac{1}{2}}$ and $\ker(T^*A^{\frac{1}{2}}) = \{0\}$, because $\ker(A^{\frac{1}{2}}) = \ker(T^*) = \{0\}$. Therefore $\overline{\operatorname{ran}(V)} = \ker(V^*)^{\perp} = \mathscr{H}$ and hence the isometry V is surjective, because the range of an isometry is a subspace. That means V is unitary. □

Corollary 3.16. *On a Hilbert space with dimension greater that one, a C_{11}-contraction has a nontrivial invariant subspace.*

Proof. Since nonscalar normal operators, in particular nonscalar unitary operators, have a nontrivial hyperinvariant subspace, it follows from Corollary 3.10 and Theorem 3.15 that a C_{11}-contraction has a nontrivial hyperinvariant subspace, whenever the unitary operator V is nonscalar. If V is scalar, say, that $V = \gamma I$ for some $\gamma \in \mathbb{C}$, such that $|\gamma| = 1$, then $A^{\frac{1}{2}}T = VA^{\frac{1}{2}} = \gamma A^{\frac{1}{2}}$. Then, because $\ker(A^{\frac{1}{2}}) = \{0\}$, we have $T = \gamma I$. Thus T is a scalar unitary operator and therefore has a nontrivial invariant subspace, because it acts on a Hilbert space with dimension greater than one. □

Corollary 3.17. *If a Hilbert space contraction has no nontrivial invariant subspace it is either a C_{00}, C_{10} or C_{01}-contraction.*

Proof. Let T be a contraction on $\mathscr{H}$. If $\{0\} \neq \ker(A) \neq \mathscr{H}$, then $\ker(A)$ is a nontrivial invariant subspace for T. Dually, if $\{0\} \neq \ker(A_*) \neq \mathscr{H}$, then $\ker(A_*)$ is a nontrivial invariant subspace for T^* so that $\ker(A_*)^\perp$ is a nontrivial invariant subspace for T.

Therefore if T has no nontrivial invariant subspace there are only three cases left, because the case that $\ker(A) = \ker(A_*) = \{0\}$, leads to a C_{11}-contraction which by 3.16 has a nontrivial invariant subspace. The remaining cases are $\ker(A) = \{0\}$ and $\ker(A^*) = \mathscr{H}$, $\ker(A) = \mathscr{H}$ and $\ker(A_*) = \{0\}$, and $\ker(A) = \ker(A_*) = \mathscr{H}$. Equivalently $T \in C_{10}$, $T \in C_{01}$ and $T \in C_{00}$. □

Theorem 3.18. *Every contraction that does not belong to the class C_{00}, has a nontrivial invariant subspace if and only if every contration which is a quasiaffine transform of a unitary operator has a nontrivial invariant subspace.*

Proof. In fact we want to find out, when a contraction not in C_{00} has a nontrivial invariant subspace. This would sharpen the conclusion of Corollary 3.17 to $T \in C_{00}$. Since T has a nontrivial invariant subspace if and only if T^* has, according to Corollary 3.17, we are now asking ourselves under which conditions a $C_{1\cdot}$-contraction has a nontrivial invariant subspace.

This is equivalent to the question if a contraction without a nontrivial invariant subspace is of class $C_{0\cdot}$, or, in other words, whether a contraction without a nontrivial invariant subspace is strongly stable.

This is equivalent to the following question:

1) When does a contraction T for which $A \neq 0$ have a nontrivial invariant subspace?

We claim that if a contraction T on $\mathscr{H}$ has no nontrivial invariant subspace and $A \neq 0$, then $\overline{\operatorname{ran}(A)} = \mathscr{H}$ and the isometry $V : \mathscr{H} \to \mathscr{H}$ is unitary.

To prove this, suppose a contraction T on $\mathscr{H}$ has no nontrivial invariant subspace. If $A \neq 0$, then Corollary 3.17 ensures that $T \in C_{10}$. Hence $ker(A) = \{0\}$ or, equivalently, $\overline{\operatorname{ran}(A)} = \mathscr{H}$. Now consider the isometry V on $\overline{\operatorname{ran}(A)} = \mathscr{H}$ with $VA^{\frac{1}{2}}x = A^{\frac{1}{2}}Tx$. The identities $T^*AT = A$ obtained by Proposition 3.12*(iv)*, $V^*V = I$, which we have because V is an isometry, and $VA^{\frac{1}{2}}x = A^{\frac{1}{2}}Tx$ lead to

$$T^*A^{\frac{1}{2}}A^{\frac{1}{2}}T = A^{\frac{1}{2}}V^*VV^*A^{\frac{1}{2}} = T^*A^{\frac{1}{2}}VV^*A^{\frac{1}{2}}T.$$

Thus $T^*A^{\frac{1}{2}}(I - VV^*)A^{\frac{1}{2}}T = 0$. Since $0 \leq (I - VV^*)$, because V^* is a contraction, we get $\|(I-VV^*)^{\frac{1}{2}}A^{\frac{1}{2}}Tx\|^2 = (T^*A^{\frac{1}{2}}(I-VV^*)A^{\frac{1}{2}}Tx, x) = 0$ for all $x \in \mathscr{H}$, so that $(I-VV^*)^{\frac{1}{2}}A^{\frac{1}{2}}T = 0$. Therefore

$$(I - VV^*)A^{\frac{1}{2}}T = 0.$$

Since the nonzero operator T has no nontrivial invariant subspace Proposition 2.5 ensures that $(I - VV^*)A^{\frac{1}{2}} = 0$. Hence $(I - VV^*) = 0$, because $\overline{\operatorname{ran}(A^{\frac{1}{2}})} = \overline{\operatorname{ran}(A)} = \mathscr{H}$. So we have that $VV^* = I$, so that the isometry V is also a coisometry or, equivalently, V is unitary.

We are now going to present two more questions, which are going to be pairwise equivalent to **1)**, and therefore prove our theorem.

2) Does a contraction which is a quasiaffine transform of a unitary operator have a nontrivial invariant subspace?

3) Does a contraction which is intertwined to $C_{1\cdot}$-contraction, have a nontrivial invariant subspace?

If **3)** has a negative answer, then there exists a contraction $T \in \mathcal{B}[\mathscr{H}]$, without a nontrivial invariant subspace, a $C_{1\cdot}$-contraction $U \in \mathcal{B}[\mathscr{K}]$ and a nonzero $X \in \mathcal{B}[\mathscr{H}, \mathscr{K}]$ such that $XT = UX$. A trivial induction shows that $XT^n = U^nX$, for every $n \in \mathbb{N}$. Since $\operatorname{ran}(X) \neq \{0\}$, there exists a nonzero $x \in \mathscr{H}$ such that $Xx \neq 0$. Hence $XT^nx = U^nXx \nrightarrow 0$ as $n \to \infty$, for $U \in C_{1\cdot}$ so that $T^nx \nrightarrow 0$ as $n \to \infty$. Therefore T is not strongly stable, which means that $A \neq 0$. Summing up, T is a contraction without a nontrivial invariant subspace for which $A \neq 0$, thus question **1)** has a negative answer. Equivalently, a positive answer to **1)** leads to a positive answer to **3)**.

A positive answer to **3)** obviously leads to a positive answer to **2)**.

If question **1)** has a negative answer, then there exists a contraction T on $\mathscr{H}$, with $A \neq 0$, which has no nontrivial invariant subspace. Therefore, as we have already showed, $\overline{\operatorname{ran}(A)} = \mathscr{H}$ and $A^{\frac{1}{2}}T = VA^{\frac{1}{2}}$, where V is a unitary operator on $\mathscr{H}$. Hence T is a contraction, without a nontrivial invariant subspace, which is a quasiaffine transform of a unitary operator. Thus question **2)** has a negative answer. Equivalently, a positive answer to **2)**, leads to a positive answer to **1)**.

□

4 Negative Conclusions

In this chapter we focus on finding an operator, which is polynomially bounded but not similar to a contraction. Therefore this is a counterexample to the conjecture that an operator is similar to a contraction if and only if it is polynomially bounded. The construction of this operator follows Chapter 10 of [2]. The proofs and the theory needed for counterexamples are rather tricky and therefore this remains the only representative of counterexamples in this script. We only present a few more famous results at the end of the chapter, without proofs.

4.1 Polynomially Boundedness

Definition 4.1. Putting $\|f\|_\infty := \sup\{|f(x)| : x \in \mathscr{H}\}$, we define:

(i) We call an operator T power bounded, if there exists $M > 0$ such that $\|T^n\| \leq M$ for all $n \in \mathbb{N}$

(ii) An operator T is called polynomially bounded, if there exists $K > 0$ such that $\|p(T)\| \leq K\|p\|_\infty$ for every polynomial p.

Definition 4.2. Let $\mathcal{P}$ denote the space of all polynomials. We define $M_n(\mathcal{A})$ as the set of all $n \times n$ matrices, with entries from a C^*-Algebra $\mathcal{A}$. Further we have:

(i) If we have a sequence of maps $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{C})$ and a map $\phi : \mathcal{A} \to \mathcal{C}$ with $\phi_n((a_{i,j})) = (\phi(a_{i,j}))$ we define the completely bounded norm through $\|\phi\|_{cb} = \sup_n \|\phi_n\|$.

(ii) We call ϕ a completely bounded map if $\|\phi\|_{cb}$ is finite.

(iii) We call an operator completely polynomially bounded, if the map $\rho : \mathcal{P}(\mathbb{D}) \to \mathscr{B}(\mathscr{H})$, with $\rho(p) = p(T)$ is completely bounded.

For further details about the completely bounded norm we refer to Chapter 1, in [2]. We now focus on the types of operators, which we need for Foguel's and Pisier's counterexamples. Let $S_{\mathscr{H}} : \ell^2(\mathscr{H}) \to \ell^2(\mathscr{H})$ be the forward shift, $S_{\mathscr{H}}(h_0, h_1, ...) := (0, h_0, h_1, ...)$, with its adjoint being the backward shift $S^*_{\mathscr{H}}(h_0, h_1, ...) = (h_1, ...)$. To simplify notation we write S for $S_{\mathscr{H}}$. Let $X \in \mathcal{B}(l^2(\mathscr{H}))$, with its entries $x_{i,j} \in \mathcal{B}(\mathscr{H})$ for $i, j \geq 0$.

Definition 4.3. 1. We shall call an operator of the form

$$F = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix} \qquad \textit{on} \qquad \ell^2(\mathscr{H}) \oplus \ell^2(\mathscr{H}),$$

the Foguel operator over $\mathscr{H}$ with symbol X.

2. Such operators are of special interest, when X has Hankel form. We say that X has Hankel form, if

$$X = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots \\ x_1 & x_2 & x_3 & \iddots \\ x_2 & x_3 & x_4 & \iddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and we write $X := (x_{i+j})_{i,j \in \mathbb{N}}$. In this case we call F Foguel-Hankel operator over $\mathscr{H}$ with symbol X.

It is easily seen that for $n \geq 2$

$$F^n = \begin{pmatrix} S^{*n} & X_n \\ 0 & S^n \end{pmatrix}, \qquad \textit{with} \qquad X_n = \sum_{j=0}^{n-1} S^{*j} X S^{n-j-1}.$$

Set $X_0 = 0$ and $X_1 = X$ and the formula holds for all $n > 0$. Note that when X has Hankel form it is self-adjoint, therefore $S^*X = XS$ and consequently $X_n = nXS^{n-1} = n(x_{i+j+n-1})$.

Since $\|S^{*n}\| = \|S^n\| = 1$, for all $n \geq 0$, we see that X is power-bounded if and only if $\sup_n \|X_n\| \leq M$, for some M.

Define a linear map $\delta : \mathcal{P} \to \mathcal{B}(l^2(\mathscr{H}))$ by setting $\delta(z^n) = X_n$ and extending linearly. When X has Hankel form, we have $\delta(p) = Xp'(S)$, where p' is the derivative of p, because $\delta(z^n) = X_n = \sum_{j=0}^{n-1} S^{*j} X S^{n-j-1} = \sum_{j=0}^{n-1} XS^{n-1} = nXS^{n-1}$. We now have that

$$p(F) = \begin{pmatrix} p(S^*) & \delta(p) \\ 0 & p(S) \end{pmatrix}$$

and since $\|p(S^*)\| = \|p(S)\| = \|p\|_\infty$, F is polynomially bounded if and only if δ is a bounded linear map.

Analogously, we have that F is completely polynomially bounded if and only if δ is a completely bounded linear map.

One final property that is useful is that δ obeys a certain derivation property. Indeed

$$\begin{pmatrix} (pq)(S^*) & \delta(pq) \\ 0 & (pq)(S) \end{pmatrix} = (pq)(F) = p(F)q(F)$$

$$= \begin{pmatrix} p(S^*) & \delta(p) \\ 0 & p(S) \end{pmatrix} \begin{pmatrix} q(S^*) & \delta(q) \\ 0 & q(S) \end{pmatrix}$$

$$= \begin{pmatrix} p(S^*)q(S^*) & p(S^*)\delta(q) + q(S)\delta(p) \\ 0 & p(S)q(S) \end{pmatrix}$$

and we have

$$\delta(pq) = p(S^*)\delta(q) + q(S)\delta(p),$$

for any polynomials p and q.

We begin with Pisier's counterexample. We start with a family of Foguel-Hankel operators that are easy to analyze.

To this end set $\mathscr{H} = \ell^2$ and let $\{E_{i,j} : i, j \in \mathbb{N}\}$ denote the matrices such that

$$E_{i,j} = \left\{ \begin{array}{cc} 1 & \text{at the } i,j\text{-th entry} \\ 0 & \text{otherwise} \end{array} \right\},$$

regarded as elements of $\mathcal{B}(\ell^2)$. To simplify notation we set $E_i = E_{i0}$. Since $E_i^* E_j = \delta_{ij} E_0$, we see that for any vector $h = (c_0, c_1, ...) \in \ell^2$, the operator $T(h) = \sum_{i=0}^{\infty} c_i E_i$ is bounded and in fact

$$\|T(h)\|^2 = \|T(h)^* T(h)\| = \left\| \sum_{i,j} \overline{c_i} c_j E_i^* E_j \right\| = \left\| \sum_i |c_i|^2 E_0 \right\| = \|h\|^2$$

Thus $T : \ell^2 \to \mathcal{B}(\ell^2)$ defines an isometric linear map that identifies a vector with the infinite column matrix.

There is one subtle distinction to be made. For while ℓ^2 is a Hilbert space $T(\ell^2)$ is an operator space. In particular, given vectors $(v_{ij}) \in \ell^2$, we have a well-defined norm for $(T(v_{ij}))$. To compute its norm, one simply substitutes the corresponding column matrix for each vector.

Theorem 4.4. *Fix a sequence $\{a_n : n \in \mathbb{N}\} \in \ell^2$ and consider the Foguel-Hankel operator F over ℓ^2 with symbol X, with X having the form $(x_{i+j}) = (a_{i+j} E_{i+j})$ for $i, j \in \mathbb{N}$. Then the following are equivalent:*

(i) *F is similar to a contraction.*

(ii) *F is polynomially bounded.*

(iii) *F is power bounded.*

(iv) $\sup_n n(\sum_{k=n-1}^{\infty} |a_k|^2)$ *is finite.*

Proof. Clearly *(i)* implies *(ii)* and *(ii)* implies *(iii)*. Recall that F is power bounded if and only if $\sup_n \|X_n\|$ is finite, where

$$X_n = nXS^{n-1} = n(a_{i+j+n-1} E_{i+j+n-1}),$$

We have that

$$\begin{aligned} X_n^* X_n &= n^2 \left(\sum_{k=0}^{\infty} \overline{a}_{i+k+n-1} a_{j+k+n-1} E_{i+k+n-1}^* E_{j+k+n-1} \right) \\ &= n^2 \left(\sum_{k=0}^{\infty} \overline{a}_{i+k+n-1} a_{j+k+n-1} \delta_{ij} E_0 \right). \end{aligned}$$

Thus, $X_n^* X_n$ is a diagonal operator matrix, whose diagonal entries are

$$n^2 \sum_{k=0}^{\infty} |a_{i+k+n-1}|^2 = n^2 \sum_{k=i+n-1}^{\infty} |a_k|^2.$$

Thus, $\|X_n^* X_n\| = n^2 \sum_{k=n-1}^{\infty} |a_k|^2$, from which the equivalence of *(iii)* and *(iv)* follows.

Finally, to prove that *(iv)* implies *(i)*, we consider the operator matrix $Y = (ja_{j+i-1} E_{i+j-1})$, where we set $E_{-1} = 0$. Assume that Y is bounded. We have that

$$YS - S^*Y = ((j+1)x_{i+(j+1)-1} E_{i+(j+1)-1}) - (ja_{(i+1)+j-1} E_{(i+1)+j-1}) = (a_{i+j} E_{i+j}) = X.$$

Thus, if we set $R = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$, then $R^{-1} = \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix}$ and we have

$$R^{-1}FR = \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix} \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} S^* & X - YS + S^*Y \\ 0 & S \end{pmatrix} = \begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix}.$$

Hence we have, that F is similar to a contraction. It remains to prove that Y is bounded. Computing

$$Y^*Y = (i\overline{a}_{i+j-1}E^*_{i+j-1})(ia_{i+j-1}E_{i+j-1}) = \left(\sum_{k=0}^{\infty} ij\overline{a}_{i+k-1}a_{k+j-1}E^*_{i+k-1}E_{k+j-1} \right),$$

we see that Y^*Y is a diagonal operator matrix, whose diagonal entries are $\sum_{k=0}^{\infty} i^2|a_{i+k-1}|^2E_0$. Thus, $\|Y^*Y\| = \sup_n n^2 \sum_{k=0}^{\infty} i^2|a_{k+n-1}|^2$ and so *(iv)* implies *(i)*. □

Theorem 4.4 shows us that the above family of operators could not possibly provide an example of a polynomially bounded operator that is not similar to a contraction. However to obtain Pisier's counterexample, we swap the above sequence of operators $\{E_i\}$ for another sequence $\{W_i\}$. The following propositions allow us to show that the equivalences of *(ii)*, *(iii)* and *(iv)* of Theorem 4.4 still hold for Foguel-Hankel operators obtained of this new sequence of operators, but their equivalence to *(i)* no longer hold.

Before we move to the first proposition, we shall present the following theorem, which we need to prove the proposition. Its proof can be found in [2], Chapter 5, page 68. Note that in the following theorem, $B_r(e^{i\theta}) := \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta}$.

Theorem 4.5 (Nehari-Page)**.** *Let $\mathscr{H}$ be a separable Hilbert space and let $A_n \in \mathcal{B}(\mathscr{H})$, $n \in \mathbb{N}$ be a sequence of operators. Then the operator-valued Hankel matrix $(A_{i+j})_{i,j\in\mathbb{N}}$ is bounded on $\ell^2(\mathscr{H})$, if and only if there exists $A_n \in \mathcal{B}(\mathscr{H})$, for $n \in \mathbb{Z}\backslash\mathbb{N}$ such that $\|B\|_\infty \equiv \sup_{r<1} \|B_r\| < \infty$. Moreover in this case there exists a particular choice of $A_n \in \mathcal{B}(\mathscr{H})$, $n < 0$ such that*

$$\|(A_{i+j})\| = \|B\|_\infty, \quad B(e^{i\theta}) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}.$$

Proposition 4.6. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces and let $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{K})$ be a bounded linear map. If $(A_{i+j}) = A$ is a bounded linear operator on $\ell^2(\mathscr{H})$, then $A_\Phi = (\Phi(A_{i+j}))$ is a bounded linear operator on $\ell^2(\mathscr{K})$. In fact $\|A_\Phi\| \le \|\Phi\|\|A\|$.*

Proof. By the Nehari-Page theorem there exists a sequence $\{A_n : n < 0\}$ such that

$$\|A\| = \sup_{r<1} \left\| \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} A_n \right\|_\infty,$$

Hence $\sup_{r<1} \left\| \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \Phi(A_n) \right\|_\infty \le \|\Phi\|\|A\|$. Applying the Nehari-Page theorem again, we have that

$$\|A_\Phi\| = \inf_B \sup_{r<1} \left\| \sum_{n=-\infty}^{-1} r^{|n|} e^{in\theta} B_n + \sum_{n=0}^{\infty} r^{|n|} e^{in\theta} \Phi(A_n) \right\|_\infty.$$

where the infimum is over all sequences $B = \{B_n : n < 0\}$ in $\mathcal{B}(\mathscr{K})$. Hence $\|A_\Phi\| \le \|\Phi\|\|A\|$. □

The above result shows that even though Φ may not be a completely bounded map, for Hankel matrices it acts completely bounded.

Proposition 4.7. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces and let $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{K})$ be a bounded linear map. Let F be a Foguel-Hankel operator over $\mathscr{H}$ with symbol $X = (A_{i+j})$ and let F_Φ be the corresponding Foguel-Hankel operator over $\mathscr{K}$ with symbol $X_\Phi = (\Phi(A_{i+j}))$. Then:*

(i) F power bounded implies F_Φ power bounded

(ii) F polynomially bounded implies F_Φ polynomially bounded.

Proof. We prove only *(ii)*, the proof of *(i)* is similar. Recall that F is polynomially bounded if and only if the map $\delta : \mathcal{P} \to \mathcal{B}(l^2(\mathscr{H})), \delta(p) = Xp'(S)$ is bounded. For the operator F_Φ we need to consider the map $\delta_\Phi(p) = X_\Phi p'(S)$. But $Xp'(S) = (B_{i+j})$ is a Hankel operator matrix, and $X_\Phi p'(S) = (\Phi(B_{i+j}))$. Hence,

$$\|\delta_\Phi(p)\| \leq \|\Phi\|\|(B_{i+j})\| = \|\Phi\|\|\delta\|\|p\|_\infty.$$

Thus, δ_Φ is a bounded map with $\|\delta_\Phi\| \leq \|\Phi\|\|\delta\|$ and hence F_Φ is polynomially bounded. □

The analogous conclusion for completely bounded operators is false, which are the source of our counterexamples. Our construction uses a particular choice of a sequence of operators related to the so-called CAR-operators:

Definition 4.8. A sequence $\{C_n : n \in \mathbb{N}\}$ of operators on a Hilbert space $\mathscr{H}$ is said to satisfy the canonical anticommutation relations (CAR), if

$$C_i C_j + C_j C_i = 0 \tag{16}$$

and

$$C_i C_j^* + C_i^* C_j = \delta_{ij} I, \tag{17}$$

for all i and j, where δ_{ij} denotes the Kronecker delta.

Given such a sequence and $h = (\alpha_0, \alpha_1, ...)$ in ℓ^2, set $\Lambda(h) = \sum_{i=0}^\infty \alpha_i C_i$. This defines a bounded operator with $\|\Lambda(h)\|^2 \leq \|h\|^2$, since

$$\Lambda(h)\Lambda(h)^* + \Lambda(h)^*\Lambda(h) = \sum_{i,j=0}^\infty \alpha_i \overline{\alpha}_j (C_i C_j^* + C_i^* C_j) = \sum_{i=0}^\infty |\alpha_i| I$$

by (17). However if we let $P = \Lambda(h)^*\Lambda(h)$, $Q = \Lambda(h)\Lambda(h)^*$, then $\|P\| = \|Q\|$ and $PQ = \Lambda(h)^*\Lambda(h)^2\Lambda(h)^* = 0$, since $\Lambda(h)^2 = 0$ by (16). Hence $\|P\| = \|Q\| = \|P + Q\| = \sum_{i=0}^\infty |\alpha_i|^2$ and it follows that $\|\Lambda(h)\| = \|h\|$. Thus, $\Lambda : \ell^2 \to \mathcal{B}(\mathscr{H})$ is an isometry, and $\Lambda(\ell^2)$ is an operator space isometric isomorph to a Hilbert space.

To construct a sequence of operators satisfying the CAR, we first need a finite sequence on a finite-dimensional space that satisfy them.

To construct these operators we start with 2×2 matrices,

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have that $V^2 = I$, $C^2 = 0$, $VC = -C$, $CV = C$,

$$C^*C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: E_{11}, \quad CC^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: E_{22}.$$

For $0 \le i \le n-1$, we define matrices $M_2 \otimes ... \otimes M_2 \cong M_{2^n}$ by setting

$$C_i = V^{\otimes(i)} \otimes C \otimes I_2^{\otimes(n-i-1)}$$

where $\otimes$ denotes the tensor product and $X^{\otimes(k)}$ the kth-time tensor product of X with itself. Note that

$$C_i^2 = (V^2)^{\otimes(i)} \otimes (C^2) \otimes I_2^{\otimes(n-i-1)} = 0 \tag{18}$$

$$C_i^* C_i = (V^*V)^{\otimes(i)} \otimes (C^*C) \otimes I_2^{\otimes(n-i-1)} = I_2^{\otimes(i)} \otimes E_{11} \otimes I_2^{\otimes(n-i-1)} \tag{19}$$

$$C_i C_i^* = (VV^*)^{\otimes(i)} \otimes (CC^*) \otimes I_2^{\otimes(n-i-1)} = I_2^{\otimes(i)} \otimes E_{22} \otimes I_2^{\otimes(n-i-1)} \tag{20}$$

while for $0 \le i < j \le n-1$, we have

$$C_i C_j = I_2^{\otimes(i)} \otimes C \otimes V^{\otimes(j-i-1)} \otimes C \otimes I_2^{\otimes(n-j-1)} = -C_j C_i \tag{21}$$

$$C_i C_j^* = I_2^{\otimes(i)} \otimes C \otimes V^{\otimes(j-i-1)} \otimes C^* \otimes I_2^{\otimes(n-j-1)} = -C_j^* C_i$$

Now (18) and (21) are seen to imply (16), while the others imply (17).

Given $h = (\alpha_0, ..., \alpha_{n-1})$, set $\Lambda(h) = \sum_{i=0}^{n-1} \alpha_i C_i$. From the above relations we have that Λ is an isometry, from an n-dimesional Hilbert space onto M_{2^n} and $\{C_i : i < n\}$ span a Hilbert space.

Next we show that $\{C_i \otimes C_i\}$ nearly span an l^1-space, that is, given scalars $a_0, a_1, ..., a_{n-1}$ we show that

$$\frac{1}{2}\sum_{i=0}^{n-1} |a_i| \le \left\| \sum_{i=0}^{n-1} a_i C_i \otimes C_i \right\| \le \sum_{i=0}^{n-1} |a_i| \tag{22}$$

Now up to a unitary equivalence, we have that $C_i \otimes C_i \cong (V \otimes V)^{\otimes(i)} \otimes (C \otimes C) \otimes I_4^{\otimes(n-i-1)}$ in $M_4 \otimes ... \otimes M_4 = M_{4^n}$. Let e_1, e_2 denote the basis for $\mathbb{C}^2$, and choose w_i, $1 \le i \le n$, constants of modulus one such that $a_i \overline{w}_i = |a_i|$. Then $x_i = (e_1 \otimes e_1 + w_i e_2 \otimes e_2)/\sqrt{2}$ will be a unit vector in $\mathbb{C}^4$ and $x = x_1 \otimes ... \otimes x_n$ are a unit vector in $\mathbb{C}^{4^n}$. Since $V \otimes V(e_i \otimes e_i) = e_i \otimes e_i$, $C \otimes C(e_1 \otimes e_1) = e_2 \otimes e_2$ and $C \otimes C(e_2 \otimes e_2) = 0$, we have that

$$\left(\sum_{i=0}^{n-1} a_i C_i \otimes C_i x, x \right) = \frac{1}{2}\sum_{i=0}^{n-1} a_i \overline{w}_i = \frac{1}{2}\sum_{i=0}^{n-1} |a_i|.$$

Thus $\| \sum_{i=0}^{n-1} a_i C_i \otimes C_i \| \ge \frac{1}{2} \sum_{i=0}^{n-1} |a_i|$ and since $\|C_i \otimes C_i\| = 1$ the other inequation follows.

So far we have only constructed, for every n, a finite sequence of $C_0, ..., C_{n-1}$ of finite matrices satisfying the CAR. To obtain infinite sequences one generally uses infinite tensor products of Hilbert spaces.

To avoid discussions of infinte tensor products we use a slightly different approach. Relabel the $2^n \times 2^n$ matrices constructed above as $C_{0,n}, ..., C_{n-1,n}$ and for $i \ge n$, set $C_{i,n} = 0$. Define bounded operators on $\bigoplus_{n=0}^{\infty} \mathbb{C}^{2^n}$, by setting $W_i = \bigoplus_{n=1}^{\infty} C_{i,n}$. It is readily verified that

(i) $\| \sum_{i=0}^{\infty} a_i W_i \|^2 \le \sum_{i=0}^{\infty} |a_i|^2$ for any $(a_0, a_1, ...) \in l^2$,

(ii) $\frac{1}{2} \sum_{i=0}^{n-1} |a_i| \le \| \sum_{i=0}^{n-1} a_i C_{i,n} \otimes W_i \| \le \sum_{i=0}^{n-1} |a_i|$,

(iii) $W_i W_j + W_j W_i = 0$,

(iv) $W_i W_j^* + W_j^* W_i = \delta_{ij} I P_i$, where P_i is the finite rank projection onto $\bigoplus_{n=1}^{i} \mathbb{C}^{2^n}$. Thus although $\{W_i : i \in \mathbb{N}\}$ does not quite satisfy the CAR, their images in the C^*-algebra obtained by quotienting out the ideal of compact operators do satisfy them.

Theorem 4.9 (Pisier). *Fix a sequence $\{a_n : n \in \mathbb{N}\}$ and consider the Foguel Hankel operator F over $\mathscr{H}$ with symbol $X = a_{i+j}W_{i+j}$, where $\{W_i : i \in \mathbb{N}\}$ are the operators constructed above. Then the following are equivalent:*

(i) F is polynomially bounded.

(ii) F is power-bounded.

(iii) $\sup_n n(\sum_{k=n-1}^{\infty} |a_k|^2)$ is finite.

However, if $(\sum_{k=0}^{\infty}(k+1)^2|a_k|^2)$ is infinite then F is not similar to a contraction.

Proof. Clearly *(i)* implies *(ii)*. To see that *(ii)* implies *(iii)*, recall that F is power bounded if, and only if, $\sup_n \|X_n\|$ is finite, where $X_n = nXS^{n-1} = n(a_{i+j+n-1}W_{i+j+n-1})$. We have that

$$X_n^*X_n + X_nX_n^* = n^2 \sum_{k=0}^{\infty} \overline{a}_{i+k+n-1}a_{j+k+n-1}[W_{i+k+n-1}^*W_{j+k+n-1} + W_{i+k+n-1}W_{j+k+n-1}^*].$$

Because $W_iW_j + W_jW_i = 0$, the off-diagonal terms are 0, and hence the operator norm of this matrix, is just the largest norm of a diagonal entry. Since each diagonal entry is a sum of positive operators, and because of the definition of W_i, the supremum occurs when $i = 0$. Thus,

$$\|X_n^*X_n + X_nX_n^*\| = n^2 \left\| \sum_{k=0}^{\infty} |a_{k+n-1}|^2[W_{k+n-1}^*W_{k+n-1} + W_{k+n-1}W_{k+n-1}^*] \right\|$$

$$= \sup_m n^2 \left\| \sum_{k=0}^{\infty} |a_{k+n-1}|^2(C_{k+n-1,m}^*C_{k+n-1,m} + C_{k+n-1,m}C_{k+n-1,m}^*) \right\| = n^2 \sum_{k=0}^{\infty} |a_{k+n-1}|^2,$$

using the fact that $C_{k,m}^*C_{k,m} + C_{k,m}C_{k,m}^* = I$ for $k < m$. Hence $\sup_n \|X_n\|$ finite implies, that $\sup_n \|X_nX_n^* + X_n^*X_n\| = \sup_n n^2 \sum_{k=0}^{\infty} |a_{k+n-1}|^2$ is finite.

To prove that *(iii)* implies *(i)* we use Proposition 4.7. Define $\Phi : \mathcal{B}(\ell^2) \to \mathcal{B}(\mathscr{H})$ by $\Phi((a_{i,j})) = \sum_{i=0}^{\infty} a_{i,0}W_i$. Since the norm of a matrix is larger than the norm of any column, we have that $\|\Phi((a_{i,j}))\| = \sum_{i=0}^{\infty} |a_{i,0}|^2 \leq \|(a_{ij})\|$ and hence $\|\Phi\| \leq 1$.

Assuming that $\sup_n n(\sum_{k=0}^{\infty} |a_{k+n-1}|^2)$ is finite, by Theorem 4.4 we have that the Foguel-Hankel operator with symbol $(a_{i+j}E_{i+j})$ is polynomially bounded and hence by Proposition 4.7 the Foguel-Hankel operator with symbol $(a_{i+j}\Phi(E_{i+j})) = (a_{i+j}W_{i+j})$ is polynomially bounded.

Finally to see that F is not similar to a contraction when $\sum_{k=0}^{\infty}(k+1)^2|a_k|^2$ is infinite, it is sufficient to show that the map $\delta : \mathcal{P} \to \mathcal{B}(\ell^2(\mathscr{H}))$ is not completely bounded for this operator. Note that the $(0,0)$-entry of $\delta(z^{i+1})$ is $(i+1)a_iW_i$. Thus, if we can show that the map $\delta_0 : \mathcal{P} \to \mathcal{B}(\mathscr{H})$, $\delta_0(1) = 0$, $\delta_0(z^{i+1}) = (i+1)a_iW_i$ is not completely bounded, then δ fails to be completely bounded, too.

If we consider the $2^n \times 2^n$ matrix-valued polynomial $P(z) = \sum_{i=0}^{n}(i+1)\overline{a}_iC_{i,n}z^{i+1}$, then $\delta_0^{2^n}(P) = \sum_{i=0}^{n}(i+1)^2|a|^2C_{i,n} \otimes W_i$. By the definition of W_i and equation (22) for the operators $C_{i,n}$, we have that

$$\|\delta_0^{2^n}(P)\| \geq \left\| \sum_{i=0}^{n}(i+1)^2|a|^2C_{i,n} \otimes C_{i,n} \right\| \geq \frac{1}{2}\sum_{i=0}^{n}(i+1)^2|a_i|^2.$$

Because we have that Λ is an isometry, we get that $\|P\|_\infty = (\sum_{i=0}^n (i+1)^2|a_i|^2)^{\frac{1}{2}}$. Hence

$$\|\delta_0\|_{cb} \geq \left(\sum_{i=0}^n (i+1)^2|a_i|^2\right)^{\frac{1}{2}},$$

and letting n tend to infinity we have that δ_0 is not completely bounded. □

The above result leads us to Pisier's counterexample.

Corollary 4.10 (Pisier). *Let $a_{2^k-1} = 2^k$ and $a_i = 0$ otherwise. Then the Foguel-Hankel operator with symbol $a_{i+j}W_{i+j}$ is polynomially bounded, but not similar to a contraction.*

Proof. We have that $\sup_n (n+1)\sum_{k=n-1}^\infty |a_k|^2 = sup_j 2^j \sum_{k=j}^\infty (2^{-k})^2 = \frac{4}{3}$, while

$$\sum_{k=0}^\infty (k+1)^2|a_k|^2 = \sum_{j=0}^\infty (2^j)^2(2^{-j}) = \infty$$

and so we are done by Theorem 4.9. □

For another example one can set $a_k = (k+1)^{-\frac{3}{2}}$. Then

$$\sup_n (n+1)\left(\sum_{k=n}^\infty |a_k|^2\right)^{\frac{1}{2}} \leq \sqrt{2},$$

but

$$\sum_{k=0}^\infty (k+1)^2|a_k|^2 = \infty$$

The same results as above hold when the sequence $\{W_i\}$ is replaced by an actual CAR-sequence $\{C_n\}$.

Necessary and sufficient conditions on the sequence $\{a_n\}$ for the Foguel-Hankel operator with symbol $(a_{i+j}C_{i+j})$ to be similar to a contraction were only recently obtained. In [6] it is shown that $\sum_{k=0}^\infty (k+1)^2|a_k|^2$ finite is a necessary and sufficient condition for similarity to a contraction. Earlier it had been shown that if $\sum_{k=0}^\infty (k+1)^2(\log(\log(k+1)))^{2+\epsilon}|a_k|^2$ is finite for any $\epsilon > 0$, then one has similarity to a contraction.

4.2 Other Counterexamples

In this section we want to present a few other counterexamples, which are important results in invariant subspace theory. We do not show the proofs, because they are quite long. We want to begin this section with Per Enflo's counterexample:

Theorem 4.11. *(Enflo) There exists a Banach space and a bounded linear operator on it without a nontrivial invariant subspace.*

The construction was first presented in 1976, but the final version can be found in [9]. A more recent counterexample is the following, presented in [10].

Theorem 4.12. *(Read) There is a continuous linear operator on ℓ_1 with no nontrivial invariant subspace.*

Theorem 4.13. *(Atzmon)[11] There is an operator on a nuclear Fréchet space without an invariant subspace.*

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