Mean ergodic semigroups of operators

Anna Kiesenhofer

Advisor: Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Martin Blümlinger

A thesis submitted in partial fulfillment of the requirements for the Degree of Bachelor of Science in Technischer Mathematik

Vienna University of Technology
Vienna, Austria

July 2011

Contents

Notation

If X is a Banach space and A is an operator[1] on X we denote by

$\mathcal{B}(X)$	...	the set of bounded linear operators on X
τ_s	...	the strong operator topology on $\mathcal{B}(X)$
τ_w	...	the weak operator topology on $\mathcal{B}(X)$
$D(A)$	...	the domain of A
$\rho(A)$	...	the resolvent set of A
$\sigma(A)$	...	the spectrum of A
$R(\lambda, A)$	...	the resolvent $(A-\lambda)^{-1}$ of A (if $\lambda \in \rho(A)$)
I	...	the identity operator on X
$[M]$	...	the linear hull of $M \subset X$.

Apart from the standard $C(\mathbb{R})$- and L^p-spaces the following spaces of functions will occur in the text:

$C_c(\mathbb{R})$	...	$\{f \in C(\mathbb{R}) \,\big|\, f \text{ has compact support}\}$
$C_0(\mathbb{R})$	...	$\{f \in C(\mathbb{R}) \,\big|\, \forall \epsilon > 0 \; \exists K \text{ compact such that } f(x) < \epsilon \; \forall x \in K^c\}$
$C_{ub}(\mathbb{R})$	...	the set of uniformly continuous bounded functions on $\mathbb{R}$
$C_b(\mathbb{R})$	...	the set of bounded functions on $\mathbb{R}$
$\mathcal{S}(\mathbb{R}^n)$	...	the Schwartz space on $\mathbb{R}^n$
$H^m(\Omega)$	...	the Sobolev space $\{f \in L^2(\Omega) \,\big|\, D^\alpha f \in L^2(\Omega) \, \forall |\alpha| \leq m\}$, where Ω is an open subset of $\mathbb{R}^n$ and, for $\alpha \in \mathbb{N}_0^n$, $|\alpha| := \sum_{i=1}^n \alpha_i$.

Finally, for any sets Y, Z and a function $f : Y \to Z$, $a \in Z$, we use the notation $[f = a] := f^{-1}(\{a\})$.

[1] If nothing else is specified we use the word "operator" as short for *linear* operator. We do not require an operator to be bounded, nor do we make any assumptions on its domain.

1 Introduction

Partial differential equations are ubiquitous in physics. Typically they relate the "time derivative" of a function $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ with certain derivatives of u "in space". The heat equation

$$\partial_t(t, x) = \Delta u(t, x),$$

is a simple example.

Physicists like to write down the solutions of such PDEs in terms of evolution operators: operators $(T(t))_{t\geq 0}$ which map the initial state $u(0, \cdot)$ to a future state $u(t, \cdot) = T(t)u(0, \cdot)$. In quantum mechanics this is particularly common, which is why we sketch the physical[2] approach for this case in a little more detail: The central equation in quantum mechanics is the Schrödinger equation; for a free particle

$$\Psi_t(t, x) = i\Delta\Psi(t, x). \tag{1}$$

We can regard Ψ as a function ψ depending only on the parameter t with values in a space of functions on $\mathbb{R}^3$ (usually $L^2(\mathbb{R}^3)$):

$$\psi : \mathbb{R} \to L^2(\mathbb{R}^3) : t \mapsto \Psi(t, \cdot).$$

Then the partial differential equation (1) becomes an ordinary (vector space-valued) differential equation:

$$\psi'(t) = -i\Delta\psi(t). \tag{2}$$

Now this equation strongly resembles the differential equation $f'(t) = af(t)$ for a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $a \in \mathbb{R}$, which has the solution $f(t) = e^{ta}f(0)$. For physicists it is clear that the solutions of (2) can be written in the same way:

$$\psi(t) = e^{ti\Delta}\psi(0),$$

where $e^{ti\Delta}$ is now an operator on $L^2(\mathbb{R}^3)$, the "time evolution operator". For mathematicians this is less clear: we do not know what $e^{ti\Delta}$ is; actually, we do not even know what Δ is, because we have not specified its domain.

The subject of this paper is the mathematical theory of families of operators similar to $(e^{i\Delta t})_{t\geq 0}$: more precisely, families of (bounded) operators $(T(t))_{t\geq 0}$ which are the evolution operators of certain vector space-valued differential equations

$$u'(t) = Au(t), \tag{3}$$

[2] We emphasize the word *physical*. The derivation is not rigorous.

where A is an operator on a Banach space X and $u : \mathbb{R}_0^+ \to X$ is a differentiable function. The precise definition of the families $(T(t))_{t\geq 0}$ we consider here (namely *strongly continuous semigroups*) is given in the main text (Definition 2.1). In essence, we only require an algebraic property similar to the one enjoyed by the exponential function, $e^{x+y} = e^x e^y$, and a continuity property .

The theory of strongly continuous semigroups has applications in various fields, for instance PDEs but also more general equations of the form (3) (e.g. delay differential equations) where A is not a classical differential operator. In this paper we study a specific aspect of semigroup theory: the asymptotic behaviour of the time averages

$$\frac{1}{r}\int_0^r T(t)dt,\ r > 0,$$

where $(T(t))_{t\geq 0}$ is a strongly continuous semigroup. We will consider convergence of these means (as $r \to \infty$) with respect to different topologies on $\mathcal{B}(X)$. As an application we will examine the behaviour of the "time averages" of solutions of certain PDEs.

In more detail, the content of the thesis is organized as follows:
In Section 2 we present some general results on operator semigroups: The motivation for studying strongly continuous semigroups is given in Section 2.1, where we examine the connection of semigroups and differential equations of the form (3). We proceed with an overview of the most important results in semigroup theory in Section 2.3.
Sections 3 to 5 constitute the core of this thesis. In Section 3 we define mean ergodic semigroups and give some equivalent characterizations and examples. The results are applied to some physically important differential equations in Section 4: the heat, wave and Schrödinger equation. In the last section, Section 5, we introduce the notion of uniform mean ergodicity and characterize the semigroups having this property. As a special case of uniformly mean ergodic semigroups we consider semigroups whose generator has compact resolvent.

The information presented here is drawn from numerous sources: The most important one is the excellent book on semigroups by K. Engel and R. Nagel, [Eng00]. For the part concerning the application of semigroups to PDEs the text relies mainly on [Paz83] and [Ber05]. As an additional reference for specific results on semigroups, PDEs and operator theory I used [Con85] and [Yos74] as well as the lecture notes by my teachers at TU Vienna: [Wor10], [Blü10] and [Jü09]. Other sources are cited in the main text.

2 Strongly continuous semigroups

In this and all subsequent sections, let X be a Banach space.

2.1 C_0-semigroups and the abstract Cauchy problem

Definition 2.1. A family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a **semigroup** (of operators) if the function

$$T:\, R_0^+ \to \mathcal{B}(X):\, t \mapsto T(t)$$

is a monoid homomorphism from $(\mathbb{R}_0^+, +, 0)$ to $(\mathcal{B}(X), \circ, I)$; in other words if $(T(t))_{t\geq 0}$ satisfies the functional equations

$$\begin{aligned} T(0) &= I \\ T(s)T(t) &= T(s+t) \text{ for all } s, t \in \mathbb{R}^+. \end{aligned} \tag{FE}$$

The semigroup $(T(t))_{t\geq 0}$ is called **strongly continuous** (or a C_0-semigroup) if T is continuous with respect to the strong operator topology on $\mathcal{B}(X)$; in other words if

$$\lim_{t\to t_0} T(t)x = T(t_0)x \text{ for all } t_0 \in \mathbb{R}^+, x \in X.$$

As mentioned in the introduction there is a connection between C_0-semigroups of operators and Banach space-valued initial value problems of the form

$$\begin{aligned} u(0) &= u_0 \in D(A) \\ \dot{u}(t) &= Au(t) \text{ for } t \geq 0, \end{aligned} \tag{ACP}$$

where A is a (possibly unbounded) linear operator with domain $D(A) \subset X$. Here we will explore this connection in more detail.

Definition 2.2. The problem of finding a solution $u : \mathbb{R}_0^+ \to X$ to (ACP) given A and u_0 is called an **abstract Cauchy problem**. Here, we understand the concept of a **solution** in the classical sense: the function $u : \mathbb{R}_0^+ \to X$ is a solution if all the expressions in (ACP) are well-defined (i.e. u is differentiable[3] and $u(t) \in D(A)\,\forall t \geq 0$) and the equalities hold.

Let us assume that the problem (ACP) above has a unique solution $u(\cdot, u_0)$ for every $u_0 \in D(A)$. (This is one of the features we expect a well-posed problem, e.g. the equations of motion of a dynamical system, to have.) Then

$$\tau(t):\, D(A) \to X:\, u_0 \mapsto u(t, u_0)$$

[3] By differentiability we understand that the limits $\dot{u}(t) := \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}, t > 0$ and $\dot{u}(0) = \lim_{h\searrow 0} \frac{u(h)-u(0)}{h}$ exist.

is a linear operator for all $t \in \mathbb{R}_0^+$. Moreover, τ satisfies the functional equation

$$\begin{aligned} \tau(0) &= I \\ \tau(s)\tau(t) &= \tau(s+t) \text{ for all } s, t \in \mathbb{R}^+. \end{aligned}$$

This resembles the definition of a semigroup given above. However, the operators $\tau(t)$ need not be bounded, so we do not necessarily obtain a semigroup in the sense of Definition 2.1. With an additional assumption concerning the "well-posedness" of the problem, however, all solutions of (ACP) can be given in terms of a semigroup associated with A:

Definition 2.3. The abstract Cauchy problem (ACP) associated with the linear operator $A : D(A) \subset X \to X$ is called **well-posed** if

- for every $u_0 \in D(A)$ there exists a unique solution $u(\cdot, u_0)$ of (ACP)
- the solution depends continuously on the data: there exists $C > 0$ such that for all $u_0 \in D(A)$

$$\sup_{t \in [0,1]} \|u(t, u_0)\| \leq C\|u_0\|.$$

Theorem 2.4. *Let $A : D(A) \subset X \to X$ be a closed operator with dense domain. Then the following properties are equivalent:*

(i) The problem (ACP) *associated with A is well-posed in the sense of Definition 2.3.*

(ii) There exists a C_0-semigroup $(T(t))_{t \geq 0}$ such that for all $u_0 \in D(A)$ the function $u(\cdot, u_0) := T(\cdot)u_0$ is a solution of (ACP).

(iii) There exists a C_0-semigroup $(T(t))_{t \geq 0}$ such that for the functions $u(\cdot, u_0) := T(\cdot)u_0$ the following holds:

$$D(A) = \{u_0 \in X \,|\, u(\cdot, u_0) : \mathbb{R}_0^+ \to X \textit{ is differentiable}\}$$

and $u(\cdot, u_0)'(0) = Au_0$ for all $u_0 \in D(A)$.[4]

Condition (iii) in the theorem above is the common way of defining the **generator of a C_0-semigroup**: Given a C_0-semigroup $(T(t))_{t \geq 0}$ its generator A is simply the operator which has the properties stated in (iii) (or, in

[4]The implication (ii)⇒(iii) is not trivial, since in (ii) we only demand $D(A) \supset \{u_0 \in X \,|\, u(\cdot, u_0) : \mathbb{R}_0^+ \to X$ is differentiable$\}$ instead of equality between these two sets. Moreover, note that in (iii) we only demand that the derivative of $u(t, u_0)$ at $t = 0$ equals $Au(t, u_0)$.

a more explicit form, in Equation (4) in the next section). From the equivalences in the theorem we see that it is of great interest to know whether an operator A generates a C_0-semigroup. If it does, the associated abstract Cauchy problem is well posed — in particular, there exists a unique solution and this solution can be written down in terms of the semigroup generated by A. Even if we cannot obtain an explicit expression for $(T(t))_{t\geq 0}$ (and hence the solution), semigroup theory helps us to obtain useful information about the qualitative behaviour of solutions, e.g. regularity and asymptotic behaviour, from the knowledge of the generator A alone.

Remark. If X is finite dimensional or, more generally, if X is arbitrary and $A : X \to X$ is bounded, then the abstract Cauchy problem (ACP) is always well posed, i.e. A generates a C_0-semigroup $(T(t))_{t\geq 0}$:

$$T(t) = \exp(At) \equiv \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}, \quad t \geq 0.$$

The "interesting" ACPs are therefore those where the operator A is unbounded.

Before having a closer look at the interplay of C_0-semigroups and their generators in Section 2.3 we discuss some general properties of C_0-semigroups:

2.2 Some properties of C_0-semigroups

In the previous section we defined the generator A of a C_0-semigroup $(T(t))_{t\geq 0}$ as

$$Ax = \lim_{t\searrow 0} \frac{T(t)x - x}{t}, \quad D(A) = \{x \in X \mid \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists}\}. \tag{4}$$

It is one of the standard results of semigroup theory (see e.g. [Eng00]) that this operator is densely defined and closed. From Theorem 2.4 it follows that the semigroup is uniquely determined by its generator; in other words, if $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are different semigroups then their generators must differ as well.

We now summarize some of the basic results concerning C_0-semigroups. The details and proofs can, for instance, be found in [Eng00].

We first turn to a result that is in some sense a generalization of the classical fundamental theorem of calculus to Banach-space valued functions, where the Riemann integral of a continuous function $f : [a, b] \to \mathbb{R}$ is replaced by the Bochner integral of the continuous function $f : [a, b] \to X$:

$$\int_a^b f(t)dt := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i^{(n)})[x_{i+1}^{(n)} - x_i^{(n)}], \quad x_i^{(n)} := a + i\frac{b-a}{n}. \tag{5}$$

From the definition it immediately follows that

$$S \int_a^b f(t)dt = \int_a^b Sf(t) \text{ for } S \in \mathcal{B}(X).$$

We will use this property in subsequent chapters.[5]

Theorem 2.5. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator A. The following holds:*

(i) If $x \in D(A)$ then $T(t)x \in D(A)\, \forall t \geq 0$ and

$$AT(t)x = T(t)Ax = \lim_{h \searrow 0} \frac{T(t+h)x - T(t)x}{h}.$$

(ii) $T(t)x - T(s)x = \int_s^t T(r)Axdr = \int_s^t AT(r)xdr = A\int_s^t T(r)xdr$.

The second important result we bring here is the following estimate for the growth of a C_0-semigroups:

Theorem 2.6. *For every C_0-semigroup $(T(t))_{t\geq 0}$ there exists constants $M > 0, \omega \in \mathbb{R}$, such that:*

$$\|T(t)\| \leq Me^{\omega t} \quad \forall t \geq 0. \tag{6}$$

Therefore, the following definition makes sense:

Definition 2.7. The infimum of all numbers $\omega \in \mathbb{R}$ such that (6) is satisfied for some $M > 0$ is called the **growth bound** of the semigroup $(T(t))_{t\geq 0}$. (We also allow growth bounds $-\infty$.)

If the growth bound ω_0 is negative, i.e. if there exists $\omega < 0$ such that (6) is satisfied, the semigroup is called **exponentially stable**. If in (6) we can take $\omega = 0$ the semigroup is called **bounded**. Finally, if we can take $M = 1$, $\omega = 0$, the semigroup is called **contractive** (or a **contraction semigroup**).

2.3 Generators of C_0-semigroups

In Section 2.1 we saw that knowing whether a certain operator A generates a C_0-semigroup provides a great deal of information about the solutions of the corresponding ACP. One of the cornerstones of semigroup theory is the following characterization of the generators of C_0-semigroups:

[5]Note, however, that this does *not* imply result (ii) in Theorem 2.5 since A might be unbounded.

Theorem 2.8 (Hille-Yoshida). *An operator A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying*

$$\|T(t)\| \leq Me^{\omega t},$$

where $M > 0, \omega \in \mathbb{R}$, if and only if

- *A is closed and densely defined.*
- *For all $\lambda > \omega$ it holds that $\lambda \in \rho(A)$ and*

$$\|R_\lambda(A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \quad \forall \lambda > \omega, n \in \mathbb{N}.$$

The Hille-Yoshida Theorem is often inconvenient to use because $\|R_\lambda(A)^n\|$ can be hard to determine. For the special case of contraction semigroups there is a simpler characterization, the Lumer-Phillips Theorem. We first define the notion of dissipativity:

Definition 2.9. An operator A on a Banach space X is called **dissipative** if for every $x \in D(A)$ with $\|x\| = 1$ there exists $x' \in X'$, $\|x'\| = 1$, such that

$$x'(x) = 1 \quad \text{and} \quad \operatorname{Re} x'(Ax) \leq 0.$$

Remark. In the definition above, if X is a Hilbert space then for $x \in D(A), \|x\| = 1$, the only $x' \in X'$ satisfying $\|x'\| = 1$ and $x'(x) = 1$ is the functional $x' = (x, \cdot)$. Therefore, A is dissipative if and only if $\operatorname{Re}(Ax, x) \leq 0$ for all $x \in D(A)$.

Theorem 2.10 (Lumer-Phillips). *Let X be a Banach space and let A be a densely defined operator on X. Then A generates a contraction semigroup if and only if A is dissipative and there exists $\lambda > 0$ such that* $\operatorname{ran}(A-\lambda) = X$.

Finally, we consider a special class of contraction semigroups, **C_0-semigroups of unitary operators**. Such semigroups will play a role in the study of some PDEs in Section 4, namely the wave equation and the Schrödinger equation.

In Section 2.1 we mentioned that the C_0-semigroup generated by a bounded operator A is given by by $T(\cdot) = \exp(A\,\cdot)$. For a self-adjoint but possibly unbounded operator iA the expression $e^{tA} = e^{-it(iA)}$ can be given meaning via the functional calculus for (unbounded) self-adjoint operators. This observation motivates the following interesting proof of Stone's Theorem, which differs from the one usual found in texts on C_0-semigroups:

Theorem 2.11 (Stone). *Let X be a Hilbert space. An operator A on X is the generator of a C_0-semigroup of unitary operators on X if and only if A is skew-adjoint*[6].

We use the following lemma to show self-adjointness of iA:

Lemma 2.12. *A densely defined operator A on a Hilbert space X is self-adjoint if A is symmetric and* $\operatorname{ran}(A-\lambda) = X$ *for some* $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Proof. The proof is based on the fact that $\ker(B-\lambda) = \{0\}$ if B is symmetric and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ ([Blü10]). Moreover, we observe that if B, D are operators, $B \subset D$, B is surjective and D is injective, then $B = D$. Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$ be such that $A - \lambda$ is surjective. The results just mentioned applied to

$$(A-\lambda, D(A)) \subset (A^* - \lambda, D(A^*))$$

imply that $D(A) = D(A^*)$, hence $A = A^*$. $\square$

Proof of Stone's Theorem. First assume that A generates a C_0-semigroup $(U(t))_{t\geq 0}$ of unitary operators. Because

$$\left(\frac{U(t)x - x}{t}, y\right) = \left(x, \frac{U(t)^{-1}y - y}{t}\right) = \left(U(t)x, \frac{y - U(t)y}{t}\right), t > 0$$

we see that $(Ax, y) = -(x, Ay)$ for all $x, y \in D(A)$. Therefore, iA is symmetric. Moreover, the Hille-Yoshida Theorem implies that $1 \in \rho(A)$, hence $i \in \rho(iA)$. From Lemma 2.12 it follows that iA is self-adjoint.

Conversely, let iA be self-adjoint. Then $B := (-iA, D(A))$ is self-adjoint as well and has a spectral measure E associated with it. The candidate for the semigroup generated by A is the family of operators $U(t) := e^{At} = e^{iBt}, t \in \mathbb{R}$, in other words the spectral integral of the (bounded!) functions $f_t : s \mapsto e^{its}$ with respect to E. Since $f \mapsto \int f dE$ is an isometric *-homomorphism from $C_b(\mathbb{R})$ to $\mathcal{B}(X)$, the operators $U(t)$ are unitary and form a semigroup. Moreover, the semigroup is strongly continuous because the functions f_t are uniformly bounded and for $t_n \searrow 0$ we have $f_{t_n} \to 1$ pointwise, hence $T(t_n) = \int f_{t_n} dE \to \int 1 dE = 1$ in τ_s.

It remains to show that the generator of the C_0-semigroup $(U(t))_{t\geq 0}$ is indeed A. From the relation

$$\left|\left(\frac{U(t)x - x}{t} - Ax, y\right)\right| \leq \int \left|\frac{e^{its} - 1}{t} - is\right| d\mu_{x,y}(s), \quad x \in D(A), y \in X$$

and the fact that $\left|\frac{e^{its}-1}{t} - is\right| \xrightarrow{t\searrow 0} 0$ uniformly in s it follows that the generator $\tilde{A}$ of $(U(t))_{t\geq 0}$ is an extension of A, $A \subset \tilde{A}$. This implies $A^* \supset (\tilde{A})^*$. By the first step of the proof $i\tilde{A}$ is self-adjoint. Therefore, $D((\tilde{A})^*) = D(\tilde{A})$. By assumption, iA is self-adjoint as well, hence $D(A) = D(A^*) \supset D((\tilde{A})^*) = D(\tilde{A})$. Therefore, $A = \tilde{A}$. $\square$

Remark. Since a C_0-semigroup is uniquely determined by its generator, the proof shows that the semigroup $(U(t))_{t\geq 0}$ generated by a skew-adjoint operator is given by $U(t) = e^{At}$, $t \geq 0$.

[6] An operator A is called skew-adjoint if $(iA, D(A))$ is self-adjoint.

2.4 Standard Examples

Two standard examples of C_0-semigroups are given by the translation and multiplication semigroups on appropriate spaces of functions. They will serve as an illustration of some of the concepts developed in subsequent chapters. Here we define the semigroups and write down their generators; for details and proofs see [Eng00].

Example 2.13 (Translation semigroup). The (right) translation semigroup $(T(t))_{t\geq 0}$ defined by

$$T(t)f = f(\cdot + t), \quad f \in X,$$

where $X = C_{ub}(\mathbb{R})$ or $X = L^p(\mathbb{R}), 1 \leq p < \infty$ is a C_0-semigroup. Its generator A is given by differentiation,

$$Af = f', \quad f \in D(A),$$

where the domain $D(A)$ is one of the following:

(a) $X = C_{ub}(\mathbb{R})$:

$$D(A) = \{f \in C_{ub}(\mathbb{R}) \,\big|\, f \text{ is differentiable and } f' \in C_{ub}(\mathbb{R})\}.$$

(b) $X = L^p(\mathbb{R}), 1 \leq p < \infty$:

$$D(A) = \{f \in L^p(\mathbb{R}) \,\big|\, f \text{ is absolutely continuous and } f' \in L^p(\mathbb{R})\}.$$

Remark. Note that we chose $X = C_{ub}(\mathbb{R})$ instead of $X = C_0(\mathbb{R})$ or $X = C_b(\mathbb{R})$ because the translation semigroup on these spaces would not be strongly continuous.

Example 2.14 (Multiplication semigroup). Let Ω be a locally compact Hausdorff space and consider the multiplication semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)f = e^{qt}f, \quad f \in X,$$

where either

(a) $X = C_0(\Omega)$ for some locally compact Hausdorff space Ω and $q \in C(\Omega)$, or

(b) $X = L^p(\Omega, \mu)$, where (Ω, Σ, μ) is a σ-finite measure space, $1 \leq p < \infty$, and q is a measurable function.

In both cases the generator is given by

$$Af = qf, \quad f \in D(A) = \{f \in X \,|\, qf \in X\}.$$

The semigroup $(T(t))_{t\geq 0}$ is bounded if and only if $q \leq 0$ (a.e.). We will need the condition of boundedness to apply some of the theorems derived in subsequent chapters; therefore, we make the additional assumption that $q \leq 0$ (a.e.).

3 Mean ergodic semigroups

Definition 3.1. Let $(T(t))_{t\geq 0}$ be a a strongly continuous semigroup. For $r > 0$ we define the **Cesàro means** $C(r) \in \mathcal{B}(X)$ as

$$C(r) := \frac{1}{r}\int_0^r T(s)ds : x \mapsto \frac{1}{r}\int_0^r T(s)xds.$$

If the limit $\lim_{r\to\infty} C(r)$ exists in the strong operator topology, the semigroup $(T(t))_{t\geq 0}$ is called **mean ergodic**.

Let $(T(t))_{t\geq 0}$ be a mean ergodic semigroup. We expect that $P := \lim_{r\to\infty} C(r)$, the "time average" of the semigroup, will itself not change with time: $T(t)P = P$ for all $t \geq 0$. This is indeed the case:

$$\begin{aligned} T(t)Px &= \lim_{r\to\infty} T(t)\frac{1}{r}\int_0^r T(s)xds = \lim_{r\to\infty}\frac{1}{r}\int_0^r T(s+t)xds = \\ &= \lim_{r\to\infty}\frac{1}{r}\int_t^{r+t} T(s)xds = \lim_{r\to\infty}\frac{1}{r}\int_0^{r+t} T(s)xds - \frac{1}{r}\int_0^t T(s)xds. \end{aligned}$$

The second term tends to zero whereas the first converges to

$$\lim_{r\to\infty}\frac{r+t}{r}\frac{1}{r+t}\int_0^{r+t} T(s)xds = \lim_{r\to\infty} C(r+t)x = Px.$$

Therefore, $T(t)Px = Px$ for all $x \in X$, as expected.

The result just derived implies that P is a projection: We have $C(r)Px = Px$ for all $r > 0, x \in X$, so by letting $r \to \infty$ we see that $P^2 = P$. The operator P is therefore called the **mean ergodic projection** associated with $(T(t))_{t\geq 0}$. In the following lemma we identify the range and kernel of P:

Lemma 3.2. *Let P be the mean ergodic projection associated with the mean ergodic semigroup $(T(t))_{t\geq 0}$. Then P is a bounded projection with*

$$\mathrm{ran} P = \mathrm{fix}(T(t))_{t\geq 0}, \quad \ker P = \overline{\left[\{x - T(t)x \,|\, x \in X, t \geq 0\}\right]}.$$

In particular, we have the following decomposition:

$$X = \mathrm{fix}(T(t))_{t\geq 0} \oplus \overline{\big[\{x - T(t)x \,\big|\, x \in X, t \geq 0\}\big]}. \tag{7}$$

Proof. We have already seen that P is a projection. Moreover, P is bounded as the τ_s-limit of bounded operators on a Banach space (this follows from the Banach-Steinhaus Theorem).

The equality $\mathrm{ran}P = \mathrm{fix}(T(t))_{t\geq 0}$ holds because for all $t \geq 0$ we have $PT(t) = P$, hence $\mathrm{ran}P \supset \mathrm{fix}(T(t))_{t\geq 0}$, and $T(t)P = P$, so $\mathrm{ran}P \subset \mathrm{fix}(T(t))_{t\geq 0}$.

We prove that $\ker P = \overline{M}$, where $M := \big[\{x - T(t)x \,\big|\, x \in X, t \geq 0\}\big]$. The relation $\ker P \supset M$ is clear, and because $\ker P$ is closed $\ker P \supset \overline{M}$. To see that $\ker P \subset \overline{M}$ assume there exists $z \in \ker P \backslash \overline{M}$. Then by the Hahn-Banach theorem we find $f \in X'$ such that

$$\{0\} = \mathrm{Re} f(\overline{M}) < \mathrm{Re} f(z).$$

Because $\mathrm{Im} f(y) = -\mathrm{Re}(if(y)) = -\mathrm{Re} f(y) = 0$ for all $y \in \overline{M}$, we have $f(\overline{M}) = \{0\}$, so

$$f = f \circ T(t) \text{ for all } t \geq 0.$$

This implies $f = f \circ P$. Therefore $f(z) = f(Pz) = 0$, which contradicts our choice of f. □

The spaces $\mathrm{fix}(T(t))_{t\geq 0}$ and $\overline{\big[\{x - T(t)x \,\big|\, x \in X, t \geq 0\}\big]}$ in Equation (7) can be formulated in terms of the generator A:

Lemma 3.3. *For a C_0-semigroup $(T(t))_{t\geq 0}$ with generator A it holds that:*

(i) $\mathrm{fix}(T(t))_{t\geq 0} = \ker A$

(ii) $\overline{\big[\{x - T(t)x \,\big|\, x \in X, t \geq 0\}\big]} = \overline{\mathrm{ran}A}$.

Proof.

(i) This is clear from the definition of A and the fact that $T(t)x - x = \int_0^t T(s)Axds$.

(ii) The inclusion "$\subset$" follows from $T(t)x - x = A\int_0^t T(s)xds \in \mathrm{ran}A$, the other follows from $Ax = \lim_{t\searrow 0} \frac{T(t)x - x}{t} \in \overline{\big[\{x - T(t)x \,\big|\, x \in X, t \geq 0\}\big]}$.

□

3.1 Criteria for mean ergodicity

We just saw that for a mean ergodic semigroup $(T(t))_{t\geq 0}$ the underlying Banach space can be decomposed as in (7). The converse is true as well if the semigroup satisfies a certain growth condition:

Proposition 3.4. *A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X satisfying*

$$\lim_{t\to\infty} \frac{\|T(t)\|}{t} = 0 \tag{8}$$

is mean ergodic if and only if

$$X = \operatorname{fix}(T(t))_{t\geq 0} + \overline{\big[\{x - T(t)x \,\big|\, x \in X, t \geq 0\}\big]}.$$

Proof. Because of Lemma 3.2 we only need to show that $(T(t))_{t\geq 0}$ is mean ergodic if the decomposition formula for X holds. If this is the case the set

$$G := \operatorname{fix}(T(t))_{t\geq 0} + M, \tag{9}$$

where $M := \big[\{x - T(t)x \,\big|\, x \in X, t \geq 0\}\big]$, is dense in X. We show that for all $z \in G$ the limit $\lim_{r\to\infty} C(r)z$ exists: Let

$$z = u + \sum_{i=1}^{n} \big(x_i - T(t_i)x_i\big) \in G$$

with $u \in \operatorname{fix}(T(t))_{t\geq 0}$, $x_i \in X$, $t_i \geq 0$. Since $C(r)u = u$ for all $r > 0$ we only need to show that $\lim_{r\to\infty} C(r)(x - T(t)x)$ exists for $x \in X, t \geq 0$:

$$\begin{aligned} C(r)(x - T(t)x) &= \frac{1}{r}\int_0^r \big(T(s)x - T(t+s)x\big)\, ds = \\ &= \frac{1}{r}\int_0^r T(s)xds - \frac{1}{r}\Big(\int_0^{r+t} T(s)xds - \int_0^t T(s)xds\Big) = \\ &= -\frac{1}{r}\int_r^{r+t} T(s)xds + \frac{1}{r}\int_0^t T(s)xds. \end{aligned}$$

Clearly, the second term goes to 0 as $r \to \infty$, and so does the first:

$$\begin{aligned} \Big\|\frac{1}{r}\int_r^{r+t} T(s)xds\Big\| &\leq \frac{1}{r}\int_r^{r+t} \|T(s)x\| ds \leq \\ &\leq \|x\|\Big(\sup_{r\leq s\leq r+t} \frac{\|T(s)\|}{s}\Big)\frac{1}{r}\int_r^{r+t} sds. \end{aligned}$$

The expression in brackets goes to 0 by assumption, while the remaining terms are bounded. Therefore $\lim_{r\to\infty} C(r)z = u$.

To show mean ergodicity of $(T(t))_{t\geq 0}$ we need the existence of $\lim_{r\to\infty} C(r)x$ for *all* $x \in X$. This is a simple consequence of the Banach-Steinhaus theorem. The family $(C(r))_{r>0}$ of bounded operators is pointwise bounded on the dense set $G \subset X$, hence uniformly bounded on X: $\mu := \sup_{r>0} \|C(r)\| < \infty$. Hence, for $x \in X$ and any sequence $r_n \to \infty$ the difference

$$\begin{aligned}\|C(r_n)x - C(r_m)x\| &\leq \|(C(r_n) - C(r_m))(x - z)\| + \|C(r_n)z - C(r_m)z\| \leq \\ &\leq 2\mu\|x - z\| + \|C(r_n)z - C(r_m)z\|\end{aligned}$$

becomes arbitrarily small if we choose $z \in G$ sufficiently close to x and n, m sufficiently large. Because X is complete, this implies the existence of $\lim_{r\to\infty} C(r)x$. □

Note that in the proposition above the condition $\lim_{t\to\infty} \frac{\|T(t)\|}{t} = 0$ cannot be omitted. A counterexample is provided by the simplest non-trivial semigroup there is:

Example 3.5. Consider the semigroup $(T(t))_{t\geq 0}$ on $\mathbb{C}$ defined by $T(t)x = e^t x$. Clearly, this semigroup is not mean ergodic:

$$C(r)x = \frac{1}{r}\int_0^r e^s x ds = \frac{e^r - 1}{r} x \to \infty$$

for all $x \in \mathbb{C}\backslash\{0\}$. However, the decomposition (7) holds because $\{x - T(t)x \,|\, x \in \mathbb{C}, t \geq 0\} = \mathbb{C}$.

For C_0-semigroups which satisfy the "growth condition", Equation (8), the previous Proposition allows us to deduce a convenient method for testing mean ergodicity:

Proposition 3.6. *A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X satisfying $\lim_{t\to\infty} \frac{\|T(t)\|}{t} = 0$ is mean ergodic if and only if the fixed space*

$$\operatorname{fix}(T(t))_{t\geq 0} = \ker A$$

separates the dual fixed space

$$\operatorname{fix}(T(t)')_{t\geq 0} = \ker A'.$$

Proof. The fixed space $\operatorname{fix}(T(t))_{t\geq 0}$ separates $\operatorname{fix}(T(t)')_{t\geq 0}$ iff the following implication holds:

$$\Big[f \in \operatorname{fix}(T(t)')_{t\geq 0} \text{ and } f|_{\operatorname{fix}(T(t))_{t\geq 0}} = 0\Big] \Rightarrow f = 0.$$

Let G be the subspace of X defined in Equation (9). Then the condition above is equivalent to

$$\Big[f \in X' \text{ and } f|_G = 0\Big] \Rightarrow f = 0.$$

By the Hahn-Banach theorem this is in turn equivalent to $X \backslash \overline{G} = \emptyset$, i.e.

$$X = \overline{G} = \overline{\operatorname{fix}(T(t))_{t\geq 0} + M} = \operatorname{fix}(T(t))_{t\geq 0} + \overline{M}.$$

Proposition 3.4 now yields the desired result. $\square$

Remark. The relation $\operatorname{fix}(T(t)')_{t\geq 0} = \ker A'$ in the proposition above does *not* follow from Lemma 3.3 (i) applied to the semigroup $(T(t)')_{t\geq 0}$. This is because $(T(t)')_{t\geq 0}$ (the so-called adjoint semigroup) need not be strongly continuous [Eng00]. Rather, the relation is a consequence of Lemma 3.3 (ii), because

$$\begin{aligned}
\operatorname{fix}(T(t)')_{t\geq 0} &= \{x' \in X' \,\big|\, x'(T(t)y) = x'(y)\, \forall y \in X, t \geq 0\} = \\
&= \{x' \in X' \,\big|\, \operatorname{fix}(T(t))_{t\geq 0} \subset \ker x'\}, \\
\ker A' &= \{x' \in X' \,\big|\, x'(Ay) = 0\, \forall y \in X, t \geq 0\} = \\
&= \{x' \in X' \,\big|\, \operatorname{ran} A \subset \ker x'\}.
\end{aligned}$$

3.2 Examples Revisited I

We apply the criterion above to analyze the translation and multiplication semigroups introduced in Section 2.4 for their mean ergodicity:

Example 3.7 (Translation semigroup). Let $(T(t))_{t\geq 0}$ be the translation semigroup on X defined in Example 2.13, $T(t)f = f(\cdot + t)$.

(a) $X = L^p(\mathbb{R}), 1 < p < \infty$:
Clearly, $\operatorname{fix}(T(t))_{t\geq 0} = \{0\}$. Identifying $L^p(\mathbb{R})'$ and $L^q(\mathbb{R})$, where $1/p + 1/q = 1$, we have

$$\begin{aligned}
\operatorname{fix}(T(t)')_{t\geq 0} &= \{g \in L^q(\mathbb{R}) \,\big|\, \int gf = \int gf(\cdot + t)\, \forall f \in L^p(\mathbb{R}), t \geq 0\} = \\
&= \{g \in L^q(\mathbb{R}) \,\big|\, g = g(\cdot - t) \text{ for all } t \geq 0\} = \{0\}.
\end{aligned}$$

By Proposition 3.6, $(T(t))_{t\geq 0}$ is mean ergodic. We will see later on (Corollary 3.11) that this holds for any bounded semigroup on a reflexive Banach space.

(b) $X = L^1(\mathbb{R})$:
Here we are dealing with a non-reflexive space, so the result just mentioned cannot be applied. Indeed, the semigroup $(T(t))_{t\geq 0}$ turns out not to be mean ergodic:

$$\begin{aligned}
\operatorname{fix}(T(t)')_{t\geq 0} &= \{g \in L^\infty(\mathbb{R}) \,\big|\, \int gf = \int gf(\cdot + t)\, \forall f \in L^p(\mathbb{R}), t \geq 0\} = \\
&= \{g \in L^\infty(\mathbb{R}) \,\big|\, g = g(\cdot - t) \text{ for all } t \geq 0\} = \\
&= \{g \in L^\infty(\mathbb{R}) \,\big|\, g \text{ is constant a.e.}\} = [\{1\}].
\end{aligned}$$

Since $\operatorname{fix}(T(t))_{t\geq 0} = \{0\}$ as above, $(T(t))_{t\geq 0}$ is not mean ergodic.

(c) $X = C_{ub}(\mathbb{R})$:
In this case the dual space X' cannot be described in a simple way (reference???????) and Proposition 3.6 is therefore inconvenient to use. However, we directly see that $(T(t))_{t\geq 0}$ is not mean ergodic on $C_{ub}(\mathbb{R})$ by constructing a function $f \in C_{ub}(\mathbb{R})$ for which

$$\lim_{r\to\infty} C(r)f = \lim_{r\to\infty} \frac{1}{r}\int_0^r f(\cdot + s)ds$$

does not exist: Let f be a function that is $+1$ on $[1, 10^1 - 1]$, -1 on $[10^1, 10^2-1]$, $+1$ on $[10^2, 10^3-1]$ etc. and linear on the intervals between. Then for $r_n = 10^n$ we have

$$\begin{aligned}\frac{1}{r_n}\int_1^{r_n} f(s)ds &= \frac{\sum_{i=0}^{n-1}(-1)^i(10^{i+1} - 1 - 10^i)}{10^n} = \\ &= 9\sum_{i=0}^{n-1}(-1)^i 10^{i-n} - \sum_{i=0}^{n-1}(-1)^{i-n} = \\ &= 9(-1)^n \sum_{j=1}^{n}(-1)^j 10^{-j} - \sum_{i=0}^{n-1}(-1)^{i-n}.\end{aligned}$$

The series $\sum_{j=1}^{n}(-1)^j 10^{-j}$ converges to a non-zero value as $n \to \infty$. Therefore the expression above (and hence $C(r_n)f$) is divergent.

Example 3.8 (Multiplication semigroup). Now we have a look at the multiplication semigroup $(T(t))_{t\geq 0}$ defined in Example 2.14, $T(t)f = e^{tq}f$.:

(a) $X = L^p(\Omega,\mu), 1 \leq p < \infty$:
The fixed space is

$$\begin{aligned}\operatorname{fix}(T(t))_{t\geq 0} &= \{f \in L^p(\Omega,\mu) \,|\, f = e^{qt}f \text{ a.e. } \forall t \geq 0\} = \\ &= \{f \in L^p(\Omega,\mu) \,|\, f = \mathbf{1}_{[q=0]}f \text{ a.e. }\} \simeq \\ &\simeq L^p([q=0], \mu|_{[q=0]})\end{aligned}$$

and the dual fixed space is

$$\begin{aligned}\operatorname{fix}(T(t)')_{t\geq 0} &= \{g \in L^q(\Omega,\mu) \,|\, \int gf = \int ge^{qt}f \text{ for all } f \in L^p(\Omega,\mu), t \geq 0\} = \\ &= \{g \in L^q(\Omega,\mu) \,|\, g = \mathbf{1}_{[q=0]}g \text{ a.e. }\} \simeq \\ &\simeq L^q([q=0], \mu|_{[q=0]}).\end{aligned}$$

Therefore, $\operatorname{fix}(T(t))_{t\geq 0}$ separates $\operatorname{fix}(T(t)')_{t\geq 0}$ and the semigroup is mean ergodic by Proposition 3.6.

(b) $X = C_0(\Omega)$:
The fixed space is

$$\operatorname{fix}(T(t))_{t\geq 0} = \{f \in C_0(\Omega) \,\big|\, f|_{[q\neq 0]} = 0\}.$$

Identifying $C_0(\Omega)'$ with the space of regular complex Borel measures on Ω, we can write the dual fixed space as

$$\begin{aligned}\operatorname{fix}(T(t)')_{t\geq 0} &= \{\mu \in C_0(\Omega)' \,\big|\, \int_\Omega f d\mu = \int_\Omega e^{tq} f d\mu \text{ for all } f \in C_0(\Omega), t \geq 0\} = \\ &= \{\mu \in C_0(\Omega)' \,\big|\, \mu([q \neq 0]) = 0\}.\end{aligned} \tag{10}$$

To see why the last equation holds, let $\nu_t \in C_0(\Omega)'$ be the Borel measure $\nu_t : A \mapsto \int_A (1 - e^{tq}) d\mu$. The first line in (10) implies that $\nu_t(f) = 0$ for all $f \in C_0(\Omega), t \geq 0$, hence $\nu_t = 0 \,\forall t \geq 0$. In particular,

$$|\nu_t|(\Omega) = \int_\Omega (1 - e^{tq}) d|\mu| = 0.$$

If $\mu([q \neq 0]) \neq 0$ then from the regularity of μ it follows that there exists a compact set $K \subset [q \neq 0]$ such that $\mu(K) \neq 0$; hence

$$\int_K (1 - e^{tq}) d|\mu| \geq \min_K (1 - e^{tq}) \, |\mu|(K) > 0,$$

a contradiction. We conclude that $\mu([q \neq 0]) = 0$. The converse implication is obvious.

We now turn to the question of whether $\operatorname{fix}(T(t))_{t\geq 0}$ separates $\operatorname{fix}(T(t)')_{t\geq 0}$. This means that for $\mu \in C_0(\Omega)'$ we have the implication

$$\Big[\mu([q \neq 0]) = 0 \text{ and } \big(\mu(f) = 0 \,\forall f \in C_0(\Omega) \text{ with } f|_{[q\neq 0]} = 0\big)\Big] \Rightarrow \mu = 0.$$

We show that this is equivalent to $[q = 0]$ being an open set: If $[q = 0]$ is open any function $g \in C_0(\Omega)$ can be written as the sum of C_0-functions $g = g\mathbf{1}_{[q\neq 0]} + g\mathbf{1}_{[q=0]}$ and a measure $\mu \in C_0(\Omega)'$ with the property on the left therefore satisfies $\mu(g) = 0 \,\forall g \in C_0(\Omega)$, hence $\mu = 0$. Conversely, assume the implication above is true. Then for $s \in [q = 0]$ the measure $\mu := \delta_s$ satisfies $\mu \neq 0, \mu([q \neq 0]) = 0$, therefore there exists $f \in C_0(\Omega)$ with $f|_{[q\neq 0]} = 0$ such that $\mu(f) = f(s) \neq 0$. The set $U := [f \neq] \subset [q = 0]$ therefore defines an open subset of $[q = 0]$ containing s. Since s was arbitrary we conclude that $[q = 0]$ is open.

In summary, we see that the multiplication group on $C_0(\Omega)$ is mean ergodic if and only if $[q = 0]$ is open. Because q is continuous by assumption, the multiplication group for $\Omega = \mathbb{R}$ is only mean ergodic if $[q = 0]$ is $\emptyset$ or $\mathbb{R}$, i.e. if $q < 0$ or $q = 0$.

3.3 Mean ergodicity of relatively weakly compact semigroups

The original definition of mean ergodicity requires convergence of $C(r_n)$ in the *strong* operator topology for *all* sequences $r_n \to \infty$. Actually, a much weaker condition is sufficient if the already familiar condition regarding the growth of the semigroup is satisfied:

Proposition 3.9. *A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X satisfying $\lim_{t\to\infty} \frac{\|T(t)\|}{t} = 0$ is mean ergodic if and only if for all $x \in X$ there exists a sequence $r_n \to \infty$ such that $(C(r_n)x)_{n\in N}$ converges in the weak topology of X.*

Proof. We only need to show the “if” part. By Proposition 3.6, we can do so by showing that for $f \in \text{fix}(T(t)')_{t\geq 0}, f \neq 0$, there exists $z \in \text{fix}(T(t))_{t\geq 0}$ such that $f(z) \neq 0$. Let $f \in \text{fix}(T(t)')_{t\geq 0}$ and let $x \in X, f(x) \neq 0$. The idea is to substitute x by a certain “time average” z of x, which we hope will be “independent of time“, $z \in \text{fix}(T(t))_{t\geq 0}$, while at the same time the value $f(x)$ does not change (the latter is plausible because $f \in \text{fix}(T(t)')_{t\geq 0}$).

Let $z := w\text{-}\lim_{n\to\infty} C(r_n)x$, which, by assumption, exists for some sequence $r_n \to \infty$.

Step 1: We show that $z \in \text{fix}(T(t))_{t\geq 0}$. In the proof of Proposition 3.4 we saw that $\lim_{r\to\infty} C(r)(T(t)x - x) = 0$ for all $t \geq 0, x \in X$. In particular,

$$w\text{-}\lim_{n\to\infty} C(r_n)T(t)x = w\text{-}\lim_{n\to\infty} C(r_n)x.$$

The left-hand side is equal to $T(t)z$ (because the norm continuous operator $T(t)$ is also weakly continuous[7]), while the right-hand side equals z. Therefore, $T(t)z = z$ for all $t \geq 0$.

Step 2: We show that $f(z) = f(x)$. Since $f \in \text{fix}(T(t)')_{t\geq 0}$ and f is linear and continuous, $f(y) = f(x)$ for all $y \in K := \overline{\text{co}} \{T(t)x \,|\, t \geq 0\}$. From the definition of the Bochner integral (5) it is clear that $C(r_n) \in K$ for all $n \in \mathbb{N}$. Therefore, z is in the weak closure of K, which coincides with K because K is convex. Hence, $z \in K$ and $f(z) = f(x)$. □

The last proposition implies mean ergodicity of an important class of C_0-semigroups, relatively weakly compact semigroups:

Definition 3.10. A C_0-semigroup $(T(t))_{t\geq 0}$ on X is called **relatively weakly compact** if for all $x \in X$ the set

$$\{T(t)x \,|\, t \geq 0\}$$

is relatively weakly compact in X.

[7] The proof is simple: If $B \in \mathcal{B}(X)$ then for $x' \in X'$ also $x' \circ B \in X'$. Therefore, if $y_n \overset{\tau_w}{\to} y$ we have $x'(By_n) = (x' \circ B)y_n \to (x' \circ B)y = x'(By)$ for all $x' \in X'$, i.e. $By_n \overset{\tau_w}{\to} By$.

Note that relatively weakly compact semigroups are bounded, since every weakly compact subset in a normed space is bounded (this can be seen by applying the Banach-Steinhaus theorem to the family of functionals $(f \mapsto f(x))_{x\in X}$ on X'). Therefore, the condition $\frac{\|T(t)\|}{t} \to 0$ is satisfied and we can apply the previous proposition to obtain the following:

Corollary 3.11. *All relatively weakly compact semigroups on X are mean ergodic. In particular, if X is reflexive, every bounded semigroup on X is mean ergodic.*

Proof. Let $(T(t))_{t\geq 0}$ be a relatively weakly compact semigroup and let $x \in X$. By the Krein-Smulian Theorem, the closed convex hull K of the weakly compact set $\{T(t)x \,|\, t \geq 0\}$ is weakly compact. By the Eberlein-Smulian Theorem, K is weakly sequentially compact. Therefore, the sequence $(C(n)x)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(C(r_n)x)_{n\in\mathbb{N}}$. Hence by Proposition 3.9 the semigroup is mean ergodic.

For a reflexive space every bounded subset is relatively weakly compact by Alagolu's Theorem. The second part of the corollary is therefore a direct consequence of the first. □

Remark. A bounded semigroup can have negative growth bound or growth bound zero. In the first case (the case of an exponentially stable semigroup) the semigroup is always mean ergodic no matter what the underlying space X, since $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$ implies that $\lim_{r\to\infty}\int_0^r T(t)dt$ exists even in the uniform operator topology, hence $\|C(r)\| \to 0$. Exponentially stable semigroups are therefore examples of so-called *uniformly mean ergodic* semigroups which we will study in more detail in Section 5.

For semigroups with growth bound zero Corollary 3.11 implies mean ergodicity if X is reflexive. If this is not the case the semigroup need not be mean ergodic — the translation semigroup on $L^1(\mathbb{R})$ or $C_{ub}(\mathbb{R})$ is such an example (Example 2.13 (ii),(iii)).

As an application of Corollary 3.11 we derive a result that is closely related to the famous *von Neumann mean ergodic theorem* [Gre08]. The latter states that for a probability space (Ω, Σ, μ) and an ergodic measure-preserving transformation ϕ on Ω, the "time averages" $\frac{1}{n}\sum_{j=0}^{n-1} f\circ\phi^j, n \in \mathbb{N}$ of any function $f \in L^2(\Omega)$ converge to the "space average" $\int_\Omega f d\mu$:

$$\frac{1}{n}\sum_{j=0}^{n-1} f\circ\phi^j \xrightarrow{L^2(\Omega)} \int_\Omega f d\mu \quad \text{as} \quad n\to\infty.$$

Example 3.12. We prove a continuous version of this result: Let (Ω, Σ, μ) be a probability space and let ϕ be a bijective measure-preserving transformation on Ω. Then for all $f \in L^2(\Omega)$ it holds that

$$\frac{1}{r}\int_0^r f\circ\phi^t dt \xrightarrow{L^2(\Omega)} \int_\Omega f d\mu \quad \text{as} \quad r\to\infty.$$

The powers ϕ^t are to be understood in terms of the functional calculus for the unitary operator $U : f \mapsto f\circ\phi$ on $L^2(\Omega)$: $f\circ\phi^t := U^t = \int_0^{2\pi} e^{its}dE(s)$ where E is the spectral measure satisfying $U = \int_0^{2\pi} e^{is}dE(s)$.

Let $T(t) = U^t$ for $t \geq 0$. The family $(T(t))_{t\geq 0}$ is a bounded C_0-semigroup on $L^2(\Omega)$. Let A be its generator. By Corollary 3.11

$$\lim_{r\to\infty} C(r)f = Pf \text{ for all } f \in L^2(\Omega),$$

where $C(r)$ are the Cesàro means of the semigroup $(T(t))_{t\geq 0}$ and P is the projection onto $\ker A$ with kernel $\overline{\operatorname{ran} A}$. Because the operator $T(t)$, $t \geq 0$, are unitary the generator A is skew-adjoint by Stone's Theorem. Therefore $\ker A = (\operatorname{ran} A^*)^\perp = (\operatorname{ran} A)^\perp$, i.e. P is an orthogonal projection. The range of P is

$$\operatorname{ran} P = \operatorname{fix}(T(t))_{t\geq 0} = \{f \in L^2(\Omega) \,|\, f\circ\phi = f \text{ a.e.}\}.$$

In the case of an ergodic transformation ϕ, the only ϕ-invariant functions are the constant functions on Ω.[8] Therefore, the projection of $f \in L^2(\Omega)$ onto $\operatorname{ran} P = [\{1\}]$ is the constant function $Pf = (\int_\Omega f\cdot 1)\cdot 1 = \int_\Omega f$. This proves the claim.

3.4 A mean ergodic theorem for semigroups of affine operators

Although our primary interest is in semigroups of *linear* operators we present here a simple result related to semigroups of *affine* operators. Such semigroups arise naturally in the study of inhomogeneous partial differential equations (or, more generally, inhomogeneous abstract Cauchy problems), see Section 4.2.1 and 4.1.2 for examples.

Formally, a C_0-semigroup of affine operators on a Banach space X is a family $(S(t))_{t\geq 0}$ of bounded affine operators on X satisfying the functional equations (FE) and the condition that

$$\mathbb{R}^+ \to X : t \mapsto S(t)x$$

[8]Let $f \in L^2(\Omega)$ be a ϕ-invariant function and let $x_0 \in \Omega$. Consider the set

$$M(x_0) := \{x \in \Omega \,|\, f(x) = f(x_0)\},$$

where we use the same symbol f to denote the equivalence class $f \in L^2(\Omega)$ and an arbitrary (but fixed) representative of f. Clearly, the set $M(x_0)$ is ϕ-invariant, $M(x_0) = \phi(M(x_0))$. In particular, $\mu(M(x_0)) = \mu(\phi(M(x_0)))$ and because ϕ is ergodic this implies $\mu(M(x_0)) \in \{0,1\}$. But since $\Omega = \bigcup_{x_0\in\Omega} M(x_0)$ there exists $y_0 \in \Omega$ such that $\mu(M(y_0)) = 1$, hence $f = f(y_0)$ a.e.

is continuous for all $x \in X$. Clearly, if $(S(t))_{t\geq 0}$ is a C_0-semigroup of affine operators then the operators

$$T(t)x := S(t)x - S(t)0,\ x \in X, t \geq 0 \tag{11}$$

form a C_0-semigroup of linear operators. The following theorem given in [Liu05] establishes a relationship between the convergence of the Cesàro means of $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$:

Theorem 3.13. *Let $(S(t))_{t\geq 0}$ be a semigroup of affine operators on X and let $(T(t))_{t\geq 0}$ be the associated semigroup of linear operators defined in* (11)*. If $(S(t))_{t\geq 0}$ has a common fixed point $x^* \in X$ and if $(T(t))_{t\geq 0}$ is mean ergodic, then the Cesàro means*

$$\tilde{C}(r)x := \frac{1}{r}\int_0^r S(t)x dt,\ r > 0, x \in X$$

converge for all $x \in X$. The limit is given by

$$\lim_{r\to\infty} \tilde{C}(r)x = Px + (I - P)x^*,$$

where P is the mean ergodic projection associated with $(T(t))_{t\geq 0}$.

Proof. The assertion follows from the simple fact that

$$\tilde{C}(r)x := C(r)(x - x^*) + \tilde{C}(r)x^*,$$

where $C(r) : y \mapsto \frac{1}{r}\int_0^r T(t)y dt$ are the Cesàro means associated with $(T(t))_{t\geq 0}$. The first term in the equation above converges to $P(x - x^*)$ whereas the second is equal to x^* for all $r > 0$. □

4 Applications to partial differential equations

In the previous sections the focus was on the mathematical properties of mean ergodic semigroups. We now apply the results derived to some of the "classical" partial differential equations occurring in physics: the heat, wave and Schrödinger equation.

For the entire section let Ω be a bounded domain in $\mathbb{R}^n$. The PDEs just mentioned involve the Laplacian Δ as a differential operator. When writing the equations in the form of ACPs we need to consider Δ as an operator on some appropriate Banach space. In Sections 4.1 and 4.2 the PDEs are for functions on $\mathbb{R} \times \Omega$; the Banach space for the ACP is $L^2(\Omega)$ and Δ is the operator

$$\Delta : D(\Delta) \to L^2(\Omega) : f \mapsto \Delta f, \quad D(\Delta) := H^2(\Omega) \cap H_0^1(\Omega).$$

The choice of $D(\Delta)$ as a subset of $H_0^1(\Omega)$ reflects the fact that we are interested in solutions which "vanish" at the boundary $\partial\Omega$.

In Section 4.3 we are interested in functions on $\mathbb{R}\times\mathbb{R}^n$; the Banach space for the ACP will be $L^2(\mathbb{R}^n)$ and

$$\Delta : D(\Delta) \to L^2(\mathbb{R}^n) : f \mapsto \Delta f, \quad D(\Delta) := H^2(\mathbb{R}^n).$$

4.1 The heat equation

Physically, the heat equation describes how a given temperature distribution u_0 in a volume $\Omega \subset \mathbb{R}^3$ evolves with time when left to itself: the change in energy inside a subset[9] $U \subset \Omega$ (which is up to a material-specific constant $\int_U u'$), equals the energy flowing into U (which is, again up to a constant, $\int_{\partial U} \nabla u = \int_U \Delta u$):

$$\int_U u' = \int_U \Delta u$$

If heat sources exists inside Ω such that for every point $x \in \Omega$ and time t a certain amount $f(x,t)$ of heat per time and volume is produced, the energy created by the sources has to be added to the energy flowing into U:

$$\int_U u' = \int_U \Delta u + \int_U f(x,\cdot)dx. \tag{12}$$

Since U was an (almost) arbitrary subset of Ω, Equation (12) implies

$$u'(t) = \Delta u(t) + f(\cdot,t) \text{ for all } t \geq 0, \tag{13}$$

which is the **inhomogeneous heat equation**. We make the (idealized) assumption that the temperature outside Ω is zero; therefore the boundary condition for continuous temperature distribution u, i.e. $u \in C(\overline{\Omega})$, is

$$u|_{\partial\Omega} = 0.$$

4.1.1 The homogeneous heat equation

We first study the homogeneous problem, Equation (13) with $f = 0$. More precisely, we look at the following ACP on $X := L^2(\Omega)$:

$$\begin{aligned} u'(t) &= \Delta u(t), \quad t \geq 0 \\ u(0) &= u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \end{aligned} \tag{14}$$

where, as mentioned at the beginning of this section, Δ is the Laplacian with domain $D(\Delta) := H^2(\Omega) \cap H_0^1(\Omega)$. Intuition tells us that the temperature of

[9] Of course, U must have all mathematical properties we need: it must be open and $\partial U \in C^1$.

a system whose surroundings are very cool will decay to a very low value as well. We analyze the asymptotic behaviour mathematically:

In a first step we show that the operator

$$A_\lambda := (\Delta + \lambda, D(\Delta))$$

generates a contraction semigroup on X for all $\lambda \in \mathbb{R}$ below a certain positive constant ω. To do so, we verify the conditions of the Lumer-Phillips Theorem (Theorem 2.10) for A_λ: Clearly, $D(A_\lambda)$ is dense in X. Moreover, if $\lambda \leq \frac{1}{C_p}$

$$\operatorname{Re}(A_\lambda u, u) = \int \overline{\Delta u}\, u + \lambda |u|^2 = -\|\nabla u\|^2 + \lambda \|u\|^2 \leq (-\frac{1}{C_p} + \lambda)\|u\|^2 \leq 0$$

for all $u \in D(A_\lambda)$. Here, $C_p > 0$ is the Poincaré constant:

$$(u, v) \leq C_p(\nabla u, \nabla v) \text{ for all } u, v \in H_0^1. \tag{15}$$

It remains to show that $A_\lambda - \mu$ is onto for some $\mu > 0$. In other words, setting $\tilde{\lambda} = \lambda - \mu$, we have to show that for all $f \in L^2(\Omega)$ the elliptic problem

$$\Delta u + \tilde{\lambda} u = f \tag{16}$$

has a solution $u \in D(A_\lambda) = H^2(\Omega) \cap H_0^1(\Omega)$. This is a well-known result of the theory of PDEs and we only sketch the argument:

Equation (16) is equivalent to

$$(\nabla u, \nabla v) - \tilde{\lambda}(u, v) = -(f, v) \text{ for all } v \in H_0^1.$$

We interpret the left hand side as a sesquilinear form $a(u, v)$. It follows from the Poincaré inequality (15) that for $\tilde{\lambda} < \frac{1}{C_p}$ this form is a scalar product on H_0^1 that is equivalent to the usual scalar product on H_0^1. Therefore, H_0^1 endowed with the scalar product $a(\cdot, \cdot)$ is a Hilbert space and using Riesz' representation theorem we conclude that the problem

$$a(u, v) = -(f, v) \text{ for all } v \in H_0^1$$

has a solution $u \in H_0^1(\Omega)$. The final step is to show that $u \in H^2(\Omega)$, which would establish that the elliptic problem (16) has a solution for all $\tilde{\lambda} < \frac{1}{C_p}$ and hence that $A - \mu$ is onto for all $\mu > 0$. We will not prove this (non-trivial) fact here, but refer to the literature on partial differential equations, e.g. [Eva98].

In conclusion, A_λ generates a contraction semigroup on $L^2(\Omega)$ for all $\lambda \leq \frac{1}{C_p} =: \omega$. In particular, since $\omega > 0$, the operator $A_0 = \Delta$ generates a contraction semigroup $(T(t))_{t \geq 0}$, the so-called **heat semigroup**. Since $(T(t)e^{\omega t})_{t \geq 0}$ is the semigroup generated by $\Delta + \omega$ we see that

$$\|T(t)\| \leq e^{-\omega t}, \tag{17}$$

i.e. $(T(t))_{t \geq 0}$ is exponentially stable. In particular, $(T(t))_{t \geq 0}$ is mean ergodic and the Cesàro means $C(r)$ tend to 0 in the norm topology on $\mathcal{B}(L^2(\Omega))$, see the remark after Corollary 3.11.

Finally, using Theorem 2.4, we reformulate our results in a more explicit way:

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^2$ and let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then the heat equation*

$$\begin{aligned} u' &= \Delta u \text{ on } \mathbb{R}^+ \\ u(0) &= u_0. \end{aligned} \tag{18}$$

considered as a Banach-space valued initial value problem on $L^2(\Omega)$ has a unique solution $u : \mathbb{R}_0^+ \to H^2(\Omega) \cap H_0^1(\Omega)$, which decays exponentially in $L^2(\Omega)$:

$$\|u(t)\|_{L^2(\Omega)} \leq e^{-\omega t}\|u_0\|_{L^2(\Omega)} \text{ for all } t \geq 0,$$

where ω is a positive constant.

Remark. To see that the heat semigroup is mean ergodic we would only have needed to show that the operator $A = \Delta$ (as opposed to all operators $A = \Delta + \lambda$, $\lambda \leq \omega$) generates a contraction semigroup and apply Corollary 3.11. This would have made the argumentation above simpler, but the general result (17) on the asymptotics of the heat semigroup is more interesting from a mathematical and physical point of view.

Finally, we remark that semigroup theory yields a number of other interesting facts about the solutions of the heat equation (or, more generally, equations of the form (14) where Δ is replaced by any strongly elliptic second-order differential operator with sufficiently smooth coefficients). In particular, if $\partial\Omega \in C^\infty$ then the solution $u : \mathbb{R}_0^+ \to H^2(\Omega)$ of (14) is infinitely often differentiable on $\mathbb{R}^+$ and $u(t) \in C^\infty(\Omega)$ for all $t > 0$. [?] However, the focus of our discussion is on asymptotic behaviour, so we will not elaborate on these aspects.

4.1.2 The inhomogeneous heat equation

At the beginning of this section we derived the inhomogeneous heat equation (13), where the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ takes into account the heat produced by sources in Ω. If this heat production is constant in time we intuitively expect that the temperature distribution u will converge to a time-independent function. This is indeed the case:

Theorem 4.2. *Let $u : \mathbb{R}_0^+ \to H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the inhomogeneous heat equation*

$$\begin{aligned} u' &= \Delta u + f \text{ on } \mathbb{R}^+ \\ u(0) &= u_0, \end{aligned}$$

considered as an abstract Cauchy problem in the Banach space $L^2(\Omega)$*, where* $\Omega \subset \mathbb{R}^n$ *is a bounded domain,* $\partial\Omega \in C^2$*,* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $f \in L^2(\Omega)$*. Then*

$$\|u(t) - w\| \leq e^{-\omega t}\|u_0\|_{L^2(\Omega)} \text{ for all } t \geq 0,$$

where w *is the solution of the stationary heat equation:*

$$0 = \Delta w + f.$$

Proof. First, note that w is well-defined because Δ generates an exponentially stable semigroup and therefore $0 \in \rho(\Delta)$ by the Hille-Yoshida Theorem. Let $v := u - w$. We have to show that $\lim_{t\to\infty} v = 0$. Computing

$$v' = u' = \Delta(u - w) + \Delta w + f = \Delta v$$

we see that v solves the homogeneous heat equation and the assertion therefore follows immediately from Theorem 4.1. □

4.2 The wave equation

Consider the second-order abstract Cauchy problem

$$\begin{aligned} u'' &= \Delta u \text{ on } \mathbb{R}^+ \\ u(0) &= u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ u'(0) &= v_0 \in L^2(\Omega) \end{aligned} \tag{19}$$

on the Hilbert space $L^2(\Omega)$. We can rewrite (19) as a *first-order* ACP on $X := H^1(\Omega) \times L^2(\Omega)$[10]:

$$\begin{aligned} U' &= AU \text{ on } \mathbb{R}^+ \\ U(0) &= \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(A), \end{aligned} \tag{20}$$

where A is the operator

$$A : D(A) \to X :$$
$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix}$$

with domain $D(A) := (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \subset X$.

[10] At first glance it would seem more natural to consider the space $L^2(\Omega) \times L^2(\Omega)$ instead of X. However, demanding more regularity in the first component will be useful for defining a convenient scalar product on X. Note that we do not lose any classical solutions of (19) by making this restriction.

If we choose an appropriate scalar product on X the space X becomes a Hilbert space and A becomes a skew-symmetric operator[11] on X. We set

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_X := (\nabla u_1, \nabla u_2) + (v_1, v_2),$$

where $(\cdot,\cdot)$ denotes the usual scalar product on $L^2(\Omega)$. With this definition

$$\begin{aligned}(AU, V)_X = \left(\begin{pmatrix} u_2 \\ \Delta u_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_X &= (\nabla u_2, \nabla v_1) + (\Delta u_1, v_2) = \\ &= (\nabla u_2, \nabla v_1) - (\nabla u_1, \nabla v_2) = -(U, AV)_X \text{ for all } U, V \in D(A).\end{aligned}$$

Therefore, A is skew-symmetric and hence iA is symmetric. This motivates us to show that iA is even self-adjoint: By Lemma 2.12 it suffices to show that $\operatorname{ran}(A - \lambda) = X$ for some $\lambda \in \mathbb{C}\backslash(i\mathbb{R})$. This is equivalent to saying that for all $(f_1, f_2)^T \in X$ the set of equations

$$u_2 - \lambda u_1 = f_1, \quad \Delta u_1 - \lambda u_2 = f_2$$

has a solution $(u_1, u_2)^T \in D(A)$. Replacing u_2 in the second equation by $f_1 + \lambda u_1$ we see that this is satisfied if

$$\Delta u_1 - \lambda^2 u_1 = f_2 + \lambda f_1 \tag{21}$$

has a solution $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. In the previous section, Section 4.1, we saw that $\rho(\Delta) \supset [0, \infty)$. Therefore, Equation (21) has a solution for all $\lambda \in \mathbb{R}$, which proves that iA is self-adjoint. By Stone's Theorem, A generates a C_0-semigroup $(T(t))_{t\geq 0}$ of unitary operators. A consequence is that the semigroup $(T(t))_{t\geq 0}$ is mean ergodic (Corollary 3.11). We will discuss this and other properties in more detail below for the general case of the *inhomogeneous* wave equation.

4.2.1 The inhomogeneous wave equation

In physical applications one often has to deal with the inhomogeneous wave equation

$$\begin{aligned} u'' &= \Delta u + f \text{ on } \mathbb{R}^+ \\ u(0) &= u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ u'(0) &= v_0 \in L^2(\Omega), \end{aligned} \tag{22}$$

where $f \in L^2(\Omega)$ is the source function (or driving force). A simple example is that of a vibrating string in the Earth's gravitational field. For small

[11] We call a densely defined operator A on a Hilbert space X skew-symmetric if $(x, Ay) = -(Ax, y)\, \forall x, y \in D(A)$, in other words if iA is symmetric.

displacements from the horizontal the motion of the string is approximately described by an inhomogeneous wave equation with the gravitational force as a source function. Intuitively, we expect that the mean displacement of the string over time will equal the displacement of a motionless string subject to the gravitational force. Using the theory of mean ergodic semigroups it will not be difficult to prove this mathematically (see Theorem 4.3 below).

As in the homogeneous case we can replace the second-order ACP (22) by a first-order problem on $X = H^1(\Omega) \times L^2(\Omega)$:

$$\begin{aligned} U' &= AU + F \text{ on } \mathbb{R}^+ \\ U(0) &= \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(A), \end{aligned} \tag{23}$$

where A is defined as in Section 4.1 and $F := (0, f)^T$.

Knowing that the solutions of the homogeneous problem can be described by a C_0-semigroup of unitary operators we easily obtain the following properties for the solutions of the general inhomogeneous equation:

Theorem 4.3. *The inhomogeneous wave equation* (22) *considered as an ACP on the Hilbert space $L^2(\Omega)$ has a unique solution $u : \mathbb{R}_0^+ \to L^2(\Omega)$. The solution has the following properties:*

(a) The "energy" function

$$E(t) := \int_\Omega (|u(t)|^2 + \|\nabla u(t)\|^2 + |u'(t)|^2),\ t \geq 0$$

is constant.

(b) The "mean value" of the solution satisfies

$$\lim_{r \to \infty} \frac{1}{r} \int_0^r u(t)dt = \phi \textit{ in } H^1(\Omega),$$

where ϕ is the solution of the stationary equation

$$0 = \Delta\phi + f.$$

Proof. We define the C_0-semigroup of affine operators $(\tilde{T}(t))_{t \geq 0}$ by

$$\tilde{T}(t)U_0 := T(t)(U_0 + A^{-1}F) - A^{-1}F.$$

Note that A^{-1} exists because $\text{ran}A = X$ as shown in Section 4.1 and $\ker A = \{0\}$ since $(v, \Delta u)^T = 0$ for $(u, v)^T \in D(A)$ implies $(u, v)^T = 0$. A simple calculation shows that the function $U : \mathbb{R}_0^+ \to X$ defined by

$$U(t) := \tilde{T}(t)U_0,$$

is a solution of (23). Because the homogeneous problem has a unique solution (Theorem 2.4) the solution of the inhomogeneous problem is unique as well.

We now show properties (a) and (b):

(a) Since $T(t)$ is unitary $\tilde{T}(t)$ is isometric. Therefore $E(t) = \|U(t)\|_X = \|\tilde{T}(t)U_0\|_X = \|U_0\|_X$ is constant.

(b) Clearly, $\Phi := -A^{-1}F$ is a stationary point for all $\tilde{T}(t), t \geq 0$. Since $\operatorname{fix}(T(t))_{t\geq 0} = \ker A = \{0\}$ the mean ergodic projection of $(T(t))_{t\geq 0}$ is 0. Therefore, by Theorem 3.13 the Cesàro means $\frac{1}{r}\int_0^r U(t)dt$ of $(\tilde{T}(t))_{t\geq 0}$ satisfy

$$\lim_{r\to\infty} \frac{1}{r}\int_0^r U(t)dt = \Phi$$

where the limit is in $X = H^1(\Omega)\times L^2(\Omega)$. In particular, $\frac{1}{r}\int_0^r u(t)dt \to \phi$ in $H^1(\Omega)$, where ϕ is the first component of Φ, i.e. $\Delta\phi = -f$.

□

4.3 The Schrödinger equation

In quantum mechanics, the state of a system at a given time t is described by a wave function $\psi(t) \in X := L^2(\mathbb{R}^n)$. The time evolution of ψ starting from some state ψ_0 is given by the Schrödinger equation

$$\begin{aligned} \psi' &= -iH\psi, \\ \psi(0) &= \psi_0 \end{aligned} \tag{24}$$

where the (densely defined) operator H is the Hamiltonian of the system.

For a single particle the typical form of H is

$$H = -\Delta + M_V, \quad D(H) = H^2(\mathbb{R}^n) \tag{25}$$

where M_V is the multiplication operator with the real function $V \in L^\infty(\mathbb{R}^n)$ (the "potential") and Δ is the Laplacian with domain $H^2(\mathbb{R}^n)$. Physically, H can be interpreted as the operator corresponding to the total energy of the particle, while $-\Delta$ corresponds to the kinetic and M_V to the potential energy.

Before discussing the mathematical properties of H we remark on what we should expect from a physical point of view: If $-iH$ generates a C_0-semigroup of operators $(U(t))_{t\geq 0}$, these operators represent the "time evolution" of the system, mapping the initial state ψ_0 to the state $\psi(t)$ at some later time $t > 0$. Time evolution should be bijective: To a state $\psi(t)$ there should correspond a unique state ψ_0 which is mapped to $\psi(t)$ under $U(t)$. Moreover, the physical interpretation of $|\psi(t)|^2$ is that of a probability density function for the position of the particle at time t: The probability that the particle is in the measurable set $U \subset \mathbb{R}^n$ is $\int_U |\psi(t)|^2$. Since the particle must be "somewhere" at all times, $\int_{\mathbb{R}^n} |\psi(t)|^2 = \|\psi(t)\| = 1$ for all $t \geq 0$. In conclusion, we expect the time evolution operators $U(t)$ to be unitary or, equivalently (Theorem 2.11), the Hamiltonian H to be self-adjoint. We verify this for the special form of H given in (25):

Theorem 4.4. *The Hamiltonian H for a single particle in a potential $V \in L^\infty(\mathbb{R}^n)$ (Equation (25)) is self-adjoint. In particular, for all $\psi_0 \in D(H) = H^2(\mathbb{R}^n)$ the corresponding Schrödinger equation (24) has a unique solution $\psi : \mathbb{R}_0^+ \to L^2(\mathbb{R}^n)$ and ψ satisfies $\|\psi(t)\| = \|\psi(0)\|$ for all $t > 0$.*

Proof. From the definition of distributional derivatives it follows that

$$(u, \Delta v)_{L^2(\mathbb{R}^n)} = (\Delta u, v)_{L^2(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n), v \in H^2(\mathbb{R}^n).$$

By density of $C_c^\infty(\mathbb{R}^n) \subset H^2(\mathbb{R}^n)$ (with respect to the usual H^2-norm), this holds for all $u, v \in H^2(\mathbb{R}^n)$. Therefore, Δ is symmetric. From the definition of the adjoint of an operator it follows that

$$\begin{aligned} D(\Delta^*) &= \{u \in L^2(\mathbb{R}^n) \,|\, D(\Delta) \to \mathbb{C} : v \mapsto (u, \Delta v)_{L^2(\mathbb{R}^n)} \text{ is bounded}\} = \\ &= \{u \in L^2(\mathbb{R}^n) \,|\, \Delta u \in L^2(\mathbb{R}^n)\} \end{aligned}$$

where in the last line we have used Riesz' Representation Theorem. Applying the Fourier transform $\mathcal{F}$, which is a unitary operator on $L^2(\mathbb{R}^n)$, to the set above, it is not difficult to show that $D(\Delta^*)$ equals $H^2(\mathbb{R}^n) \equiv D(\Delta)$[12]; hence $\Delta = \Delta^*$. Because the multiplication operator $(M_V, D(\Delta))$ with the real potential $V \in L^\infty(\mathbb{R}^n)$ is bounded and symmetric, the Hamiltonian $(-\Delta + M_V, D(\Delta))$ is self-adjoint as well. □

In particular, it follows from Corollary 3.11 that the semigroup generated by $-iH$ is mean ergodic. Therefore, the limit of the time averages of wave functions $\frac{1}{r}\int_0^r \psi(t)dt$ exists in $L^2(\mathbb{R}^n)$. If the potential V is zero (i.e., the

[12]For a multi-index $\alpha \in \mathbb{N}_0^n$ define the following function on $\mathbb{R}^n$:

$$m_\alpha : \mathbb{R}^n \to \mathbb{R} : x \mapsto x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

If $u \in L^2(\mathbb{R}^n)$ such that $\Delta u \in L^2(\mathbb{R}^n)$ then $\mathcal{F}\Delta u = -\sum_{i=1}^n m_{2e_i}\mathcal{F}u \in L^2(\mathbb{R}^n)$. Since

$$|m_\alpha(x)| \leq |\sum_{i=1}^n m_{2e_i}(x)| = \sum_{i=1}^n x_i^2 \quad \text{for all } \alpha \in \mathbb{N}_0^n, |\alpha| \leq 2, x \in ([-1,1]^n)^c$$

this implies $m_\alpha \mathcal{F}u \in L^2(\mathbb{R}^n)$ for all $\forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq 2$. Therefore,

$$\begin{aligned} (u, D^\alpha v)_{L^2(\mathbb{R}^n)} &= (\mathcal{F}u, i^{|\alpha|} m_\alpha \mathcal{F}v)_{L^2(\mathbb{R}^n)} = (\overline{i^{|\alpha|}} m_\alpha \mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} = \\ &= (\mathcal{F}^{-1}\overline{i^{|\alpha|}} m_\alpha \mathcal{F}u, v)_{L^2(\mathbb{R}^n)} \text{ for all } v \in C_c(\mathbb{R}^n), |\alpha| \leq 2. \end{aligned}$$

This shows that $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq 2$, hence $u \in H^2(\mathbb{R}^n)$.

particle is "free"), then $\ker H = \ker \Delta = \{0\}$, which implies

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r \psi(t)dt = 0 \text{ in } L^2(\mathbb{R}^n). \tag{26}$$

This corresponds to the intuitive idea that the particle travels freely through space and has no preferred location: The "common part" of wave functions at different times t_1, t_2, i.e. the overlap $\int_{R^n} \psi(t_1)^*\psi(t_2)$, becomes arbitrarily small. The connection with the mean ergodic property (26) is shown in the following proposition:

Proposition 4.5. *Let ψ be the solution of the Schrödinger equation* (24) *for a free particle, i.e. with Hamiltonian $H = -\Delta$. Then there exists a sequence of non-negative numbers $t_n \to \infty$ such that the net of overlaps*

$$\int_{R^n} \psi(t_n)^*\psi(t_m), \quad n, m \in \mathbb{N}$$

has 0 *as an accumulation point.*

Proof. If the assertion were wrong there would exist $T > 0$ and $\epsilon > 0$ such that

$$(\psi(t_1), \psi(t_2))_{L^2(\mathbb{R}^n)} > \epsilon \quad \forall t_1, t_2 > T.$$

Let $r > T$. The integral of the continuous function $\psi : \mathbb{R}_0^+ \to L^2(\mathbb{R}^n)$ is

$$\int_T^r \psi(t)dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} \psi\Big(T + \frac{r-T}{n} i\Big) \frac{r-T}{n}.$$

Because the sequence in the last equation converges in $L^2(\mathbb{R}^n)$, the squared norm is

$$\Big\| \int_T^r \psi(t)dt \Big\|^2 = (r-T)^2 \lim_{n\to\infty} \frac{1}{n^2} \Big\| \sum_{i=0}^{n-1} \psi\Big(T + \frac{r-T}{n} i\Big) \Big\|^2 =$$

$$= (r-T)^2 \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Big(\psi\big(T + \frac{r-T}{n} i\big), \psi\big(T + \frac{r-T}{n} j\big)\Big)_{L^2(\mathbb{R}^n)} >$$

$$> (r-T)^2 \epsilon.$$

But $\lim_{r\to\infty} \frac{1}{r^2} \big\| \int_0^r \psi(t)dt \big\|^2 = 0$ by Equation (26), a contradiction. □

5 Uniformly mean ergodic semigroups

In this section we discuss a different form of mean ergodicity, which requires convergence of the Cesàro means $C(r) = \frac{1}{r}\int_0^r T(t)dt$ not in the strong but in the uniform operator topology on X:

Definition 5.1. A C_0-semigroup $T(t)_{t\geq 0}$ on a Banach space X is called **uniformly mean ergodic** if the limit $\lim_{r\to\infty} C(r)$ exists in the operator norm.

In this section we only consider *bounded* C_0-semigroups. In the remark after Corollary 3.11 we have already mentioned that a bounded semigroup can either be exponentially stable, i.e. has negative growth bound, or has growth bound zero. In the first case we saw that the semigroup is always uniformly mean ergodic, in the second case it can be uniformly mean ergodic, mean ergodic but not uniformly mean ergodic or not mean ergodic at all (see Example 5.4).

5.1 Characterization of uniformly mean ergodic semigroups

The main result of this section, Theorem 5.3, is a characterization of bounded uniformly mean ergodic semigroups. Among others, we will see that a bounded semigroup is uniformly mean ergodic if and only if

$$\lim_{\lambda\searrow 0} \lambda R(\lambda, A)$$

exists in the operator norm. Note that the expression above makes sense since $(0,\infty)\subset\rho(A)$ for any bounded semigroup by the Hille-Yoshida Theorem. The result implies that semigroups whose generator is invertible, i.e. $0\in\rho(A)$ (which is the case for all exponentially stable semigroups), are uniformly mean ergodic.

Another criterion for uniform mean ergodicity, which will also be proved in Theorem 5.3, is that $\mathrm{ran} A$ is closed. Since a uniformly mean ergodic semigroup is mean ergodic and therefore $X = \ker A + \overline{\mathrm{ran} A}$ by Theorem 3.4, we see that for uniformly mean ergodic semigroups we have the decomposition

$$X = \ker A + \mathrm{ran} A. \tag{27}$$

Conversely, if this relation holds, then $X = \ker A + \overline{\mathrm{ran} A}$; hence, the semigroup is mean ergodic. Because in this case the sum $\ker A + \overline{\mathrm{ran} A}$ is direct, we conclude from

$$\ker A \oplus \mathrm{ran} A = \ker A \oplus \overline{\mathrm{ran} A}$$

that $\mathrm{ran} A = \overline{\mathrm{ran} A}$, so the semigroup is uniformly mean ergodic. In sum, Theorem 5.3 implies that a bounded C_0-semigroup is uniformly mean ergodic if and only if (27) holds.

Lemma 5.2 and Theorem 5.3 are both collections of properties that are equivalent to uniform mean ergodicity. For the proof it is convenient to bring them separately:

Lemma 5.2. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup with generator A and Cesàro means $C(r), r > 0$. For an operator B on X we denote by $B_|$ the part of B in $\overline{\mathrm{ran}A}$, i.e. the operator*

$$B_| : D(B_|) \to \overline{\mathrm{ran}A} : x \mapsto Bx,$$
$$D(B_|) := \{x \in D(B) \cap \overline{\mathrm{ran}A} \,\big|\, Bx \in \overline{\mathrm{ran}A}\}.$$

The following statements are equivalent:

(a) $\|C(r)_|\| \to 0$.

(b) $A_|$ *is invertible, i.e.* $0 \in \rho(A_|)$.

(c) $\mathrm{ran}A$ *is closed in* X,

Proof. (c) $\Rightarrow$ (a). If $y = Ax, x \in X$, then

$$rC(r)y = \int_0^r T(s)Axds = T(r)x - x.$$

Because $(T(t))_{t\geq 0}$ is bounded this implies

$$\sup_{r>0} \|rC(r)y\| < \infty \text{ for all } y \in \mathrm{ran}A.$$

Since $\mathrm{ran}A$ is closed by assumption it follows from the Banach-Steinhaus Theorem that the operators $(rC(r)_|)_{r>0}$ are uniformly bounded, hence

$$\lim_{r\to\infty} \|C(r)_|\| = 0.$$

(a) $\Rightarrow$ (b). The range of $A_|$ equals $\mathrm{ran}A$; therefore $\sigma(A_|) = \sigma_p(A_|) \cup \sigma_c(A_|)$. If $0 \in \sigma_c(A_|)$ the inverse $(A_|^{-1}, \mathrm{ran}A)$ is a well-defined operator on $\overline{\mathrm{ran}A}$, but it cannot be continuous, because from the closedness of A it would follow that $\mathrm{ran}A_|$ is closed[13]. Therefore, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A_|)$ such that $Ax_n \to 0$ but $\|x_n\| = 1, \forall n \in \mathbb{N}$. The same is true in the case $0 \in \sigma_p(A_|)$. So if $0 \notin \rho(A_|)$ we conclude that

$$\begin{aligned}\|C(r)x_n - x_n\| &= \Big\|\frac{1}{r}\int_0^r \big(x_n + \int_0^s T(t)Ax_n dt\big)ds - x_n\Big\| \leq \\ &\leq \frac{1}{r}\int_0^r \int_0^s \|T(t)Ax_n\|dtds \leq \|Ax_n\| \sup_{t\geq 0}\|T(t)\|\frac{r}{2},\end{aligned}$$

hence $\lim_{n\to\infty} C(r)x_n - x_n = 0$. Therefore $\|C(r)_|\| \geq 1 \forall r > 0$, which contradicts property (a).
(b) $\Rightarrow$ (c). If $A_|$ is invertible then $\mathrm{ran}A_| = \mathrm{ran}A = \overline{\mathrm{ran}A}$. □

[13] Let $(Ax_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{ran}A$ with limit $y \in X$. If $(A_|^{-1}, \mathrm{ran}A)$ is continuous then $x_n = A^{-1}(Ax_n) \to A^{-1}(y) \in D(A)$. But if $(x_n)_{n\in\mathbb{N}}$ converges to an element in $D(A)$ and $(Ax_n)_{n\in\mathbb{N}}$ converges to $y \in X$, the closedness of A implies $y \equiv \lim_{n\to\infty} Ax_n = A(\lim_{n\to\infty} x_n) \in \mathrm{ran}A$, hence $\mathrm{ran}A = \overline{\mathrm{ran}A}$.

Theorem 5.3. *For a bounded C_0-semigroup $(T(t))_{t\geq 0}$ with generator A the following properties are equivalent:*

(a) $(T(t))_{t\geq 0}$ *is uniformly mean ergodic.*

(b) $\lim_{\lambda\searrow 0} \lambda R(\lambda, A)$ *exists in the operator norm.*

(c) $\mathrm{ran}A$ *is closed in* X.

Proof. (a) $\Rightarrow$ (b). Let P be the mean ergodic projection associated with $(T(t))_{t\geq 0}$. For $\lambda \in \rho(A), x \in X, y \in D(A)$ we have $R(\lambda, A)x = y$ if and only if

$$x = (A - \lambda)y = (A - \lambda)(I - P)y + \lambda P y,$$

i.e. if and only if $(I - P)y = R(\lambda, A)(I - P)x$ and $Py = \frac{1}{\lambda}Px$. Therefore, we can write the resolvent $R(\lambda, A)$ of A as

$$R(\lambda, A) = R(\lambda, A_|)(I - P) + \frac{1}{\lambda}P,$$

where, as before, $A_|$ is the part of A in $\overline{\mathrm{ran}A} = \ker P$. Since $(T(t))_{t\geq 0}$ is uniformly mean ergodic Lemma 5.2 implies that $0 \in \rho(A_|)$ and therefore $\lambda \mapsto R(\lambda, A_|)$ is analytic in a neighborhood of 0. In particular, $\lim_{\lambda\searrow 0} \lambda R(\lambda, A)$ exists (and is equal to P).

(b) $\Rightarrow$ (c). Let $x \in D(A), y = Ax$. Then for $\lambda > 0$

$$\lambda R(\lambda, A)y = \lambda(A - \lambda + \lambda)R(\lambda, A)x = \lambda[x + \lambda R(\lambda, A)x],$$

hence $\lim_{\lambda\searrow 0} \lambda R(\lambda, A)y = 0$ for all $y \in \mathrm{ran}A$. Since $\lambda R(\lambda, A)$ converges in norm this implies

$$\lim_{\lambda\searrow 0} \|\lambda R(\lambda, A)|_{\overline{\mathrm{ran}A}}\| = 0.$$

From the identity $AR(\lambda, A) = \lambda R(\lambda, A) + I$ and the result above it follows that $AR(\lambda, A)_|$ is invertible for sufficiently small $\lambda > 0$. In particular,

$$\overline{\mathrm{ran}A} = \mathrm{ran}(AR(\lambda, A)_|) \subset \mathrm{ran}A.$$

(c) $\Rightarrow$ (a). From Lemma 5.2 it follows that $\lim_{r\to\infty} \|C(r)_|\| = 0$. Therefore, it suffices to show that $X = \ker A + \mathrm{ran}A$. Also by Lemma 5.2, $A_|$ is invertible. Hence, for $x \in D(A)$ there exists $y \in D(A_|) = D(A) \cap \mathrm{ran}A$ such that $Ay = Ax$. Writing $x = (x - y) + y$ we see that $D(A) \subset \ker A + \mathrm{ran}A$. Because $D(A)$ is dense in X, it follows that $X = \ker A + \mathrm{ran}A$. □

5.2 Examples Revisited II

We apply Theorem 5.3 to discuss the mean ergodic properties of the already familiar translation and multiplication semigroups:

Example 5.4. Exponentially stable C_0-semigroups are always uniformly mean ergodic. The interesting case is when the semigroup has growth bound 0, as is the case for the translation and multiplication semigroups introduced in Example 2.13 and 2.14.

(a) The translation semigroups on $L^1(\mathbb{R})$ and $C_{ub}(\mathbb{R})$ are not mean ergodic (see Example 3.7), hence not uniformly mean ergodic.

(b) The multiplication semigroup

$$T(t)f = e^{tq}f$$

on $C_0(\mathbb{R})$ where $q \in C_0(\mathbb{R}), q < 0$, is mean ergodic (see Example 3.8), but not uniformly mean ergodic: For $n \in \mathbb{N}$ let h_n be a continuous function satisfying

$$\mathbf{1}_{[-n,n]} \leq h_n \leq \mathbf{1}_{[-n-1,n+1]}.$$

Then $Ah_n = qh_n \in \operatorname{ran}A$ and $\lim_{n\to\infty} Ah_n = q \in C_0(\mathbb{R})$. But $q \notin \operatorname{ran}A$ (because $1 \notin C_0(\mathbb{R})$); therefore $\operatorname{ran}A$ is not closed and by Theorem 5.3 the semigroup is not uniformly mean ergodic.

(c) The translation semigroup on $L^p(\mathbb{R}), 1 < p < \infty$ is mean ergodic (see Example 3.7), but not uniformly mean ergodic: Define the function

$$g(x) = \sin\left(\frac{1}{x}\right)\mathbf{1}_{[\frac{1}{\pi},\infty)}(x).$$

Note that $g \notin L^p(\mathbb{R})$. Let $a_1 < a_2 < \cdots$ be the zeroes of g. The functions $g_n := g\mathbf{1}_{[1,a_n]}, n \geq 1$, are elements of $D(A) = \{f \in L^p(\mathbb{R}) \mid f \text{ is absolutely continuous and } f' \in L^p(\mathbb{R})\}$. Their images

$$f_n(x) := Ag_n(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\mathbf{1}_{[\frac{1}{\pi},a_n]}(x) \in \operatorname{ran}A.$$

converge in $L^p(\mathbb{R})$:

$$f(x) = \lim_{n\to\infty} f_n(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\mathbf{1}_{[\frac{1}{\pi},\infty)}(x) \in L^p(\mathbb{R}).$$

But $f \notin \operatorname{ran}A$, because the only absolutely continuous functions g_C satisfying $g_C' = f$ are given by $g_C := g + C, C \in \mathbb{R}$, which are not in L^p, hence not in $D(A)$.

(d) The multiplication semigroup

$$T(t)f = e^{tq}f$$

on $L^p(\Omega,\mu), 1 \leq p < \infty$ where $q < 0$ is a measurable function, is mean ergodic (see Example 3.8). If $\frac{1}{q} \in L^\infty(\Omega,\mu)$ then $(T(t))_{t\geq 0}$ is uniformly mean ergodic, because $f = q\frac{1}{q}f = A\frac{1}{q}f \in \operatorname{ran}A$ for all $f \in L^p(\Omega, \mu)$, hence $\operatorname{ran}A = L^p(\Omega, \mu)$ is closed.

If $q < 0$ is an arbitrary measurable function the semigroup need not be uniformly mean ergodic: For instance, if $(\Omega, \mu) = (\mathbb{R}, \lambda)$ and $q \in L^p(\mathbb{R})$ then $g_n = q\mathbf{1}_{[-n,n]} \in \operatorname{ran}A$ and $\lim_{n\to\infty} g_n = q \in L^p(\mathbb{R})$ exists, but $\lim_{n\to\infty} g_n \notin \operatorname{ran}A$ (because $1 \notin L^p(\mathbb{R})$).

The examples above show that a C_0-semigroup with growth bound zero can be uniformly mean ergodic (d), mean ergodic but not uniformly mean ergodic (b, c) or not mean ergodic at all (a).

5.3 Generators with compact resolvent

We apply the theorem above to prove uniform mean ergodicity of a special class of semigroups.

Definition 5.5. We say that an operator A with $\rho(A) \neq \emptyset$ has compact resolvent if there exists $\lambda \in \rho(A)$ such that the resolvent $R(\lambda, A)$ is compact.

Note that A has compact resolvent if and only if $R(\lambda, A)$ is compact for *all* $\lambda \in \rho(A)$. This follows from the resolvent identity

$$R(\lambda, A) - R(\mu, A) = (\lambda - \mu)R(\lambda, A)R(\mu, A), \quad \lambda, \mu \in \rho(A)$$

and the fact $R(\lambda, A)R(\mu, A)$ is compact if $R(\lambda, A)$ is.

Corollary 5.6. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup. If the generator A of $(T(t))_{t\geq 0}$ has compact resolvent then $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*

Proof. Let $\lambda > 0$. From the theory of compact operators we know that $\operatorname{ran}(R(\lambda, A) + \mu)$ is closed for all $\mu \neq 0$, in particular for $\mu = \frac{1}{\lambda}$. Therefore, the identity $\frac{1}{\lambda}AR(\lambda, A) = R(\lambda, A) + \frac{1}{\lambda}$ implies that $\operatorname{ran}(AR(\lambda, A)) = \operatorname{ran}A$ is closed as well. By Theorem 5.3, the semigroup $(T(t))_{t\geq 0}$ is uniformly mean ergodic. $\square$

Clearly, compactness of the resolvent of A is not necessary for uniform mean ergodicity: On any Banach space X the operator $A = 0$ generates a uniformly mean ergodic semigroup, but does not have compact resolvent if X is infinite dimensional.

Corollary 5.6 is useful in the context of partial differential equations: The natural domain of many differential operators on $L^2(\Omega)$ (for instance the operator Δ appearing in Section 4.1) is a subset of some Sobolev space $H^k(\Omega), k > 1$. The Rellich-Kondrachov Theorem together with Corollary 5.6

and Lemma 5.7 below implies that the semigroup generated by such an operator is uniformly mean ergodic (Corollary 5.8 below):

Lemma 5.7. *Let A be an operator on X with domain $D(A)$ and $\rho(A) \neq \emptyset$. On $D(A)$ we define the norm*

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in D(A).$$

Then A has compact resolvent if and only if the canonical injection

$$\iota : (D(A), \|\cdot\|_A) \hookrightarrow X \tag{28}$$

is compact.

Proof. We formulate the two properties in a more explicit way: Let $\lambda \in \rho(A)$. The operator A has compact resolvent if and only if the set

$$\begin{aligned} M_1 &:= \{(A-\lambda)^{-1}(x) \,|\, x \in X, \|x\| < 1\} = \\ &= \{y \in D(A) \,|\, \|(A-\lambda)y\| < 1\} \end{aligned}$$

is precompact in X. The function ι is compact if and only if the set

$$M_2 := \{x \in D(A) \,|\, \|x\|_A < 1\}$$

is precompact in X. The proof is complete if we can show that $\|\cdot\|_A$ and $\|(A-\lambda)(\cdot)\|$ are equivalent norms on $D(A)$. This can be verified easily:

$$\|(A-\lambda)x\| \le \|Ax\| + |\lambda|\|x\| \le \max\{1,|\lambda|\}\|x\| \le \max\{1,|\lambda|\}\|x\|_A$$

and

$$\begin{aligned} \|x\|_A = \|x\| + \|Ax\| &\le \|x\| + \|(A-\lambda)x\| + |\lambda|\|x\| \le \\ &\le \Big(1 + (1+|\lambda|)\|(A-\lambda)^{-1}\|\Big)\, \|(A-\lambda)x\|. \end{aligned}$$

□

Corollary 5.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary, $\partial\Omega \in C^1$. Let A be a closed operator on $H^k(\Omega)$, $k \geq 0$, with domain $D(A) \subset H^{k+1}(\Omega)$. If $\rho(A) \neq \emptyset$ then A has compact resolvent.*

The proof is based on the Rellich-Kondrachov Theorem, which we recall below (for a proof see e.g. [Ngô] and [Eva98]):

Theorem 5.9 (Rellich-Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^n$ and $k \in \mathbb{N}_0$ be as above. Then the Sobolev space $H^{k+1}(\Omega)$ is compactly embedded in $H^k(\Omega)$.*

Proof of Corollary 5.8. We show that the inclusion map ι defined in (28) is compact. We can write ι as the composition of the inclusion maps ι' and ι'' given below:

$$\iota : (D(A), \|\cdot\|_A) \overset{\iota''}{\hookrightarrow} H^{k+1}(\Omega) \overset{\iota'}{\hookrightarrow} H^k(\Omega).$$

From the Rellich-Kondrachov Theorem it follows that ι' is compact. Moreover, ι'' is closed: If $x_n \to x$ in $(D(A), \|\cdot\|_A)$ and $x_n \to y$ in $H^{k+1}(\Omega)$, then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to both x and y in $H^k(\Omega)$ because $\|\cdot\|_{H^k(\Omega)} \leq \|\cdot\|_A$ and $\|\cdot\|_{H^k(\Omega)} \leq \|\cdot\|_{H^{k+1}}$; hence $x = y$. Since A is closed, $(D(A), \|\cdot\|_A)$ is a Banach space. By the Closed Graph Theorem ι'' is bounded and therefore $\iota = \iota' \circ \iota''$ is compact. From Lemma 5.7 it follows that A has compact resolvent. □

Example 5.10. From Corollary 5.8 it follows immediately that whenever the operator $(\Delta + \lambda, H^2(\Omega) \cap H_0^1(\Omega))$ generates a C_0-semigroup, this semigroup is uniformly mean ergodic. In particular, the heat semigroup ($\lambda = 0$) is uniformly mean ergodic, but this is clear anyway because it is exponentially stable (Equation (17)).

The semigroup associated with the wave equation, Equation (20), is uniformly mean ergodic as well. This can be seen directly from Equation (21), which implies that for all $\lambda \in \rho(A) \supset \mathbb{R}$ the resolvent of the generator A can be written as the product $R_\lambda(A) = R_\lambda(A)_1 \times R_\lambda(A)_2$, where

$$\begin{aligned} R_\lambda(A)_1 : (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \to L^2(\Omega) &\overset{R_{\lambda^2}(\Delta)}{\to} H^1(\Omega) : \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto f_2 + \lambda f_1 =: g &\mapsto R_{\lambda^2}(\Delta)(g) \end{aligned}$$

and

$$\begin{aligned} R_\lambda(A)_2 : (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \to H^1(\Omega) &\overset{\iota}{\to} L^2(\Omega) : \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto f_1 + \lambda u_1 =: u_2 &\mapsto u_2 \end{aligned}$$

are compositions of a compact and a continuous operator, hence compact. Alternatively, the compactness of $R_\lambda(A)$ can be seen from the fact that $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \subset H^2(\Omega) \times H^1(\Omega)$ is compactly embedded in $X = H^1(\Omega) \times L^2(\Omega)$ and a similar argumentation as in the proof of Corollary 5.8.

References

[Ber05] A.R. Bernal. *Introduction to semigroup theory for partial differential equations.* 2005. Lecture notes, Universidad Complutense de Madrid.

[Blü10] M. Blümlinger. *Funktionalanalysis 2.* 2010. Lecture notes, TU Vienna.

[Con85] J.B. Conway. *A Course in Functional Analysis.* Graduate Texts in Mathematics. Springer Verlag, 1985.

[Eng00] R. Engel, K.J. Nagel. *One-parameter semigroups for linear evolution equations.* Graduate Texts in Mathematics. Springer Verlag, 2000.

[Eva98] L.C. Evans. *Partial Differential Equations.* American Mathematical Society, 1998.

[Gre08] B.J. Green. *Ergodic theory.* 2008. Lecture notes, University of Cambridge.

[Jü09] A. Jüngel. *Partielle Differentialgleichungen.* 2009. Lecture notes, TU Vienna.

[Liu05] K.K. Liu, W.J. Tan. *Mean ergodic theorems for C_0 semigroups of continuous linear operators.* Journal of Mathematical Analysis and Applications, 288(2), 2005.

[Ngô] Q.A. Ngô. *An extension of the Rellich-Kondrachov theorem.* Online: http://anhngq.wordpress.com/2010/05/05/an-extension-of-the-rellich-kondrachov-theorem/. Accessed on May 29, 2011.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer, 1983.

[Wor10] M. Woracek, H. Kaltenbäck. *Funktionalanalysis.* 2010. Lecture notes, TU Vienna.

[Yos74] K. Yosida. *Functional Analysis.* Springer Verlag, 1974.