Generalisations of Semigroups of Operators

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0.1 Preface

The theory of semigroups of operators introduced by Hille and Yoshida is the basis of this work. The main thought is to weaken the definition of a semigroup and therefore get a generalisation of the situation. The loss of information in relation to the "well-known" case is reflected in the injective operator $P(0)$. As in semigroup theory, one considers an Abstract Cauchy Problem for an operator $A : dom(A) \subset X \to X$,

$$u'(t) = Au(t), \qquad u(0) = c,$$

for $t \in [0, \infty)$ where u is a Banach space valued function.
Finally, the focus is on *exponentially tamed* pre-semigroups which can be identified with strongly continuous semigroups on a Banach subspace of the considered Banach space X.

0.2 Notation, Definitions and Elementary Results

First we make some remarks and introduce some notation.

- Let X always denote a Banach space with norm $\|.\|$.
- An operator is always linear.
- $\mathcal{B}(X)$ is the set of all linear bounded operators from X in X.
- A function $F : [0, \infty) \to \mathcal{B}(X)$ is called **strongly continuous** if

$$\lim_{h \to 0} \|(F(t+h) - F(t))x\| = 0$$

for all $t \in [0, \infty)$ and each fixed $x \in X$
(for $t = 0$ we have the limit from the right side $h \to 0^+$).

- $C([a, b]; X)$ $(a, b \in \mathbb{R})$ is the vector space of continuous functions $f : [a, b] : I \to X$ normed by $\|f\|_\infty := \sup_{t \in [a,b]} \|f(t)\|$. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $C([a, b]; X)$. By the definition of the $\|.\|_\infty$-norm $(f_n(t))_{n \in \mathbb{N}}$ is Cauchy in X and therefore converges to $f(t)$ for each fixed $t \in [a, b]$. Let h be sufficiently small, then

$$\begin{aligned}\|f(t+h) - f(t)\| &\leqslant \|f(t+h) - f_n(t+h)\| + \|f_n(t+h) - f_n(t)\| + \|f_n(t) - f(t)\| \\ &< \epsilon,\end{aligned}$$

and hence f is continuous using continuity of f_n. So $C([a, b]; X)$ is complete.

- $C_b([0, \infty); X)$ denotes the vector space of all bounded uniformly continuous functions $f : [0, \infty) \to X$ with norm $\|f\|_\infty := \sup_{t \geqslant 0} \|f(t)\|$. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $C_b([0, \infty); X)$. By definition of the norm, $(f_n(t))_{n \in \mathbb{N}}$ is Cauchy in X and therefore converges to $f(t)$ for each fixed $t \geqslant 0$. Since f_n is bounded, f is bounded. Clearly, convergence in $\|.\|_\infty$ is nothing else but uniform convergence, i.e.

$$\forall \epsilon' > 0 \;\; \exists N_{\epsilon'} \in \mathbb{N} : \|f_n(t) - f(t)\| < \epsilon' \quad \forall t \geqslant 0, n > N_{\epsilon'}.$$

Using this and uniform continuity of f_n (for a fixed n), i.e.

$$\forall \bar{\epsilon} > 0 \;\; \exists \delta_{\bar{\epsilon},n} > 0 : \|f_n(t+h) - f_n(t)\| < \bar{\epsilon} \quad \forall t \geqslant 0, |h| < \delta_{\bar{\epsilon},n},$$

we get for $\epsilon > 0$

$$\begin{aligned}\|f(t+h) - f(t)\| &\leqslant \|f(t+h) - f_n(t+h)\| + \|f_n(t+h) - f_n(t)\| + \|f_n(t) - f(t)\| \\ &< 2\epsilon' + \bar{\epsilon} < \epsilon,\end{aligned}$$

for an arbitrarily fixed $n > N_{\frac{\epsilon}{3}}$ and for all $|h| < \delta_{\frac{\epsilon}{3},n}$. Obviously, h and n are independent of $t \in [0, \infty)$, hence f is uniformly continuous. Therefore, $C_b([0, \infty); X)$ is a Banach space.

- $C_0(\mathbb{R})$ is the space of all bounded, continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{x\to\pm\infty} f(x) = 0$, i.e.

$$\forall \epsilon > 0 \quad \exists M_\epsilon > 0 : \quad |f(x)| < \epsilon \qquad \forall\, |x| > M_\epsilon \tag{1}$$

  With the norm $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ it is a Banach space. To see this, we consider the sequence $f_n \in C_0(\mathbb{R})$, $n \in \mathbb{N}$ with $f_n \to f$ for $n \to \infty$ in $\|.\|_\infty$. Convergence in the $\|f\|_\infty$-norm implies pointwise convergence. Because of (1) and continuity of f_n it follows

$$|f(x)| \leqslant |f(x) - f_n(x)| + |f_n(x)| < 2\epsilon,$$

  for all $|x| > M_{n,\epsilon}$ and an arbitrary $n \geqslant N_\epsilon \in \mathbb{N}$. By the pointwise convergence, boundedness and continuity of f are clear (in analogue to $C_b([0,\infty); X)$, see Notation, Definitions and Elementary Results), hence $f \in C_0(\mathbb{R})$.

- $C_{00}(\mathbb{R})$ is the linear subspace of $C_0(\mathbb{R})$ of functions with compact support, where the support *supp*(f) of a function f is defined as

$$supp(f) := \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}.$$

  It is easy to see that $C_{00}(\mathbb{R})$ lies dense in $C_0(\mathbb{R})$. For that, consider $f \in C_0(\mathbb{R})$ and define:

$$f_n(x) := \begin{cases} f(x) & |x| \leqslant n \\ 0 & |x| > n \end{cases}$$

  It is obvious that the discontinuity of f_n at $x = \pm n$ can be eliminated by a C^∞-function that "connects" $f(\pm n)$ with 0 on an interval $[-n-\epsilon, -n]$ ($[n, n+\epsilon]$ respectively). Then, f_n clearly belongs to $C_{00}(\mathbb{R})$. Furthermore,

$$\|f_n - f\|_\infty = \sup_{|x|>n} |f(x)| \to 0,$$

  for $n \to \infty$, hence $C_{00}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.
  Note that with this definition, $\overline{C_{00}(\mathbb{R})} \supset C_0(\mathbb{R})$.

- The **strong derivative** of a function $f : [a,b] \to X$ at $t \in (a,b)$ is defined as

$$\frac{d}{dt} f(t) = f'(t) := \lim_{h\to 0} \frac{1}{h}(f(t+h) - f(t),$$

  if the limit exists. The strong derivative at the boundary points is defined through the limit from the right hand side for $t = a$ (strong right derivative) and through the limit from the left hand side for $t = b$ respectively. As for $\mathbb{R}$-valued functions we have (see [Kal08b]): If $f' = 0$ on $[a,b]$, then f is constant on $[a,b]$.

- For functions $f : [a,b] \to X$ a Banach space valued Riemann integral $\int_a^b f(s)\, ds$ can be defined in the same way as for $\mathbb{R}$-valued functions by Riemann-sums. See

[Kal08b] for details. Thus, many results and rules concerning the integral (e.g. linearity,..) are similar. We want to point out that for $T \in \mathcal{B}(X)$:

$$\int Tf(s)\ ds = T \int f(s)\ ds.$$

This integral concept also includes improper Riemann-integrals. Such an improper integral is defined as

$$\int_a^\infty f(s)\ ds = \lim_{\beta \to \infty} \int_a^\beta f(s)\ ds,$$

where the limit is in the norm $\|.\|$ of the Banach space. Since

$$\left\| \int_a^b f(s)\ ds \right\| \leqslant \int_a^b \|f(s)\|\ ds$$

(which follows easily by definition of Riemann sums and triangle inequality), a sufficient condition for the existence of this limit is the convergence of

$$\lim_{\beta \to \infty} \int_a^\beta \|f(s)\|\ ds$$

in $\mathbb{R}$. The case "$-\infty$" is completely analogue.

- $I : X \to X : x \mapsto x$ denotes the identity operator.

- The operator norm of a bounded operator $T : X \to X$ is

$$\|T\|_{\mathcal{B}(X)} = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

  We will write only $\|.\|$, if it is clear that the object is an operator and if it is obviously on which space the map is defined.

- For an operator A defined on a subset of X, $dom(A)$ denotes the domain. Furthermore $A(dom(A))$ denotes the image of A.

- For an operator $A : dom(A) \to X$ and a subspace $Y \subset X$, the **part of A in Y**, A_Y, is defined as the operator with

$$dom(A_Y) = \{x \in dom(A) : x \in Y \wedge Ax \in Y\}, \qquad A_Y x = Ax.$$

- An operator $A : dom(A) \to X$ is called **closed**, if for all sequences $(x_n)_{n \in \mathbb{N}}$, $x_n \in dom(A)$ for all $n \in \mathbb{N}$, with

$$x_n \to x \in X \qquad \text{and} \qquad Ax_n \to y \in X,$$

  it follows

$$x \in dom(A) \qquad \text{and} \qquad Ax = y.$$

- **Closed Graph Theorem**: Let X, Y be Banach spaces and let $A : X \to Y$ be an operator ($dom(A) = X!$). The following assertions are equivalent

1. A is closed,
2. A is continuous, i.e. $A \in \mathcal{B}(X)$.

- $C^1([a,b];X)$ denotes the vector space of all functions $f:[a,b] \to X$ which are continuously (strong) differentiable, with norm $\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$, with $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|$. To show that this space is complete, we consider $C^1([a,b];X)$ as

$$\{(f;g) \in C([a,b];X) \times C([a,b];X) : g = f'\}\,. \tag{2}$$

  Since this is a subset of the Banach space $C([a,b];X) \times C([a,b];X)$ with the norm $\|.\|_\infty + \|.\|_\infty$, it suffices to show that the set in (2) is closed. It is equivalent to show that the differentiation operator

$$\begin{aligned}&D : dom(D) \to C([a,b];X) : f \mapsto f',\\ &dom(D) := \{f \in C([a,b];X) : f' \text{ exists and is continuous}\} \subset C([a,b];X)\end{aligned}$$

  is closed. Let $f_n \in dom(D), n \in \mathbb{N}$, $f_n \to f \in C([a,b];X)$ and $Df_n \to g \in C([a,b];X)$ (limits in $C([a,b];X)$). Since convergence in $C([a,b];X)$ implies uniform convergence, $g = \lim_{n\to\infty} f_n'$ is continuous. Furthermore, uniform convergence gives us, ($t, t_0 \in [a,b]$ and $t > t_0$)

$$\int_{t_0}^t g(s)\ ds = \lim_{n\to\infty} \int_{t_0}^t f_n'(s)\ ds.$$

  By the fundamental theorem of calculus we get

$$\begin{aligned}\int_{t_0}^t g(s)\ ds + \lim_{n\to\infty} f_n(t_0) &= \lim_{n\to\infty} \Big(\int_{t_0}^t f_n'(s)\ ds + f_n(t_0)\Big)\\ &= \lim_{n\to\infty} f_n(t)\\ &= f(t).\end{aligned}$$

  The left hand side is differentiable at t since g is continuous, and therefore $f' = g$. Hence, D is closed and $C^1([a,b];X)$ is a Banach space.

- **Principle of uniform boundedness theorem**: Let X, Y be Banach spaces and $\{T_i : i \in I\}$ a family of bounded operators. If the family is bounded pointwisely, i.e. for all $x \in X$ there exists a $M_x > 0$ so that

$$\sup_{i\in I} \|T_i x\| \leqslant M_x,$$

  then there exists a $M > 0$, so that

$$\sup_{i\in I} \|T_i\| < M < \infty.$$

- For a closed operator A, the **resolvent set** $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ for which the operator $(\lambda I - A) : dom(A) \to X$ is bijective. For $\lambda \in \rho(A)$, the **resolvent** $R_{\lambda,A}$ denotes $(\lambda I - A)^{-1}$ which is necessarily also closed and therefore in $\mathcal{B}(X)$ since

A is closed (by the Closed Graph Theorem). We point out that $(\lambda I - A)$ is not necessarily bounded for $\lambda \in \rho(A)$. Apparently the following relations hold true:

$$R_{\lambda,A}(\lambda I - A)x = x \quad x \in dom(A),$$
$$(\lambda I - A)R_{\lambda,A}x = x \quad x \in X.$$

The existence of a map $R_{\lambda,A} : X \to dom(A)$, which satisfies these two equations, is also obviously sufficient for $\lambda \in \rho(A)$.

$$R_{\lambda,A}(\lambda I - A)x = (\lambda I - A)R_{\lambda,A}x,$$

for all $x \in dom(A)$.

- **LEMMA 0.1** *For a closed operator $A : dom(A) \subset X \to X$ which commutes with $B : X \to X$, i.e. $Bx \in dom(A)$ and*

$$BAx = ABx \qquad \forall x \in dom(A),$$

it follows that the resolvent $R_{\lambda,A}$ commutes with B, i.e.

$$BR_{\lambda,A}x = R_{\lambda,A}Bx \qquad \forall x \in X, \lambda \in \rho(A).$$

PROOF: By definition of $R_{\lambda,A}$ and using the assumption we see

$$\begin{aligned} BAy = ABy \qquad & \forall y \in dom(A) \\ \Leftrightarrow \lambda By - BAy = \lambda By - ABy \quad & \forall y \in dom(A) \\ \Leftrightarrow B(\lambda I - A)y = (\lambda I - A)By \quad & \forall y \in dom(A) \\ \Leftrightarrow R_{\lambda,A}B(\lambda I - A)y = By \qquad & \forall y \in dom(A) \\ \Leftrightarrow R_{\lambda,A}Bx = BR_{\lambda,A}x \qquad & \forall x \in X. \end{aligned}$$

■

- **LEMMA 0.2** *Let $A : X \to X$, $B : X \to X$ be operators. Let B be injective and AB=BA. Then, $A(BX) \subset BX$ and*

$$AB^{-1}x = B^{-1}Ax,$$

for all $x \in BX$.

PROOF: Commutativity gives $A(BX) \subset BX$. Using this and $x \in BX$, it follows

$$\begin{aligned} Ax &= Ax \\ \Rightarrow ABB^{-1}x &= BB^{-1}Ax \\ \Rightarrow AB^{-1}x &= B^{-1}Ax. \end{aligned}$$

■

- **LEMMA 0.3** *For operators* $A : dom(A) \subset X \to X$, $B : dom(B) \subset X \to X$ *with surjective* A , *injective* B *the relation* $A \subset B$, *i.e.*

$$dom(A) \subset dom(B) \quad \wedge \quad Ax = Bx \qquad \forall x \in dom(A),$$

implies

$$A = B.$$

PROOF: It suffices to show that $dom(A) = dom(B)$. $dom(A) \subset dom(B)$ is fulfilled by assumption. Let x be in $dom(B)$. Since A is surjective, there exists a $y \in dom(A)$ so that $Ay = Bx$. By assumption $A \subset B$. Therefore, $y \in dom(B)$ and $Ay = By$. Thus $Bx = By$. The injectivity of B leads to $x = y \in dom(A)$, that is $dom(B) \subset dom(A)$ and hence $dom(A) = dom(B)$. ■

Chapter 1

Pre-Semigroups

DEFINITION 1.1 *A family* $\{P(t)\}_{t\geqslant 0}$ *of operators is called* ***pre-semigroup****, if*

1. $P : [0,\infty) \to \mathcal{B}(X)$ *is strongly continuous,*
   i.e. $\lim_{h\to 0} \|P(t+h)x - P(t)x\| = 0 \qquad \forall x \in X, \forall t \in [0,\infty)$
2. $P(0) : X \to X$ *is injective*
3. $P(t-u)P(u)$ *is independent of* u *for all* $0 \leqslant u \leqslant t$

This definition is a generalisation of strongly continuous semigroups of operators (C_0-semigroups). Point (3.) in the given form is not really convenient for the following statements and their proofs. That is why we reformulate it in the next lemma.

LEMMA 1.2 *For a family* $\{P(t)\}_{t\geqslant 0}$ *of operators the following points are equivalent:*

- $P(t-u)P(u)$ *is independent of* u *for* $0 \leqslant u \leqslant t$
- $P(t-u)P(u) = P(0)P(t)$ *for* $0 \leqslant u \leqslant t$
- $P(0)P(u+s) = P(s)P(u)$ *for all* $u, s \geqslant 0$

PROOF: (1.) $\Leftrightarrow$ (2.): One direction follows by setting $u = t$. The other implication is trivial.
(2.) $\Leftrightarrow$ (3.): Set $t = s + u$. ■

The last point of this lemma,

$$P(0)P(u+s) = P(s)P(u) \qquad u, s \geqslant 0 \tag{ADD}$$

reflects some kind of additivity of the pre-semigroup and immediately implies the commutativity of the operators $P(s)$,

$$P(s)P(u) = P(u)P(s) \qquad u, s \geqslant 0 \tag{COM}$$

REMARK 1.3 In the property $P(0)P(u+s) = P(s)P(u)$ we can see the connection and the difference to "normal" C_0-semigroups: It is the injective operator $P(0)$ which controlls the additivity of $P(.)$. Now we have noticed that it is just $P(0)$ which generalises the situation of a strongly continuous semigroup. That is the reason why pre-semigroups are sometimes called "C-semigroups" where C denotes the injective operator $P(0)$. Probably this definition is not really suitable since this can be easily confused with C_0-semigroups. That is why for example in [deL94] the term "C-regularized-semigroup" is introduced. The notation "pre-semigroups" has been adopted from [Kan95].

The next lemma shows a basic property of a pre-semigroup.

LEMMA 1.4 *For a pre-semigroup* $\{P(t)\}_{t\geqslant 0}$ *the family of operators* $\{P(s) : s \in [a, b]\}$ *is uniformly bounded for each compact interval* $[a, b]$ *in* $[0, \infty)$*, i.e. there exists a* $M > 0$*:*

$$\|P(s)\| < M \quad \forall s \in [a, b].$$

PROOF: We have to show that $\{P(s) : s \in [a, b]\}$ is bounded pointwisely. Since the norm $\|.\| : X \to [0, \infty)$ is continuous, it follows from strong continuity of the pre-semigroup that $\|P(.)x\| : [0, \infty) \to [0, \infty)$ is continous for all $x \in X$. Such a function clearly has a maximum on a compact interval. Hence for each $x \in X$ there exists a M_x, so that $sup_{s\in[a,b]} \|P(s)x\| < M_x$. With the Principle of uniform boundedness (see Notation, Definitions and Elementary Results) the proof is completed. ■

Example **1.5** Consider the Banachspace $X = C_0(\mathbb{R})$ and the the family of operators $\{P(t)\}_{t\geqslant 0}$, defined through

$$P(t)f(x) = e^{-x^2+tx} f(x), \tag{1.1}$$

for $x \in \mathbb{R}$. We will see that this is a pre-semigroup. For that, we have to check the conditions of DEFINITION 1.1.

1. $P : [0, \infty) \to \mathcal{B}(X)$ is strongly continuous.

Fix $t \geqslant 0$. First, we have to assure that $P(t)f$ is in $C_0(\mathbb{R})$. This is clear since $\lim_{x\to\pm\infty} e^{-x^2+tx} = 0$ and $f \in C_0(\mathbb{R})$. The parabel $x \mapsto -x^2 + tx$ has its maximum $\frac{t^2}{4}$ at $x_m = \frac{t}{2}$. Therefore,

$$\begin{aligned}
\|P(t)f\|_\infty &= \sup_{x\in\mathbb{R}} \left| e^{-x^2+tx} f(x) \right| \\
&\leqslant \sup_{x\in\mathbb{R}} \left| e^{-x^2+tx} \right| \sup_{x\in\mathbb{R}} |f(x)| \\
&= e^{\frac{t^2}{2}} \|f\|_\infty,
\end{aligned}$$

hence $P(t) \in \mathcal{B}(X)$. Fix $f \in C_0(\mathbb{R})$. For strong continuity we have to show that for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that

$$\sup_{x\in\mathbb{R}} \left| e^{-x^2+(t+h)x} f(x) - e^{-x^2+tx} f(x) \right| < \epsilon, \tag{1.2}$$

for all $|h| < \delta_\epsilon$. First we consider functions $f \in C_{00}(\mathbb{R})$. Since f has a compact support K the left hand side in line (1.2) reads

$$\sup_{x\in K}\left|e^{-x^2+(t+h)x}f(x) - e^{-x^2+tx}f(x)\right| < \sup_{x\in K}\left|e^{-x^2+tx}f(x)\right| \sup_{x\in K}\left|e^{hx}-1\right| = S_K \sup_{x\in K}\left|e^{hx}-1\right|,$$

where S_K denotes the maximum of $e^{-x^2+tx}f(x)$ on K. From monotony of $y \mapsto e^y$ we get, with $K_{max} = \max\{|x| : x \in K\}$,

$$\sup_{x\in K}\left|e^{hx}-1\right| \leqslant \left|\sup_{x\in K}\left|e^{|h||x|}\right| - 1\right| = \left|e^{|h|K_{max}} - 1\right| \to 0,$$

for $h \to 0$. Hence we have strong continuity for $f \in C_{00}(\mathbb{R})$. Let $f \in C_0(\mathbb{R})$. Since $C_{00}(\mathbb{R})$ lies dense in $C_0(\mathbb{R})$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $C_{00}(\mathbb{R})$ such that,

$$\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N}: \quad \|f - f_n\|_\infty < \epsilon \quad \forall n \geqslant N_\epsilon.$$

Therefore we can write

$$\begin{aligned} &\|P(t+h)f - P(t)f\|_\infty = \\ &= \|P(t+h)f - P(t+h)f_n + P(t+h)f_n - P(t)f_n + P(t)f_n - P(t)f\|_\infty \\ &\leqslant \|P(t+h)(f-f_n)\|_\infty + \|P(t)(f-f_n)\|_\infty + \|P(t+h)f_n - P(t)f_n\|_\infty \\ &\leqslant (\|P(t+h)\| + \|P(t)\|)\, \|f-f_n\|_\infty + \|P(t+h)f_n - P(t)f_n\|_\infty . \end{aligned}$$

Because of strong continuity for functions in $C_{00}(\mathbb{R})$ we have

$$\|P(t+h)f_n - P(t)f_n\|_\infty < \epsilon,$$

for $|h| < \delta_\epsilon$. Using LEMMA 1.4 we know that $\|P(t+h)\|$ and $\|P(t)\|$ are bounded (independent of h) by a constant $M > 0$. Therefore,

$$(\|P(t+h)\| + \|P(t)\|)\, \|f-f_n\|_\infty + \|P(t+h)f_n - P(t)f_n\|_\infty < \epsilon$$

for an arbitrary $n \geqslant N_{\epsilon/2(2M)}$ and $|h| < \delta_{\epsilon/2}$. Hence, the family of operators is strongly continuous.

2. $P(0)$ is injective because for $P(0)f = P(0)g$ with $f, g \in C_0(\mathbb{R})$ we have for all $x \in \mathbb{R}$

$$\begin{aligned} P(0)f(x) &= P(0)g(x) \\ e^{-x^2}f(x) &= e^{-x^2}g(x) \\ \Leftrightarrow f(x) &= g(x). \end{aligned}$$

3. Clearly, for all $f \in C_0(\mathbb{R})$ and all $x \in \mathbb{R}$ the following holds true

$$P(0)P(s+t)f(x) = e^{-x^2}e^{-x^2+(s+t)x}f(x) = e^{-x^2+sx}\left[e^{-x^2+tx}f(x)\right] = P(s)P(t)f(x),$$

hence $P(0)P(s+t) = P(s)P(t)$.

This example will accompany us throughout this work. Actually it can be weakened. Instead of the assumption that the functions tend to zero for $x \to \pm\infty$, we can just require $\lim_{x\to\pm\infty} f(x) = b_f$ for a $b_f \in \mathbb{R}$. Note, that then $P(t)f$ is still in $C_0(\mathbb{R})$.

For the definition of the "generator" of a pre-semigroup we need the right derivative of a Banach space valued function (in analogue to $\mathbb{R}$, see Notation, Definitions and Elementary Results).

DEFINITION 1.6 *Let $\{P(t)\}_{t\geqslant 0}$ be a pre-semigroup and $x \in X$. The strong right derivative $P'^{+}(t)x \in X$ of $P(.)x$ at t is defined as*

$$\lim_{h\to 0^+} \frac{1}{h}\,[P(t+h)x - P(t)x], \tag{1.3}$$

if the limit exists.

Now we can define an operator for the pre-semigroup connected with the derivative at zero.

DEFINITION 1.7 *Let $\{P(t)\}_{t\geqslant 0}$ be a pre-semigroup.*
Define the operator $A : dom(A) \to X$ by:

- *$dom(A) = \{x \in X : P'^{+}(0)x$ exists in X and belongs to $P(0)X\}$*
- *$Ax = P(0)^{-1}P'^{+}(0)x$*

*A is called the **generator** of the pre-semigroup $\{P(t)\}_{t\geqslant 0}$. We also say "A generates the pre-semigroup $\{P(t)\}_{t\geqslant 0}$".*

Because of the injectivity of $P(0)$, the generator is well-defined. The linearity follows, clearly, from the linearity of $P(t)$ for all $t \in [0,\infty)$. For $P(0) = I$ this definition obviously equals the definition of the generator for semigroups.

***Example* 1.8** Consider again the pre-semigroup from Example 1.1. For $f \in C_0(\mathbb{R})$ we regard the strong right derivative of $P(.)f$ at zero.

$$P'^{+}(0)f = \lim_{h\to 0^+} \frac{1}{h}\big(P(h)f - P(0)f\big)$$

Assume that $f \in dom(A)$. Since point evaluations are continuous on $C_0(\mathbb{R})$, we obtain for $x \in \mathbb{R}$ with de L'Hospital

$$\begin{aligned}\big(P'^{+}(0)f\big)(x) = \lim_{h\to 0^+} \frac{1}{h}\big(P(h)f(x) - P(0)f(x)\big) &= \lim_{h\to 0^+} \frac{1}{h}\big(e^{-x^2+hx}f(x) - e^{-x^2}f(x)\big)\\ &= e^{-x^2}f(x)\lim_{h\to 0^+}\frac{e^{hx}-1}{h}\\ &= e^{-x^2}f(x)x.\end{aligned}$$

In the definition of $dom(A)$ we demand $P'^{+}(0)f$ to be in the image of $P(0) : g \mapsto (x \mapsto e^{-x^2}g(x))$, therefore our function f in $dom(A)$ satisfies $(x \mapsto xf(x)) \in C_0(\mathbb{R})$.
Conversely, let $f \in C_0(\mathbb{R})$ with $(x \mapsto xf(x)) \in C_0(\mathbb{R})$. We show that for such f, $P'^{+}(0)f$ is $(x \mapsto e^{-x^2}xf(x))$. That is, for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that

$$\sup_{x\in\mathbb{R}}\left|e^{-x^2}f(x)\frac{e^{hx}-1}{h} - e^{-x^2}xf(x)\right| < \epsilon, \tag{1.4}$$

for $h < \delta_\epsilon$. We see that $\left|e^{-x^2} x f(x)\right|$ is sufficiently small, say $< \epsilon/2$, for $x \to \pm\infty$. Moreover for $h < 1$ we have

$$\begin{aligned}
\left|e^{-x^2} f(x) \frac{e^{hx}-1}{h}\right| &= \left|e^{-x^2} f(x) x \sum_{i=1}^{\infty} \frac{(xh)^i}{i!}\right| \\
&\leqslant \left|e^{-x^2} x f(x)\right| \sum_{i=1}^{\infty} \frac{(|x|)^i}{i!} \\
&\leqslant e^{-x^2} e^{|x|} \left|x f(x)\right| .
\end{aligned}$$

This expression is also $< \epsilon/2$ for $|x|$ sufficiently large. Hence we can define $S_\epsilon > 0$ so that (1.4) $< \epsilon$ for $|x| > S_\epsilon$ and $h < 1$. For the remaining x we calculate

$$\begin{aligned}
\sup_{|x| \leqslant S_\epsilon} \left|e^{-x^2} f(x) \big(\frac{e^{hx}-1}{h} - x\big)\right| &\leqslant \|f\|_\infty \sup_{|x| \leqslant S_\epsilon} \left|\int_0^x (e^{hv}-1)\ dv\right| \\
&\leqslant \|f\|_\infty S_\epsilon \sup_{|v| \leqslant S_\epsilon} \left|e^{hv}-1\right| \\
&\leqslant \|f\|_\infty S_\epsilon \sup_{|\tau| \leqslant h S_\epsilon} \left|e^{\tau}-1\right| \to 0,
\end{aligned}$$

for $h \to 0^+$. Hence, $P'^+(0)f = (x \mapsto e^{-x^2} x f(x))$. Since $(x \mapsto x f(x)) \in C_0(\mathbb{R})$, $P'^+(0)f$ is in the image of $P(0)$, and therefore, $f \in dom(A)$. Altogether we have (since $P(0)^{-1}(x \mapsto e^{-x^2} x f(x)) = (x \mapsto x f(x))$),

$$Af(x) = x f(x), \qquad dom(A) = \{f \in C_0(\mathbb{R}) : (x \mapsto x f(x)) \in C_0(\mathbb{R})\} .$$

After we have noticed that the operator A is well defined, we want to know "if this map is reasonable in a certain sense". One question is about the domain of A: The pre-semigroup is per definitionem "only" (strongly) continuous. This does not really imply that there exists a (strong right) derivative. For example, we want to analyse how big the domain is.

The following theorem shows some basic results of the generator.

THEOREM 1.9 *For the generator A of a given pre-semigroup $\{P(t)\}_{t \geqslant 0}$. The following assertions hold true.*

1. $x \in dom(A) \Rightarrow P(t)x \in dom(A)$ *for all* $t \geqslant 0$
2. $AP(t)x = P(t)Ax$ *for all* $x \in dom(A)$
3. $P(.)x \in C^1([0,\infty);X)$ *for* $x \in dom(A)$,

   $$AP(t)x = \lim_{h \to 0} \frac{1}{h}(P(t+h)x - P(t)x) = \frac{d}{dt} P(t)x$$

   for all $x \in dom(A), t \in [0,\infty)$
4. *For* $x \in X$*:*

   $$\int_0^t P(s)x\ ds \in dom(A)$$

5. *A is closed and* $P(0)X \subseteq \overline{dom(A)}$*;*

PROOF: Let be $t \geqslant 0$, $h > 0$ and $x \in dom(A)$.
1. We use that the $P(s), s \geqslant 0$ commute to obtain

$$\frac{1}{h}\big(P(h)\,[P(t)x] - P(0)\,[P(t)x]\,\big) = \frac{1}{h}\big(P(t)[P(h)x - P(0)x]\big). \tag{1.5}$$

By the continuity of $P(t)$ and because of $x \in dom(A)$ the right hand side tends to the strong right derivative of $P(t)P(.)x$ at 0 for $h \to 0^+$, hence

$$\lim_{h\to 0}\frac{1}{h}\big(P(h)\,[P(t)x] - P(0)\,[P(t)(x)]\,\big) = P(t)[P'^+(0)x].$$

In particular, the limit for $h \to 0^+$ on the left hand side, i.e. the strong right derivative $P'^+(0)\,[P(t)x]$, exists. With the definition of A and again with the commutativity of the operators $P(s), s \geqslant 0$ we get

$$P'^+(0)[P(t)x] = P(t)\lim_{h\to 0^+}\frac{1}{h}(P(h)x - P(0)x) \tag{1.6}$$

$$= P(t)P(0)Ax \tag{1.7}$$

$$= P(0)P(t)Ax \tag{1.8}$$

Therefore, $P'^+(0)P(t)x \in P(0)X$ and hence $P(t)x \in dom(A)$.
2. Furthermore with the definiton of the operator A and using (1.8) it follows

$$A\,[P(t)x] = P(0)^{-1}\left[P'^+(0)P(t)x\right] = P(0)^{-1}P(0)P(t)Ax = P(t)Ax.$$

3. We use $P(0)P(t+h) = P(t)P(h)$ (ADD) to obtain

$$\frac{1}{h}\big(P(t)[P(h)x - P(0)x]\big) = \frac{1}{h}\big(P(0)[P(t+h)x - P(t)x]\big). \tag{1.9}$$

Letting $h \to 0^+$, we observe that, with the same argument as in 1. ($x \in dom(A)$ and $P(t)$ continuous), the strong right derivative of $P(0)P(.)x$ at $t \geqslant 0$,

$$\lim_{h\to 0^+}\frac{1}{h}P(0)(P(t+h)x - P(t)x) = [P(0)P(.)]'^+\,(t)x,$$

exists and equals $P(t)P(0)Ax = P(0)P(t)Ax$. We show that this is also the strong left derivative of $P(0)P(.)x$ for $t > 0$. With the triangle inequality we see

$$\left\|\frac{1}{h}\,[P(0)P(t) - P(0)P(t-h)]\,x - P(0)P(t)Ax\right\| \leqslant$$
$$\leqslant \left\|\frac{1}{h}\,[P(t) - P(t-h)]\,P(h)x - P(0)P(t)Ax\right\| +$$
$$+ \left\|\big(P(t-h) - P(t)\big)\frac{1}{h}\,[P(h)x - P(0)x]\right\|.$$

Using triangle inequality again, we get that this expression is less or equal to

$$\begin{aligned}
&\left\| \frac{1}{h} \left[P(t) - P(t-h) \right] P(h)x - P(0)P(t)Ax \right\| + \\
&+ \left\| \big(P(t-h) - P(t) \big) \left\{ \frac{1}{h} \left[P(h)x - P(0)x \right] - P(0)Ax \right\} \right\| + \\
&+ \left\| P(t-h)P(0)Ax - P(t)P(0)Ax \right\| .
\end{aligned}$$

Now we show that all three terms on the right hand side converge to zero for $h \to 0^+$. Because of (ADD) the first term can be written as

$$\left\| \frac{1}{h} \big(P(0)P(t+h)x - P(0)P(t)x \big) - P(0)P(t)Ax \right\| .$$

For $h \to 0^+$ this converges to 0, since $P(0)P(t)Ax$ is the strong right derivative of $P(0)P(.)x$ at t as shown before. LEMMA 1.4 can be applied on the interval $[t-h,t]$ and gives us a constant $M > 0$, so that for the second term we get

$$\begin{aligned}
&\left\| \big(P(t-h) - P(t) \big) \left\{ \frac{1}{h} \left[P(h)x - P(0)x \right] - P(0)Ax \right\} \right\| \leqslant \\
&\leqslant \| P(t-h) - P(t) \| \left\| \frac{1}{h} \left[P(h)x - P(0)x \right] - P(0)Ax \right\| \leqslant \\
&\leqslant 2M \left\| \frac{1}{h} \left[P(h)x - P(0)x \right] - P(0)Ax \right\| \to 0,
\end{aligned}$$

for $h \to 0^+$ by definition of the strong right derivative and A. The third term converges to 0 since $P(.)P(0)Ax$ is continuous.
Therefore, $P(0)P(.)x$ is differentiable for all $t \in (0,\infty)$,$x \in dom(A)$ and its derivative equals $P(0)P(t)Ax$. That is,

$$P(0)P(t)Ax = \lim_{h \to 0} \frac{1}{h} (P(0)P(t+h)x - P(0)P(t)x) = [P(0)P(.)x]'(t).$$

Obviously, the left hand side is continuous (as a function in t and fixed x), since $P(0)$ is continuous and because of strong continuity of $P(.)$. Therefore, $P(0)P(.)x$ is continuously differentiable and hence we can use the fundamental theorem of calculus,

$$\int_t^{t+h} [P(0)P(.)x]'(s)\ ds = \int_t^{t+h} P(0)P(s)Ax\ ds = P(0)P(t+h)x - P(0)P(t)x.$$

The fact that $P(0)$ bounded yields (see: Notation, Definitions and Elementary Results)

$$P(0) \int_t^{t+h} P(s)Ax\ ds = P(0)[P(t+h)x - P(t)x],$$

and by injectivity of $P(0)$ we have

$$\int_t^{t+h} P(s)Ax\ ds = P(t+h)x - P(t)x. \tag{1.10}$$

The integrand is obviously continuous, so dividing by h and letting $h \to 0$ directly gives us (by the fundamental theorem of calculus)

$$P(t)Ax = \lim_{h\to 0} \frac{1}{h}(P(t+h)x - P(t)x) = [P(.)x]'(t).$$

Clearly, also for $t = 0$ the strong right derivative of $P(.)x$ exists and equals $P(0)Ax$. Since $P(.)Ax$ is continuous on $[0,\infty)$ (because $P(.)$ is strongly continuous), $P(.)x \in C^1([0,\infty)\,;X)$. With point 2. of this theorem we obtain

$$AP(t)x = \lim_{h\to 0} \frac{1}{h}(P(t+h)x - P(t)x).$$

4. For a fixed $t > 0$, we consider the strong right derivative of $P(.)\left[\int_0^t P(s)xds\right]$ at 0:

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{h}\left[P(h) - P(0)\right]\int_0^t P(s)x\ ds &= \lim_{h\to 0^+} \frac{1}{h}\int_0^t [P(0)P(h+s) - P(0)P(s)]x\ ds \\ &= \lim_{h\to 0^+} \frac{1}{h}P(0)[\int_h^{t+h} P(u)x\ du - \int_0^t P(s)x]\ ds \\ &= P(0)\lim_{h\to 0^+} \frac{1}{h}\left[\int_t^{t+h} P(u)x\ du - \int_0^h P(s)x\ ds\right] \\ &= P(0)\left[P(t) - P(0)\right]x,\end{aligned}$$

where we used the fundamental theorem of calculus again. So the strong right derivative exists and belongs to $P(0)X$.
5. With (1.10) from point 3. we have

$$P(t)x - P(0)x = \int_0^t P(s)Ax\ ds \tag{1.11}$$

First we show that A is closed. Let $x_n \to x$ with $x_n \in dom(A)$ and $Ax_n \to y$. (1.11) and boundedness of $P(t)$ for all $t \in [0,\infty)$ yield to

$$(P(h) - P(0))x = \lim_{n\to\infty}((P(h) - P(0))x_n) = \lim_{n\to\infty}\int_0^h P(s)Ax_n\ ds.$$

Because $\|P(s)\|$ is bounded uniformly on the compact interval $[0,h]$ (LEMMA 1.4), the limit is uniformly, hence can be permuted with the integral (see [Kal08b]).

$$(P(h) - P(0))x = \int_0^h \lim_{n\to\infty} P(s)Ax_n\ ds = \int_0^h P(s)y\ ds. \tag{1.12}$$

Dividing by h and letting $h \to 0^+$ we get (with $P(.)y$ being continuous and the fundamental theorem of calculus)

$$P'^+(0)x = \lim_{h\to 0}\frac{1}{h}\big(P(h)x - P(0)x\big) = \lim_{h\to 0}\frac{1}{h}\int_0^h P(s)y\ ds = P(0)y.$$

Hence $P'^{+}(0)x \in P(0)X$ and further $x \in dom(A)$ by definition of A and $Ax = y$.
Finally we show that $P(0)X \subseteq \overline{dom(A)}$. Let $x \in X$. From 4., we see that $t^{-1}\int_0^t P(s)xds \in dom(A)$. Letting $t \to 0^+$ and using again the fundamental theorem of calculus we get

$$P(0)x = \lim_{t\to 0} t^{-1}\int_0^t P(s)x\ ds.$$

Thus $P(0)x \in \overline{dom(A)}$. ■

We see that there is a connection between the domain of A and the image of $P(0)$. The bigger P(0)X is, the bigger will be $dom(A)$. In the case that $P(0)$ is bijective, it follows that $dom(A)$ is dense in X. Furthermore, the property $P(0)P(u+s) = P(u)P(s)$ is responsible for the fact that $dom(A)$ is invariant for $P(t)$ and the commutativity of A and $P(t)$. A main result is the differentiability of $P(t)x$ for $x \in dom(A)$. This will be used in chapter 2.

REMARK 1.10 An obvious question is "What happens if we have a pre-semigroup $\{P(t)\}_{t\geqslant 0}$ and an injective, bounded operator G and we consider the family of operators $\{W(t) := GP(t)\}_{t\geqslant 0}$"? Can we expect this family to be a pre-semigroup? If we look at the assumptions in DEFINITION 1.1, clearly, strong continuity is preserved by boundedness of G and injectivity of $GP(0)$ is trivial. Concerning the additivity property, $P(0)P(t+s) = P(t)P(s)$, we get

$$W(0)W(s+t) = GP(0)GP(s+t),$$

where we see that $GP(t) = P(t)G$ for all $t \geqslant 0$ is a sufficient condition so that

$$GP(0)GP(s+t) = GGP(0)P(s+t) = GGP(t)P(s) = GP(s)GP(t) = W(s)W(t)$$

Therefore, additionally we have to require that the operator G commutes with $P(t)$ for all $t \geqslant 0$. In this case, the domain of the generator of $\{W(t) := GP(t)\}_{t\geqslant 0}$ includes the domain of A_P, the generator of $\{P(t)\}_{t\geqslant 0}$, since the boundedness of G gives us

$$\left\|\frac{1}{h}\big(GP(h)x - GP(0)\big) - GP'^{+}(0)x\right\| < \|G\| \left\|\frac{1}{h}\big(P(h)x - P(0)\big) - P'^{+}(0)x\right\|.$$

Finally, this thoughts inspire the idea to choose $G = P(0)^{-1}$. Unfortunately, in general we can not expect continuity of the inverse of $P(0)$. Although we will see in Chapter 3 that in some situations this is possible.

The situation is that we have a pre-semigroup which gives us the generator A. Especially in connection with the ACP (see next chapter) and the uniqueness of its solution we are interested in a uniqueness of the generator. The following technical lemma will be useful for conclusions on the uniqueness of the pre-semigroup for a given generator. For that, we state the a **product rule** for Banach space-valued functions.

LEMMA 1.11 *Let $W(.) : [a,b] \to \mathcal{B}(X)$ be a strongly continuous function with $W(.)x \in C^1([a,b];X)$ for all x in a linear subspace $U \subset X$. Furthermore, let $v : [a,b] \to U$ be in $C^1([a,b];X)$. Then,*

$$\big(W(.)v(.)\big)'(t) = W(t)v'(t) + W'(t)v(t), \tag{1.13}$$

where $W'(t)x := [W(.)x]'(t)$.

PROOF: Regard the function g defined as follows:

$$g : [a, b] \to X : s \mapsto W(s)v(s)$$

We consider $g'(s)$, $s \in [a, b]$, with elementary rearrangements we get

$$\begin{aligned}
g'(s) &= \lim_{h\to 0} \frac{1}{h}[W(s+h)v(s+h) - W(s)v(s)] && (1.14)\\
&= \lim_{h\to 0} \frac{1}{h}[W(s+h)v(s+h) - W(s+h)v(s) + W(s+h)v(s) - W(s)v(s)] && (1.15)\\
&= \lim_{h\to 0} W(s+h)\frac{1}{h}[v(s+h) - v(s)] + \lim_{h\to 0} \frac{1}{h}[W(s+h) - W(s)]v(s) && (1.16)\\
&= \lim_{h\to 0} W(s+h)\big(\frac{1}{h}[v(s+h) - v(s)] - v'(s)\big) + \lim_{h\to 0} W(s+h)v'(s)+ && (1.17)\\
&\quad + \lim_{h\to 0} \frac{1}{h}[W(s+h) - W(s)]v(s). && (1.18)
\end{aligned}$$

Due to the strong continuity of $W(.)$ and the principle of uniform boundedness theorem (compare: LEMMA 1.4), $\|W(s+h)\|$ is bounded (by a constant S) for h in a compact interval. Therefore, we can write

$$\left\| W(s+h)\big(\frac{1}{h}[v(s+h) - v(s)] - v'(s)\big) \right\| \leqslant S \left\| \frac{1}{h}[v(s+h) - v(s)] - v'(s) \right\|,$$

where the right hand side clearly tends to 0 for $h \to 0$, since $v(.) \in C^1([a, b]; X)$. Hence, the first term in (1.17) is $0 \in X$. The second term,

$$\lim_{h\to 0} W(s+h)v'(s) = W(s)v'(s),$$

since $W(.)$ is strongly continuous. Finally,

$$\lim_{h\to 0} \frac{1}{h}[W(s+h) - W(s)]v(s) = W'(s)v(s),$$

because $v(s) \in U$ and $W(.)x \in C^1([a, b]; X)$ for $x \in U$. Altogether,

$$g'(s) = W(s)v'(s) + W'(s)v(s),$$

which proves the lemma. (For $s = a$ or $s = b$ the limits above are to be considered for $h \to 0^+$ or $h \to 0^-$) ■

We point out that $W'(t)v(t)$ is not the composition of the operators "$W'(t)$" and $v(t)$.

LEMMA 1.12 *Let $\{P(t)\}_{t\geqslant 0}$ be a pre-semigroup generated by A. Let $v : [0, \infty) \to dom(A)$ be in $C^1([0, \infty); X)$ with $v' = Av$ and $v(0) = P(0)c$ for $c \in dom(A)$. Then,*

$$P(.)c = v(.)$$

PROOF: We fix $t > 0$. Clearly the function $h_{t,x} : [0,t] \to X : s \mapsto P(t-s)x$ is in $C^1([0,t];X)$ for $x \in dom(A)$, since $P(.)x \in C^1([0,t];X)$ for $x \in dom(A)$ (see THEOREM 1.9). Therefore, the assumptions for LEMMA 1.11, where $W(.) = P(t-.)$ and $U = dom(A)$, are satisfied. Hence for $f_t := P(t-.)v(.) : [0,t] \to X$ and $s \in [0,t]$

$$f_t'(s) = \big(P(t-.)v(.)\big)'(s) = P(t-s)v'(s) + P(t-s)'v(s).$$

We know $P(s)'x = AP(s)x$ for $x \in dom(A)$ by THEOREM 1.9, which yields $P(t-s)'x = -AP(t-s)x$ for $x \in dom(A)$. Together with our assumption $v' = Av$ we get

$$f_t'(s) = P(t-s)Av(s) - AP(t-s)v(s) = 0 \in X,$$

since A and $P(r)$ commute for all $r \geqslant 0$ (see THEOREM 1.9). From $f_t' = 0$ and the theory of Riemann integrals of Banach space-valued functions (see: Notation, Definitions and Elementary Results) it follows that f_t is constant. Especially, $f_t(0) = f_t(t)$ and with the definition of f_t we get

$$P(t)v(0) = P(0)v(t).$$

Due to the assumption $v(0) = P(0)c$ and the commutativity of the operators $P(s), s \geqslant 0$ (COM) this leads to

$$P(0)P(t)c = P(0)v(t).$$

Because P(0) is injective, the claim is proven.

■

Now we can easily show a result on pre-semigroups with the same generator :

THEOREM 1.13 *Let $\{P(t)\}_{t\geqslant 0}, \{W(t)\}_{t\geqslant 0}$ be pre-semigroups generated by A. If in addition $P(0) = W(0)$, then $P(t)x = W(t)x$ for all $t \geqslant 0$ and all $x \in dom(A)$.*

PROOF: Let $v(.) := W(.)c$ for $c \in dom(A)$. By LEMMA 1.12, $P(.)c = W(.)c$. Clearly, this is true for all c in $dom(A)$. ■

We see that a generator characterizes the pre-semigroup at least on its domain. Again the image of $P(0)$ plays an important role in the quality of the uniqueness. For a bijective $P(0)$ (as in the semigroup situation) a generator has a unique pre-semigroup because then $dom(A)$ is dense and due to the continuity of the $P(t), W(t)$, we get $P(t) = W(t)$.

Chapter 2

The Abstract Cauchy Problem

In this chapter we concentrate on a main application of semigroups and pre-semigroups. From THEOREM 1.9 we know that $P'(t) = AP(t)$ for a pre-semigroup $\{P(t)\}_{t \geqslant 0}$ generated by A. This can be seen as a motivation for analysing the following type of differential equations.

DEFINITION 2.1 *Let $A : dom(A) \to X$ be an operator and $c \in dom(A)$. Then $u \in C^1([0,\infty)\,;X)$ with $u(t) \in dom(A)$ for all $t \geqslant 0$ is a solution for the* ***Abstract Cauchy Problem ACP****, if:*

$$\frac{d}{dt}u = Au \qquad \text{and} \qquad u(0) = c, \tag{2.1}$$

where $\frac{d}{dt}u$ denotes the strong derivative of u. We denote c as the ***initial value****.*

The following examples are very special cases for X and the operator A. Although, their solutions, which we get from ordinary theory of differential equations, have abilities of (pre-)semigroups.

Example **2.2** Let be $X = \mathbb{R}^n$.
$n = 1$: In this case we have the simple one dimensional differential equation ($A \cong a \in \mathbb{R}$)

$$u' = au, \; u(0) = C.$$

With the solution $u(t) = Ce^{at}$.
$n > 2$: Here we get a linear system of differential equations with the matrix A

$$\frac{d}{dt}u = \left(\frac{d}{dt}u_i\right)_{i=1,\dots,n} = (A_{ij}u_j)_{i=1,\dots,n}$$

The solution is given by the matrix exponential $u(t) = e^{tA}$, where $e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$.

From Theorem (1.9) we get solutions for an ACP through a pre-semigroup:

COROLLARY 2.3 *Let $\{P\,(t)\}_{t \geqslant 0}$ be a pre-semigroup and let A be its generator. For $c \in dom(A)$, $u(.) = P(.)c$ is the unique solution of the ACP,*

$$\frac{d}{dt}u = Au, \; u(0) = P(0)c. \tag{2.2}$$

PROOF: From THEOREM 1.9 point 2. we know that $P(t)c \in dom(A)$ for all $t \geqslant 0$. Point three of this theorem gives us $P(.)c \in C^1([0,\infty), X)$ and

$$\frac{d}{dt}u = Au.$$

Apparently, $u(0) = P(0)c$. The uniqueness of the solution follows directly from LEMMA 1.12. Let $v : [0,\infty) \to dom(A)$ be any further solution of (2.2). Then,

$$P(.)c = v(.),$$

by LEMMA 1.12. ■

***Example* 2.4** For $X = C_0(\mathbb{R})$ and the operator $A : dom(A) \to C_0(\mathbb{R})$, $dom(A) = \{f \in C_0(\mathbb{R}) : (x \mapsto xf(x)) \in C_0(\mathbb{R})\}$, $Af(x) = xf(x)$, we have the following partial differential equation

$$\frac{d}{dt}u(t,x) = x \cdot u(t,x), \qquad u(0,x) = e^{-x^2}g(x), \tag{2.3}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}$ and where $g \in dom(A)$. Actually, in the sense of DEFINITION 2.1, $\frac{d}{dt}u(t,x)$ has to be understood as $\left(\frac{d}{dt}u(t,.)\right)(x)$ where we have the strong derivative in $X = C_0(\mathbb{R})$. Here, clearly, if the strong derivative exists, it equals the partial (pointwise) derivative $\frac{d}{dt}u(t,x)$. Therefore a strong solution (in the sense of DEFINITION 2.1) is also a solution of (2.3). We know already from EXAMPLE 1.1 that A is the generator for the pre-semigroup $\{P(t)\}_{t\geqslant 0}$,

$$P(t)f(x) = e^{-x^2+tx}f(x).$$

Therefore, by COROLLARY 2.3 a solution for (2.3) is given by

$$u(t,x) = e^{-x^2+tx}g(x),$$

where $u(t,.) \in C_0(\mathbb{R})$ for all $t \geqslant 0$. The uniqueness is at least given for the situation of the strong solution.

From COROLLARY 2.3 we get a solution for the ACP implicated by a given pre-semigroup. The initial value is in $P(0)dom(A)$. This solution is unique. In other words, we have a unique solution, if we know the pre-semigroup. Furthermore we are interested in the "other direction": If a function $u = P(.)c$ is a solution of the ACP for an operator A and $c \in dom(A)$, is $P(.)$ a pre-semigroup? The following theorem answers this question for a stiuation with comparatively strong assumptions.

THEOREM 2.5 *For a a closed operator A consider following situation:*

- $\{P(t)\}_{t\geqslant 0}$ *is a family of bounded operators, which is strongly continuous;*
- $P(0)$ *is injective;*
- A *commutes with* $P(s)$ *for* $s \geqslant 0$*;*

- *$P(.)c$ solves ACP (2.2) for all $c \in dom(A)$;*

If either $dom(A)$ is dense in X or the resolvent set of A is non-emtpy, then $P(.)$ is a pre-semigroup generated by an extension of A.

PROOF: It remains to show that $P(t-u)P(u)$ is independent of u for all $0 \leqslant u \leqslant t$. Regard the derivative of $P(t-u)P(u)c$ with respect to u for $c \in dom(A)$. We use LEMMA 1.11 (with the functions $W(.) := P(t-.)$,$v(.) := P(.)c$ and $U := dom(A)$). As $\big(P(t-.)x\big)'(u) = -AP(u)x$ for $x \in dom(A)$ we have

$$\frac{d}{du}P(t-u)P(u)c = P(t-u)AP(u)c - AP(t-u)P(u)c = 0, \tag{2.4}$$

for all $0 \leqslant u \leqslant t$ since A and $P(t-u)$ commute by assumption. Hence $P(t-u)P(u)c = P(t)P(0)c$ for all $c \in dom(A)$ and for all $u \geqslant t$. Let $dom(A)$ be dense in X. Because $P(t-u)P(u)$ and $P(t)P(0)$ are continuous and coincide on the dense set $dom(A)$, they coincide on X. Now consider the situation where the resolvent set of A is not empty. Let λ be an element in $\rho(A)$. Regard the resolvent $R_{\lambda,A} = (\lambda I - A)^{-1} : X \to dom(A)$. Because of the injectivity of $R_{\lambda,A}$, it suffices to show that

$$R_{\lambda,A}P(t-u)P(u)x = R_{\lambda,A}P(0)P(t)x \qquad \forall x \in X$$

Because of the assumption $P(.)Ax = AP(.)x$ for all $x \in dom(A)$, $R_{\lambda,A}$ commutes with the operators $P(s)$, $s \in [0,\infty)$, i.e.

$$R_{\lambda,A}P(s)x = P(s)R_{\lambda,A}x \qquad \forall x \in X.$$

(see LEMMA 0.1 in Notation, Definitions and Elementary Results). By (2.4) and $R_{\lambda,A}x \in dom(A)$, $P(t-u)P(u)R_{\lambda,A}x$ is constant with respect to u. This yields

$$\begin{aligned} R_{\lambda,A}P(t-u)P(u)x &= P(t-u)P(u)R_{\lambda,A}x \\ &= P(0)P(t)R_{\lambda,A}x \\ &= R_{\lambda,A}xP(0)P(t), \end{aligned}$$

which proves the present case. Denote the generator of the pre-semigroup $P(t)_{t\geqslant 0}$ by A_P. Since $P(.)x$ is solution of the ACP for $x \in dom(A)$, $P'^{+}(0)x$ exists and

$$P'^{+}(0) = \frac{d}{dt}(P(.)x)(0) = AP(0)x = P(0)Ax \in P(0)X,$$

where the last equality follows from the assumption that A commutes with $P(.)$. We obtain $dom(A_p) \supset dom(A)$. From definition of A_P for $x \in dom(A)$ we get

$$P(0)A_Px = P'^{+}(0)x = [P(.)x]'(0) = AP(0)x = P(0)Ax,$$

which verifies that A_P is an extension of A, because $P(0)$ is injective. ■

***Example* 2.6** Let us again consider the family of operators from EXAMPLE 1.1. We know already that $\{P(t)\}_{t\geqslant 0}$ is a pre-semigroup, but, as an example, we want to use THEOREM 2.5 to proof that it is a pre-semigroup. Therefore, we show that the image of $P(0)$ lies dense in $C_0(\mathbb{R})$.
Regard a function $g \in C_{00}(\mathbb{R})$ with compact support $K \subset \mathbb{R}$. Since $(x \mapsto e^{x^2})$ is bounded on K, $(x \mapsto e^{x^2}g(x))$ is also in $C_{00}(\mathbb{R})$, in particular in $C_0(\mathbb{R})$. Therefore, g has the form

$$g(x) = e^{-x^2}(e^{x^2}g(x))$$

Thus $g \in P(0)C_0(\mathbb{R})$. Hence, $C_{00}(\mathbb{R})$ lies in the image of $P(0)$. Since $C_{00}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, $P(0)C_0(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

REMARK 2.7 In the last theorem, the condition "resolvent set $\rho(A)$ non-empty" can be weakened. Like in [deL94] one can define a special resolvent set for A by taking $P(0)$ into consideration. $\rho_{P(0)}(A)$ is defined as the set of all $\lambda \in \mathbb{C}$ with $(\lambda I - A)$ is injective and $P(0)X \subseteq (\lambda I - A)[dom(A)]$. Clearly, $\rho(A)$ is a subset of $\rho_{P(0)}(A)$. This resolvent set is not only in this context the more natural one, since there is an explicit connection to the operator $P(0)$. Note that for bijective $P(0)$, $\rho_{P(0)}(A)$ is the usual resolvent set.

The last theorems do not answer the question "When do we get a pre-semigroup solution?". Basicly we can not even expect to get such a solution since A can not be expected to be a generator of a pre-semigroup without further information. For that we introduce an operator B which is defined on a subset of $dom(A)$. This B has some abilities that guarantee the existence of a pre-semigroup that is generated by an extension of A.

THEOREM 2.8 *Let A,B be closed operators, which are related as follows.*

- $dom(B) \subset dom(A)$;
- $0 \in \rho(B)$
- $\exists \lambda \in \rho(A), \lambda > 0 : \; R_{\lambda,A}Bx = BR_{\lambda,A}x \; \forall x \in dom(B)$

Then following assertions are equivalent:

1. *The ACP for A has a unique solution for each initial value $c \in dom(B)$.*
2. *There exists a pre-semigroup $\{P(t)\}_{t\geqslant 0}$ generated by an extension A_P of A, such that $P(0) = (\lambda I - A)B^{-1}$ and A commutes with $P(s)$ for all $s \geqslant 0$.*

PROOF: Notice that $B^{-1} = R_{0,B}$ exists since $0 \in \rho(B)$ by assumption.
1. $\Rightarrow$ 2.
Let $u_c \in C^1([0,\infty);X)$ be the unique solution of the ACP for the initial value $c \in dom(B)$. We have to construct a pre-semigroup $P(.)$. For $x \in X$, $B^{-1}x \in dom(B)$ and hence the expression

$$P(.)x := (\lambda I - A)u_{B^{-1}x}(.) = \lambda u_{B^{-1}x}(.) - u'_{B^{-1}x}(.), \tag{2.5}$$

is well defined. We are going to show that $P(.)$ is a pre-semigroup which is generated by an extension of A. This includes following tasks:

- $P(.)$ is strongly continuous;
  This follows from the term on the right side in (2.5), since $u_{B^{-1}x}(.) \in C^1([0,\infty))$.

- $P(0)$ is injective,
  because $P(0)x = (\lambda I - A)B^{-1}x$ and $\lambda \in \rho(A)$.

- $P(t) \in \mathcal{B}(X)$ for all $t \in [0,\infty)$;
  $P(t) : X \to X$ is linear, due to $B^{-1} \in \mathcal{B}(X)$ and the uniqueness of the solution of the ACP for a given initial value: Let $x, y \in X$ and $k \in \mathbb{C}$
  (it is clear that $u_{B^{-1}x} + u_{B^{-1}ky}$ solves the ACP for $u(0) = B^{-1}x + kB^{-1}y$)

$$\begin{aligned} P(t)(x+ky) &= (\lambda I - A)u_{B^{-1}x+B^{-1}ky}(t) \\ &= (\lambda I - A)\big(u_{B^{-1}x}(t) + ku_{B^{-1}y}(t)\big) \\ &= P(t)x + kP(t)y. \end{aligned}$$

For the boundedness of $P(t)$ we regard the operator

$$W(.) : X \to C^1([0,a]\,;X) : x \mapsto W(.)x = u_{B^{-1}x}(.),$$

and $a \geqslant 0$ fixed. Here, $C^1([0,a]\,;X)$ is equipped with the norm $\|u\|_{C^1} = \|u(t)\|_\infty + \|u'(t)\|_\infty$ (see: Notation, Definitions & Elementary Results).
We show that W(.) is closed. Let $x_n \to x$ in X, and $W(.)x_n \to y$ in $C^1([0,a]\,;X)$. From the definition of $W(.)$ and the convergence of $W(.)x_n$ in the $\|.\|_{C^1}$-Norm (this implies pointwise convergence) it follows for fixed $t \leqslant a$ that

$$AW(t)x_n = Au_{B^{-1}x_n}(t) = u'_{B^{-1}x_n}(t) = [W(.)x_n]'(t) \to y'(t).$$

We now regard the sequences $W(t)x_n \to y(t)$ and $AW(t)x_n \to y'(t)$. Using the fact that A is closed, we get $y(t) \in dom(A)$ and $y'(t) = Ay(t)$. Furthermore,

$$y(0) = \lim_{n\to\infty} W(0)x_n = \lim_{n\to\infty} u_{B^{-1}x_n}(0) = \lim_{n\to\infty} B^{-1}x_n = B^{-1}x.$$

The uniqueness of the solution of the ACP with the initial value $B^{-1}x \in dom(B)$ yields $y = u_{B^{-1}x} = W(.)x$ on $[0,a]$ (the second equality holds per definitionem). So W(.) is closed. As $C^1([0,a]\,;X)$ is a Banach space, by the Closed Graph Theorem W(.) is even bounded, i.e. $\exists M > 0$: $\|W(.)x\|_{C^1} \leqslant M\,\|x\|\ \forall x \in X$. Since $\lambda > 0$ it follows from (2.5) for a fixed $t \leqslant a$ that

$$\begin{aligned} \|P(t)x\| = \|(\lambda I - A)W(t)x\| &\leqslant \|\lambda W(t)x\| + \|[W(.)x]'(t)\| \\ &\leqslant (\lambda+1)\big(\,\|W(t)x\| + \|[W(.)x]'(t)\|\,\big) \\ &\leqslant (\lambda+1)\,\|W(.)x\|_{C^1} \\ &\leqslant (\lambda+1)M\,\|x\|\,. \end{aligned}$$

Since a can be chosen big enough for each t, so that $t \leqslant a$, $P(t)$ is in $\mathcal{B}(X)$ for all $t \in [0,\infty)$.

- $AP(t)x = P(t)Ax$ for all $x \in dom(A)$ and $t \in [0, \infty)$;

$$\begin{aligned} P(t)Ax &= AP(t)x \qquad \forall x \in dom(A) && (2.6)\\ \Leftrightarrow P(t)(\lambda I - A)x &= (\lambda I - A)P(t)x \qquad \forall x \in dom(A) && (2.7)\\ \Leftrightarrow R_{\lambda,A}P(t)c &= P(t)R_{\lambda,A}c, \qquad \forall c \in X && (2.8)\end{aligned}$$

it suffices to show (2.8). For that we consider $y(.) := R_{\lambda,A}u_{B^{-1}c}(.)$ for $c \in X$. Then (using that an operator commutes with its resolvent, see LEMMA 0.1)

$$\frac{d}{dt}y = R_{\lambda,A}\frac{d}{dt}u_{B^{-1}c} = R_{\lambda,A}Au_{B^{-1}}c = Ay.$$

Because $BR_{\lambda,A}d = R_{\lambda,A}Bd$ for all $d \in dom(B)$ by assumption, it follows

$$\begin{aligned} BR_{\lambda,A}d =& R_{\lambda,A}Bd \quad \forall d \in dom(B) && (2.9)\\ \Leftrightarrow R_{\lambda,A}d =& B^{-1}R_{\lambda,A}Bd \quad \forall d \in dom(B) && (2.10)\\ \Leftrightarrow R_{\lambda,A}B^{-1}c =& B^{-1}R_{\lambda,A}c \quad \forall c \in X. && (2.11)\end{aligned}$$

Hence,

$$y(0) = R_{\lambda,A}B^{-1}c = B^{-1}R_{\lambda,A}c \in dom(B).$$

So y solves ACP with initial value $R_{\lambda,A}B^{-1}c$. By uniqueness of the solution, it follows

$$R_{\lambda,A}u_{B^{-1}c}(.) = u_{R_{\lambda,A}B^{-1}c}(.). \qquad (2.12)$$

Now we prove (2.8). Using the definition of $P(t)x$ in (2.5), (2.12) and the commutativity of the resolvents (2.11), we get

$$\begin{aligned} R_{\lambda,A}P(t)x = u_{B^{-1}x}(t) &= (\lambda I - A)R_{\lambda,A}u_{B^{-1}x}(t)\\ &= (\lambda I - A)u_{R_{\lambda,A}B^{-1}x}(t)\\ &= (\lambda I - A)u_{B^{-1}R_{\lambda,A}x}(t)\\ &= P(t)R_{\lambda,A}x.\end{aligned}$$

- $P(t-u)P(u)$ is indepent of u for $0 \leqslant u \leqslant t$ and $P(.)$ is generated by an extension of A; Let $c \in dom(A)$. Clearly, we can write $c = R_{\lambda,A}d$ for a certain $d \in X$. With (2.11) and (2.12) it follows that

$$\begin{aligned} P(.)c &= (\lambda I - A)u_{B^{-1}c}(.)\\ &= (\lambda I - A)u_{B^{-1}R_{\lambda,A}d}(.)\\ &= (\lambda I - A)u_{R_{\lambda,A}B^{-1}d}(.)\\ &= u_{B^{-1}d}(.).\end{aligned}$$

Hence, $P(.)c$ solves the ACP since the initial value $P(0)c = B^{-1}d \in dom(B)$ for each $c \in dom(A)$. So together with the points above, the conditions of theorem (2.5) are fulfilled ($\rho(A)$ is non-empty by assumption). This theorem completes the proof of this direction.

2. $\Rightarrow$ 1.
Let $P(.)$ be a pre-semigroup generated by an extension A_P of A, with $P(0) = (\lambda I - A)B^{-1}$ and Furthermore, let A commute with $P(s)$ for all $s \geqslant 0$. We want to show that the ACP

$$u' = Au; \quad u(0) = c, \tag{2.13}$$

has a unique solution for each $c \in dom(B)$. From COROLLARY 2.3 it is clear that for $d \in dom(A) \subseteq dom(A_P)$ the function $P(.)d$ is the unique solution of the ACP

$$u' = A_P u; \quad u(0) = P(0)d.$$

As $d \in dom(A)$ we have $Ad = A_P d$. A_P commutes with $P(s)$ for all $s \geqslant 0$ by THEOREM 1.9. Because A commutes with $P(s)$ for all $s \geqslant 0$ by assumption, it follows that $AP(.)d = A_P P(.)d$. Hence, $P(.)d$ solves ACP (2.13) with initial value $c := P(0)d$ uniquely. Since $dom(A) = R_{\lambda,A}X$ and $R_{\lambda,A}$ commutes with the operators $P(s), s \geqslant 0$ (see LEMMA 0.1 in Notation, Definitions and Elementary Results), it follows from $P(0) = (\lambda I - A)B^{-1}$ that

$$P(0)dom(A) = P(0)R_{\lambda,A}X = R_{\lambda,A}P(0)X = R_{\lambda,A}(\lambda I - A)B^{-1}X = dom(B),$$

which shows $c \in dom(B)$. Hence, for a given $c \in dom(B)$, $P(0)^{-1}c \in dom(A)$ and $u = P(.)P(0)^{-1}c$ is the unique solution of (2.13).
■

Chapter 3

Exponentially tamed pre-semigroups

In the following we will analyse pre-semigroups with an additional property. This reduction will give us more power in creating a similar situation as there is in the theory of strongly continuous semigroups.

DEFINITION 3.1 *A pre-semigroup $\{P(t)\}_{t\geqslant 0}$ is **exponentially tamed**, if there exists $\omega > 0$ so that*

$$f_x : [0,\infty) \to X : t \mapsto e^{-\omega t}P(t)x,$$

is bounded and uniformly continuous for all $x \in X$.

In the theory of common semigroups such a relation emerges as a property of strongly continuous semigroups. There we have constants $M, a > 0$ so that $\|P(t)\| \leqslant Me^{at}$ for all $t \in [0,\infty)$. Using this (and properties of a semigroup) we see

$$\begin{aligned} \left\|e^{-a(t+h)}P(t+h)x - e^{-at}P(t)x\right\| &\leqslant e^{-at}\|P(t)\|\left\|e^{-ah}P(h)x - x\right\| \\ &\leqslant M\left\|e^{-ah}P(h)x - x\right\| \quad \forall t \in [0,\infty), x \in X, \end{aligned}$$

which implies the uniform continuity of $t \mapsto e^{-\omega t}P(t)x$, where $\omega = a$. Concerning the boundedness we have

$$\left\|e^{-at}P(t)x\right\| \leqslant e^{-at}\|P(t)\|\|x\| \leqslant e^{-at}Me^{at}\|x\| = M\|x\|,$$

for all $t \in [0,\infty)$ and each fixed $x \in X$. Therefore DEFINITION 3.1 is also a generalisation of the situation of a strongly continuous semigroup.

REMARK 3.2 We want to point out that for a pre-semigroup which is exponentially tamed, $M := \sup_{t\geqslant 0} e^{-\omega t}\|P(t)\|$ exists. This follows directly from the principle of uniform boundedness theorem, since $\sup_{t\geqslant 0}\|e^{-\omega t}P(t)x\| \leqslant M_x$ for all $x \in X$ by definition.

DEFINITION 3.3 *For an exponentially-tamed pre-semigroup $\{P(t)\}_{t\geqslant 0}$, let Y be the vector space*

$$Y := \left\{x \in X : f_x(t) \in P(0)X \quad \forall t \geqslant 0,\ P(0)^{-1}f_x \in C_b([0,\infty);X)\right\},$$

normed by $\|x\|_Y := \|P(0)^{-1}f_x\|_b = sup_{t\geqslant 0}\|e^{-\omega t}P(0)^{-1}P(t)x\|$.

REMARK 3.4 Y is clearly a vector space because of linearity (in x) of $P(0)^{-1}f_x$ and since $C_b([0,\infty);X)$ is a vector space.
An element of Y has to fulfill two strong conditions concerning the operator $P(0)^{-1}$. First of all $e^{-\omega t}P(t)x$ has to be in $P(0)X$ for all $t \geqslant 0$ so that the term is well-defined. Further $P(0)^{-1}$ has to support uniform continuity and boundedness of $f_x : [0,\infty) \to X : t \mapsto e^{-\omega t}P(t)x$. At this point it is not clear how strong these requests are, and how big this restriction for x in X is. We will analyse this later on.
As $\|x\| = \|P(0)^{-1}e^0 P(0)x\| \leqslant \sup_{t\geqslant 0} \|P(0)^{-1}e^{-\omega t}P(t)x\|$ clearly $\|.\| \leqslant \|.\|_Y$ on Y.

The fact that $C_b([0,\infty);X)$ is a Banach space, gives us even more.

LEMMA 3.5 *Let $\{P(t)\}_{t\geqslant 0}$ be a exponentially-tamed pre-semigroup. Then the normed space $(Y, \|.\|_Y)$ is a Banach space.*

PROOF: Let $\{x_n\}$ be a Cauchy sequence in $(Y,\|.\|_Y$. From $\|.\| \leqslant \|.\|_Y$ it follows that $\{x_n\}$ is also Cauchy in X and hence has a limit $x \in X$.
From definition of $(Y,\|.\|_Y)$ we know that the sequence of functions $P(0)^{-1}f_{x_n}$ is Cauchy in $C_b([0,\infty);X)$, hence converges to $g \in C_b([0,\infty);X)$, i.e. $P(0)^{-1}f_{x_n} \overset{C_b}{\to} g$. This convergence (in $\|.\|_{C_b}$) especially implies pointwise convergence in X, i.e. $P(0)^{-1}f_{x_n}(t) \overset{X}{\to} g(t)$ for all fixed $t \geqslant 0$. By continuity of $P(0)$ we get

$$P(0)P(0)^{-1}f_{x_n}(t) = f_{x_n}(t) = e^{-\omega t}P(t)x_n \overset{X}{\to} P(0)g(t).$$

Since $P(t)$ (for fixed $t \geqslant 0$) is continuous, $e^{-\omega t}P(t)$ is continuous and therefore (with $x_n \overset{X}{\to} x$, as mentioned above)

$$f_{x_n}(t) \overset{X}{\to} e^{-\omega t}P(t)x = f_x(t) = P(0)g(t).$$

So $P(0)^{-1}e^{-\omega t}P(t)x = P(0)^{-1}f_x(t) = g(t)$ for all $t \geqslant 0$, hence $P(0)^{-1}f_x \in C_b([0,\infty);X)$ and $P(0)^{-1}f_{x_n} \overset{C_b}{\to} P(0)^{-1}f_x$. Therefore x_n converges to x in $\|.\|_Y$ and $x \in Y$. ■

Our target is to construct a strongly continuous semigroup on this dedicated space Y, where we want the generator of the semigroup to correspond to the generator A of the given pre-semigroup. In this context the phrase *part of A in Y* will be used (see: Notation, Definitions and Elementary Results). Before, we state a lemma concerning the Laplace transform of a pre-semigroup.

LEMMA 3.6 *For a given exponentially-tamed pre-semigroup $\{P(t)\}_{t\geqslant 0}$ with generator A, the integral*

$$L_P(\lambda)x := \int_0^\infty e^{-\lambda t}P(t)x\ dt, \tag{3.1}$$

*exists for $\lambda > \omega$ and $x \in X$. The **Laplace transform** $L_P : (\omega,\infty) \to \mathcal{B}(X) : \lambda \mapsto L(\lambda)$ satisfies*

$$L_P(\lambda)\big[(\lambda I - A)x\big] = P(0)x, \tag{3.2}$$

for $\lambda > \omega$ and $x \in dom(A)$. In particular, if $\{P(t)\}_{t\geqslant 0}$ is semigroup, then $(\omega,\infty) \subset \rho(A)$.

PROOF: Because of REMARK 3.2 the following holds true

$$\left\|e^{-\lambda t}P(t)x\right\| \leqslant e^{-\lambda t}\left\|P(t)\right\|\left\|x\right\| \leqslant e^{-\lambda t}Me^{\omega t}\left\|x\right\| = M\left\|x\right\|e^{(\omega-\lambda)t},$$

for $x \in X, t \geqslant 0$. Therefore the integral $\int_0^\infty \left\|e^{-\lambda t}P(t)x\right\| dt$ exists for $\lambda > \omega$ since $\int_0^\infty M\left\|x\right\|e^{(\omega-\lambda)t}dt = \frac{M}{\lambda-\omega}\left\|x\right\|$ clearly exists. Hence and since $(t \mapsto e^{-\lambda t}P(t)x)$ is continuous, (3.1) exists for $\lambda > \omega$ (see: Notation, Definitions and Elementary Results).
Further we show that $L_P(\lambda)$ is in $\mathcal{B}(X)$ for $\lambda > \omega$. Linearity is trivial from the definition of the integral. As seen above, we have

$$\begin{aligned}\left\|L_P(\lambda)\right\| &= \left\|\int_0^\infty e^{-\lambda t}P(t)x\ dt\right\| \\ &\leqslant \int_0^\infty \left\|e^{-\lambda t}P(t)x\right\|\ dt \\ &\leqslant \int_0^\infty M\left\|x\right\|e^{(\omega-\lambda)t}\ dt \\ &= \frac{M}{\lambda-\omega}\left\|x\right\|,\end{aligned}$$

which gives us the boundedness.
Let $\lambda > \omega$ and $x \in dom(A)$. Consider

$$\begin{aligned}L_P(\lambda)(\lambda I - A)x &= \int_0^\infty e^{-\lambda t}P(t)(\lambda I - A)x\ dt \\ &= \int_0^\infty [\lambda e^{-\lambda t}P(t)x - e^{-\lambda t}AP(t)x]\ dt,\end{aligned}$$

where we use that A and $P(t)$ commute. With the product rule, LEMMA 1.11, we see that the integrand equals $-[e^{-\lambda t}P(t)x]'$. This yields

$$\begin{aligned}L_P(\lambda)(\lambda I - A)x &= -\int_0^\infty [e^{-\lambda t}P(t)x]'\ dt \\ &= -\left[e^{-\lambda t}P(t)x\right]\Big|_0^\infty \\ &= P(0)x.\end{aligned}$$

Let $x \in X$. For the term $(\lambda I - A)L_P(\lambda)x$ we consider the strong right derivative of $P(.)[L_P(\lambda)x]$ at zero. With $P(s) \in \mathcal{B}(X), s \geqslant 0$, (COM) and (ADD) it follows,

$$\begin{aligned}\frac{1}{h}\big(P(h) - P(0)\big)[L_P(\lambda)x] &= \frac{1}{h}\big(P(h) - P(0)\big)\int_0^\infty e^{-\lambda t}P(t)x\ dt \\ &= \frac{1}{h}\Big(\int_0^\infty e^{-\lambda t}[P(0)P(h+t) - P(0)P(t)]x\ dt \\ &= \frac{1}{h}\int_h^\infty e^{-\lambda(s-h)}P(0)P(s)x\ ds - \frac{1}{h}\int_0^\infty e^{-\lambda t}P(0)P(t)x\ dt \\ &= \frac{e^{h\lambda}-1}{h}\int_0^\infty e^{-\lambda s}P(0)P(s)x\ ds - \frac{e^{h\lambda}}{h}\int_0^h e^{-\lambda s}P(0)P(s)x\ ds \\ &= \frac{e^{h\lambda}-1}{h}P(0)\int_0^\infty e^{-\lambda s}P(s)x\ ds - e^{h\lambda}\frac{1}{h}P(0)\int_0^h e^{-\lambda s}P(s)x\ ds,\end{aligned}$$

where the last equality holds since $P(0)$ is continuous and since

$$\int_0^\infty e^{-\lambda s} P(s)x \, ds = \lim_{\beta \to \infty} \int_0^\beta e^{-\lambda s} P(s)x \, ds$$

exists (as showed in beginning of this proof). Consider the limit of the first term in the last equality,

$$\begin{aligned} \lim_{h \to 0^+} \frac{e^{h\lambda} - 1}{h} P(0) \int_0^\infty e^{-\lambda s} P(s)x \, ds &= \lim_{h \to 0^+} \frac{e^{h\lambda} - 1}{h} P(0) L_P(\lambda)x \\ &= \lambda P(0) L_P(\lambda)x. \end{aligned}$$

Since $(s \mapsto e^{-\lambda s} P(s)x)$ is continuous, it follows from the fundamental theorem of calculus, that

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h e^{-\lambda s} P(s)x \, ds = P(0)x.$$

Therefore, the limit of the second term is

$$\lim_{h \to 0^+} \frac{e^{h\lambda}}{h} P(0) \int_0^h e^{-\lambda s} P(s)x \, ds = P(0)P(0)x.$$

Hence,

$$P'^{+}(0)[L_P(\lambda)x] = \lambda P(0) L_P(\lambda)x - P(0)P(0)x.$$

Therefore the strong right derivative exists and belongs to $P(0)X$. Thus

$$A L_P(\lambda)x = \lambda L_P(\lambda)x - P(0)x.$$

This gives us

$$(\lambda I - A) L_P(\lambda)x = P(0)x,$$

for $x \in X$. If $\{P(t)\}_{t \geqslant 0}$ is even a semigroup, hence especially $P(0) = I$. Therefore, we have, with the calculation from above,

$$(\lambda I - A) L_P(\lambda)x = L_P(\lambda)(\lambda I - A)x = x,$$

for $x \in dom(A)$. Hence $(\omega, \infty) \subset \rho(A)$. ■

THEOREM 3.7 *Let $\{P(t)\}_{\geqslant 0}$ be a exponentially-tamed pre-semigoup with generator A. Then, $\{T(t)\}_{t \geqslant 0}$ with*

$$T(.) := P(0)^{-1} P(.),$$

is a strongly continuous semigroup on the Banach space $(Y, \|.\|_Y)$ which satisfies

$$\|T(t)\|_{\mathcal{B}(Y)} \leqslant e^{\omega t},$$

for all $t \geqslant 0$. The generator of $\{T(t)\}_{t \geqslant 0}$ equals the part of A in Y, A_Y.

PROOF: Let x be in Y and fix $s \geqslant 0$. Clearly, $P(0)^{-1}e^{-\omega t}P(s)x$ exists by definition of Y, hence $T(s)x = P(0)^{-1}P(s)x$ exists by linearity. Further, we have to show that $T(s)x$ is in Y, which is $P(0)^{-1}f_{T(s)x} \in C_b([0\infty); X)$. By definition, for $t \geqslant 0$ we have

$$\begin{aligned} P(0)^{-1}f_{T(s)x}(t) &= P(0)^{-1}e^{-\omega t}P(t)T(s)x \\ &= P(0)^{-1}e^{-\omega t}P(t)P(0)^{-1}P(s)x. \end{aligned}$$

$P(t)P(0)^{-1}P(s)x$ exists by the argument from above. $P(0)^{-1}P(t)P(s)x$ is equal to $P(0)^{-1}P(0)P(t+s)x = P(t+s)x$ by the properties of pre-semigroups (ADD) and therefore exists, too. Hence and because of commutativity (COM) of the operators $P(s)$, $s \in [0,\infty)$, $P(t)$ and $P(0)^{-1}$ commute (see: Notation, Definitions and Elementary Results: LEMMA 0.2) and so

$$P(t)P(0)^{-1}P(s)x = P(t+s)x. \tag{3.3}$$

This gives

$$P(0)^{-1}f_{T(s)x}(t) = P(0)^{-1}e^{-\omega t}P(t+s)x = e^{-\omega s}P(0)^{-1}e^{-\omega(t+s)}P(t+s)x.$$

Hence $P(0)^{-1}f_{T(s)x}$ belongs to $C_b([0,\infty); X)$ since $t \mapsto P(0)^{-1}e^{-\omega t}P(t)x \in C_b([0\infty); X)$ (s is fixed). So we have shown that $T(s)x$ is in Y. As a composition of linear operators, $T(s) : Y \to Y$ is also linear. The boundedness of $T(s) : Y \to Y$ is proved as follows. By definition, properties of pre-semigroups and (3.3) we get

$$\begin{aligned} \|T(s)x\|_Y &= \sup_{t\geqslant 0} \left\|P(0)^{-1}e^{-\omega t}P(t)P(0)^{-1}P(s)x\right\| \\ &\leqslant \sup_{t\geqslant 0} \left\|P(0)^{-1}e^{-\omega t}P(t+s)x\right\| \\ &= e^{\omega s}\sup_{t\geqslant 0} \left\|P(0)^{-1}e^{-\omega(t+s)}P(t+s)x\right\| \\ &\leqslant e^{\omega s}\|x\|_Y . \end{aligned}$$

This also shows $\|T(t)\|_{\mathcal{B}(Y)} \leqslant e^{\omega t}$ for $t \geqslant 0$.
The semigroup properties are obviously satisfied by definition: $T(0) = P(0)^{-1}P(0) = I$ and for $x \in Y, s, t \geqslant 0$

$$\begin{aligned} T(s+t)x &= P(0)^{-1}P(s+t)x \\ &= P(0)^{-1}P(0)^{-1}P(s)P(t)x \\ &= P(0)^{-1}P(s)P(0)^{-1}P(t)x \\ &= T(s)T(t)x, \end{aligned}$$

where we use that $P(s)$ and $P(0)^{-1}$ commute (as above, see: Notation, Definitions and Elementary Results: LEMMA 0.2).
Next, we verify the C_0-property. It suffices to show the strong continuity of $T(.)$ at zero, clearly in the $\|.\|_Y$-norm. For that we consider the uniform continuity of $P(0)^{-1}f_x$ for $x \in Y$. For $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that for all $|h| < \delta_\epsilon$

$$\sup_{t\geqslant 0} \left\|P(0)^{-1}e^{-\omega(t+h)}P(t+h)x - P(0)^{-1}e^{-\omega t}P(t)x\right\| < \epsilon \tag{3.4}$$

With the definition and (3.3), it follows elemtarily

$$\begin{aligned}\|T(h)x-x\|_Y =& \sup_{t\geqslant 0}\left\|P(0)^{-1}e^{-\omega t}P(t)[P(0)^{-1}P(h)x-x]\right\| \\ =& \sup_{t\geqslant 0}\left\|P(0)^{-1}e^{-\omega t}P(t+h)x-P(0)^{-1}e^{-\omega t}P(t)x\right\| \\ \leqslant& \sup_{t\geqslant 0}e^{\omega h}\left\|P(0)^{-1}e^{-\omega(t+h)}P(t+h)x-P(0)^{-1}e^{-\omega t}P(t)x\right\| + \\ &+\sup_{t\geqslant 0}\left\|(e^{\omega h}-1)P(0)^{-1}e^{-\omega t}P(t)x\right\|.\end{aligned}$$

From (3.4) we get

$$\|T(h)x-x\|_Y \leqslant e^{\omega h}\epsilon + (e^{\omega h}-1)\,\|x\|_Y\,.$$

This gives us strong continuity: For fixed $x \in Y$ and a given ϵ' choose $\epsilon < \epsilon'$ in (3.4). Clearly, $e^{\omega h}$ converges to 1 for $h \to 0$, hence $e^{\omega h}\epsilon \to \epsilon$ and $(e^{\omega h}-1)\,\|x\|_Y \to 0$ and therefore $\|T(h)x-x\|_Y < \epsilon'$ holds true for sufficiently small h.
It remains to show that the generator A_T of the semigroup $T(.)$ equals, the part A_Y of A in Y. Let $x \in dom(A_T) \subseteq Y$, then $\lim_{t\to 0}\frac{1}{h}[T(h)x-x]$ exists in $\|.\|_Y$, hence in $\|.\|$ due to the fact that $\|.\|_Y \geqslant \|.\|$. By definition of T we have $P(.) = P(0)T(.)$. $P(0)$ is in $\mathcal{B}(X)$ and, therefore, (following limits are in $\|.\|$)

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}[P(h)x-P(0)x] =& \lim_{h\to 0}P(0)\frac{1}{h}[T(h)x-x] \\ =& P(0)\lim_{t\to 0}\frac{1}{h}[T(h)x-x] \\ =& P(0)A_Tx.\end{aligned}$$

Hence $P'^{+}(0)$ exists and lies in P(0)X. So $x \in dom(A)$ and $Ax = P(0)^{-1}P(0)A_Tx = A_Tx$. Thus $A_T \subset A$, and even $A_T \subset A_Y$ since A_T is defined on a subset of Y.

We regard the operator $(\lambda I - A)$ for an apropriate $\lambda \in \mathbb{R}$: From LEMMA 3.6 we know that for $\lambda > \omega$ the Laplace transform $L_P(\lambda) \in \mathcal{B}(X)$ exists. The relation (3.2), $L_P(\lambda)(\lambda I - A)x = P(0)x$, implies that if $(\lambda I - A)x = 0$, then $P(0)x = 0$ and therefore $x = 0$ since $P(0)$ is injective. Consequently $(\lambda I - A)$ is injective, hence $(\lambda I - A_Y)$ is also injective.
LEMMA 3.6 can also be applied with the semigroup $T(.)$ (on Y). Thus $\lambda > \omega$ is in $\rho(A_T)$ which means that $(\lambda I - A_T)^{-1} : X \to dom(A_T)$ is bijective. In particular $(\lambda I - A_T)$ is surjective. We know already $A_T \subset A_Y$, hence

$$\lambda I - A_T \subset \lambda I - A_Y,$$

where the map on the left hand side is surjective and injective on the right hand side. Such a relation implies already that

$$\lambda I - A_T = \lambda I - A_Y,$$

(see: Notation, Definitions and Elementary Results LEMMA 0.3). Obviously this is equivalent to

$$A_T = A_Y,$$

which completes the proof. ■

REMARK 3.8 This result is surprising in a way. But how big is the space Y?! We have already mentioned in REMARK 3.4 that the elements of Y have to fulfill some strong requirements and therefore depending especially on the operator $P(0)$. Let us consider the special case where $P(0)$ is bijective. By the Closed Graph theorem, $P(0)^{-1} \in \mathcal{B}(X)$ and therefore $P(0)^{-1} f_x$ is clearly bounded and uniformly continuous for all $x \in X$ since $f_x \in C_b([0, \infty); X)$ by definition. That is, $Y = X$.

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