A Counterexample to Banach's Basis Problem

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Introduction

Due to the restriction to (finite) linear combinations classical vector space bases are not always suitable for the analysis of infinite dimensional spaces. Therefore, it is natural in some way to consider generalized basis concepts.

Note, that all vector spaces in this paper are spaces over the field $\mathbb{F}$, where $\mathbb{F}$ denotes the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Whenever reference is made to some topological property, the norm topology is implied.

Definition 0.1 A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space $(X, \|.\|_X)$ is a called a *Schauder basis* of X if for every $x \in X$ there exists a unique sequence $(\alpha_n)_{n\in\mathbb{N}}$ of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$, i.e. such that $\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} \alpha_n x_n \right\|_X = 0$.

Throughout this paper we make the convention that a basis for a Banach space shall be a Schauder basis, unless explicit reference is made to a vector space basis.

Propostion 0.2 *Suppose that $(x_n)_{n\in\mathbb{N}}$ is a basis for a Banach space X. Then $(x_n)_{n\in\mathbb{N}}$ is linearly independent. In particular, every banach space with a basis is infinite dimensional.*

Proof: Suppose that an element x of X could be written in two different ways as a finite linear combination of the terms of $(x_n)_{n\in\mathbb{N}}$, i.e. for $n, m \in \mathbb{N}, (\alpha_i)_{i=1}^{n} \in \mathbb{F}^n, (\beta_i)_{i=1}^{m} \in \mathbb{F}^m$ satisfying $\alpha_n \neq 0 \neq \beta_m$ and $(\alpha)_{i=1}^{n} \neq (\beta_i)_{i=1}^{m}$ we had $x = \sum_{i=1}^{n} \alpha_i x_i = \sum_{j=1}^{m} \beta_i x_i$. Then we had $\sum_{i=1}^{\infty} \tilde{\alpha}_i x_i = \sum_{i=1}^{\infty} \tilde{\beta}_i x_i$, for $(\tilde{\alpha}_i)_{i\in\mathbb{N}} := (\alpha_1, \dots, \alpha_n, 0, 0, \dots)$ and $(\tilde{\beta}_i)_{i\in\mathbb{N}} := (\beta_1, \dots, \beta_m, 0, 0, \dots)$. This is a contradiction to the uniqueness of expansions of vectors in terms of a basis in X.

□

1 Coordinate Functionals

Our first aim is to show that the continuity of the coordinate functionals - a property that has been required in Schauder's original defintion - follows from the rest of Definition 0.1.

Definition 1.1 Let X be a Banach space with basis $(x_n)_{n\in\mathbb{N}}$. For each $m \in \mathbb{N}$ the maps $x_m^* : X \to \mathbb{F} : \sum_{n=1}^{\infty} \alpha_n x_n \mapsto \alpha_m$ and $P_m : X \to X : \sum_{n=1}^{\infty} \alpha_n x_n \mapsto \sum_{n=1}^{m} \alpha_n x_n$ are called the m^{th} *coordinate functional* and the m^{th} *natural projection* associated with $(x_n)_{n\in\mathbb{N}}$, respectively.

Remark 1.2 Due to the uniqueness of expansions of each vector in terms of a basis required in definition 0.1 it is instantly verified, that the coordinate functionals actually are linear and the natural projections are projections.

For the sake of convenience we will not work with the original norm of the underlying Banach space, but with the following one.

Lemma 1.3 *Let* $(X, \|.\|_X)$ *be a Banach space with basis* $(x_n)_{n\in\mathbb{N}}$*. Then the norm* $\|.\|$ *defined by the formula* $\|\sum_{n=1}^{\infty} \alpha_n x_n\| = \sup_{m\in\mathbb{N}} \|\sum_{n=1}^{m} \alpha_n x_n\|_X$ *is a Banach space norm equivalent to the norm* $\|.\|_X$ *(i.e. they induce the same topology) satisfying* $\|x\| \geq \|x\|_X$ *for all* $x \in X$*.*

Proof: We prove this lemma in four steps. In step one and step two we show, that $\|.\|$ actually is a norm and the claimed inequality, respectively. In step three we find a limit in X for a random Cauchy sequence with respect to $\|.\|$ in X. Finally, in step four we complete the proof by using the open mapping theorem to show the equivalence of the two norms.

Note that in this proof convergence of series in X is *always* meant to be with respect to $\|.\|_X$.

1. Since the other two requirements of the definition of a norm follow instantly from the fact that $\|.\|_X$ is a norm, we will confine ourselves to showing that the triangle inequality holds for $\|.\|$. For this purpose pick two vectors x and y in X having the

expansions $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and $y = \sum_{n=1}^{\infty} \tilde{\alpha}_n x_n$. Then by the linearity of P_m

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \alpha_n x_n + \sum_{n=1}^{\infty} \tilde{\alpha}_n x_n \right\| &= \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \alpha_n x_n + \sum_{n=1}^{m} \tilde{\alpha}_n x_n \right\|_X \\ &\leq \sup_{m\in\mathbb{N}} \left(\left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X + \left\| \sum_{n=1}^{m} \tilde{\alpha}_n x_n \right\|_X \right) \\ &\leq \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X + \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \tilde{\alpha}_n x_n \right\|_X \\ &= \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| + \left\| \sum_{n=1}^{\infty} \tilde{\alpha}_n x_n \right\| = \|x\| + \|y\| \end{aligned}$$

2. For a vector x in X with the expansion $x = \sum_{n=1}^{\infty} \alpha_n x_n$ we obtain from the continuity of the norm $\|.\|_X$

$$\|x\| = \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| = \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X \geq \lim_{n\to\infty} \left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X = \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|_X .$$

3. We want to show next, that an arbitrary Cauchy sequence $(b_i)_{i\in\mathbb{N}} = (\sum_{n=1}^{\infty} \beta_{n,i} x_n)_{i\in\mathbb{N}}$ with respect to $\|.\|$ in X converges towards $\sum_{n=1}^{\infty} \beta_n x_n$ in X, for $\beta_n := \lim_{i\to\infty} \beta_{n,i}$, $n \in \mathbb{N}$. In order to see that the sequence $(\beta_{n,i})_{i\in\mathbb{N}}$ is Cauchy and hence convergent in $\mathbb{F}$ for each $n \in \mathbb{N}$ let j, k, n be in $\mathbb{N}$ and $n \geq 2$. Then we have

$$\begin{aligned} |\beta_{1,j} - \beta_{1,k}| \, \|x_1\|_X &= \|(\beta_{1,j} - \beta_{1,k}) \cdot x_1\|_X \\ \leq \sup_{m\in\mathbb{N}} \left\| \sum_{l=1}^{m} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\|_X &= \left\| \sum_{l=1}^{\infty} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\| \end{aligned}$$

and

$$\begin{aligned} |\beta_{n,j} - \beta_{n,k}| \, \|x_n\|_X &= \|(\beta_{n,j} - \beta_{n,k}) \cdot x_n\|_X \\ &= \left\| \sum_{l=1}^{n} (\beta_{l,j} - \beta_{l,k}) \cdot x_l - \sum_{l=1}^{n-1} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\|_X \\ &\leq \left\| \sum_{l=1}^{n} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\|_X + \left\| \sum_{l=1}^{n-1} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\|_X \\ &\leq \sup_{m\in\mathbb{N}} \left\| \sum_{l=1}^{m} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\|_X + \sup_{m\in\mathbb{N}} \left\| \sum_{l=1}^{m} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\|_X \\ &= 2 \cdot \left\| \sum_{l=1}^{\infty} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\| . \end{aligned}$$

As $(\sum_{n=1}^{\infty} \beta_{n,i} x_n)_{i\in\mathbb{N}}$ is Cauchy so must be $(\beta_{n,i})_{i\in\mathbb{N}}$ for each $n \in \mathbb{N}$. Thus the sequence $(\beta_n)_{n\in\mathbb{N}}$ is well-defined.

Since $(\sum_{n=1}^{\infty} \beta_{n,i} x_n)_{i\in\mathbb{N}}$ is Cauchy with respect to $\|.\|$ we may choose $i(\epsilon) \in \mathbb{N}$ for each fixed $\epsilon > 0$ such that for $i, j, M \in \mathbb{N},\ i, j \geq i(\epsilon)$

$$\begin{aligned}\frac{\epsilon}{3} &> \left\|\sum_{n=1}^{\infty} \beta_{n,j} x_n - \sum_{n=1}^{\infty} \beta_{n,i} x_n\right\| = \sup_{m\in\mathbb{N}} \left\|\sum_{n=1}^{m} \beta_{n,j} x_n - \sum_{n=1}^{m} \beta_{n,i} x_n\right\|_X \\ &\geq \left\|\sum_{n=1}^{M} \beta_{n,j} x_n - \sum_{n=1}^{M} \beta_{n,i} x_n\right\|_X\end{aligned}$$

The inequality holds true for $j \to \infty$. This way we obtain for all $i \geq i(\epsilon)$

$$\left\|\sum_{n=1}^{M} \beta_n x_n - \sum_{n=1}^{M} \beta_{n,i} x_n\right\|_X \leq \frac{\epsilon}{3}. \tag{1.1}$$

We will complete the proof of this step by showing, that $(\sum_{n=1}^{\infty} \beta_{n,i} x_n)_{i\in\mathbb{N}}$ converges towards $\sum_{n=1}^{\infty} \beta_n x_n$. Therefore, it is necessary to show, that $\sum_{n=1}^{\infty} \beta_n x_n$ exists. Due to the completeness of $(X, \|.\|_X)$ it is sufficient to proof that $\sum_{n=1}^{\infty} \beta_n x_n$ is Cauchy with respect to $\|.\|_X$.

Suppose that $m_1, m_2 \in \mathbb{N},\ m_2 \geq m_1 > 1$. Using (1.1) we have

$$\begin{aligned}&\left\|\sum_{n=m_1}^{m_2} \beta_n x_n - \sum_{n=m_1}^{m_2} \beta_{n,i} x_n\right\|_X \qquad (1.2)\\ =& \left\|\sum_{n=m_1}^{m_2} \beta_n x_n - \sum_{n=m_1}^{m_2} \beta_{n,i} x_n \pm \sum_{n=1}^{m_1-1} \beta_n x_n \pm \sum_{n=1}^{m_1-1} \beta_{n,i} x_n\right\|_X \\ =& \left\|\sum_{n=1}^{m_2} \beta_n x_n - \sum_{n=1}^{m_2} \beta_{n,i} x_n - \sum_{n=1}^{m_1-1} \beta_n x_n + \sum_{n=1}^{m_1-1} \beta_{n,i} x_n\right\|_X \\ \leq& \left\|\sum_{n=1}^{m_2} \beta_n x_n - \sum_{n=1}^{m_2} \beta_{n,i} x_n\right\|_X + \left\|\sum_{n=1}^{m_1-1} \beta_n x_n - \sum_{n=1}^{m_1-1} \beta_{n,i} x_n\right\|_X \\ \leq& \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}. \qquad (1.3)\end{aligned}$$

As the series $\sum_{n=1}^{\infty} \beta_{n,i} x_n$ is the expansion of a vector in X in terms of $(x_n)_{n\in\mathbb{N}}$, it must be convergent and so must be $\left(\left\|\sum_{n=1}^{N} \beta_{n,i} x_n\right\|_X\right)_{N\in\mathbb{N}}$. Therefore we can choose $m(\epsilon) \in \mathbb{N}$ such that

$$\left\|\sum_{n=m_1}^{m_2} \beta_{n,i} x_n\right\|_X < \frac{\epsilon}{3} \tag{1.4}$$

for $m_2, m_1 \in \mathbb{N},\ m_2 \geq m_1 > m(\epsilon)$. Finally (1.3) and (1.4) give us the Cauchy criterion for our series $\sum_{n=1}^{\infty} \beta_n x_n$:

$$\begin{aligned} \left\| \sum_{n=m_1}^{m_2} \beta_n x_n \right\|_X &= \left\| \sum_{n=m_1}^{m_2} \beta_n x_n \pm \sum_{n=m_1}^{m_2} \beta_{n,i} x_n \right\|_X \\ \leq \ \left\| \sum_{n=m_1}^{m_2} \beta_n x_n - \sum_{n=m_1}^{m_2} \beta_{n,i} x_n \right\|_X &+ \left\| \sum_{n=m_1}^{m_2} \beta_{n,i} x_n \right\|_X \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

and the convergence of $\sum_{n=1}^{\infty} \beta_n x_n$ is proven.

Since inequality (1.1) still holds when taking the supremum over all $M \in \mathbb{N}$ we see

$$\left\| \sum_{n=1}^{\infty} \beta_n x_n - \sum_{n=1}^{\infty} \beta_{n,i} x_n \right\| = \sup_{M \in \mathbb{N}} \left\| \sum_{n=1}^{M} \beta_n x_n - \sum_{n=1}^{M} \beta_{n,i} x_n \right\|_X \leq \frac{\epsilon}{3},$$

which completes the proof of this step.

4. The identity map $I : (X, \|.\|) \to (X, \|.\|_X)$ is a bijective, linear and due to the inequality, proofed in step 2, continuous operator. The open mapping theorem[1] ensures, that $I^{-1} : (X, \|.\|_X) \to (X, \|.\|)$ is continuous too.

□

Theorem 1.4 *Let $(X, \|.\|_X)$ be a Banach space with basis $(x_n)_{n\in\mathbb{N}}$. Then all natural projections and all coordinate functionals associated with $(x_n)_{n\in\mathbb{N}}$ are continuous.*

Proof: Fix m in $\mathbb{N}$ and a member of X having the expansion $\sum_{n=1}^{\infty} \alpha_n x_n$. Define a sequence $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ by

$$\tilde{\alpha}_n = \begin{cases} \alpha_n, & n \leq m \\ 0, & \text{else.} \end{cases}$$

Then the unique expansion of $P_m \left(\sum_{n=1}^{\infty} \alpha_n x_n\right)$ in terms of $(x_n)_{n\in\mathbb{N}}$ is given by $\sum_{n=1}^{\infty} \tilde{\alpha}_n x_n$. Now the continuity of the natural projection P_m for $(x_n)_{n\in\mathbb{N}}$ follows from

$$\begin{aligned} \left\| P_m \left(\sum_{n=1}^{\infty} \alpha_n x_n \right) \right\| &= \left\| \sum_{n=1}^{\infty} \tilde{\alpha}_n x_n \right\| = \sup_{M\in\mathbb{N}} \left\| \sum_{n=1}^{M} \tilde{\alpha}_n x_n \right\|_X = \sup_{M=1,\ldots,m} \left\| \sum_{n=1}^{M} \tilde{\alpha}_n x_n \right\|_X \\ &= \sup_{M=1,\ldots,m} \left\| \sum_{n=1}^{M} \alpha_n x_n \right\|_X \leq \sup_{M\in\mathbb{N}} \left\| \sum_{n=1}^{M} \alpha_n x_n \right\|_X = \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|. \end{aligned}$$

[1] see e.g. [1] theorem 12.1 and 12.5

For each $m > 1$ the coordinate functional x_m^* associated with $(x_n)_{n\in\mathbb{N}}$ is continuous as it is the composition of continuous maps

$$\sum_{n=1}^{\infty} \alpha_n x_n \mapsto (P_m - P_{m-1}) \left(\sum_{n=1}^{\infty} \alpha_n x_n \right) = \alpha_m x_m \mapsto \alpha_m$$

and so it is for $m = 1$

$$\sum_{n=1}^{\infty} \alpha_n x_n \mapsto P_1 \left(\sum_{n=1}^{\infty} \alpha_n x_n \right) = \alpha_1 x_1 \mapsto \alpha_1.$$

□

Remark 1.5 Consequently each coordinate functional associated with a basis in a Banach space X is a member of the continuous dual space X' of X. Though it turns out, that in general the sequence of coordinate functionals does not have to be a basis for X', it can be shown, that it is allways a so called *basic sequence*, i.e. a basis for the closed linear hull of the collection of all coordinate functionals.

2 Banach's Basis Problem

2.1 Motivation

In the following proposition we will see, that every Banach Space having a basis is separable. The question whether the converse is true, i.e. whether every infinite dimensional separable Banach space has a basis, is known as the classical *basis problem* for Banach spaces. It remained open for forty years until Per Enflo found a counterexample in 1973.

Below we will give the construction of a closed subspace of l^p, for $2 < p < \infty$, that fails to have a basis, due to A.M. Davie [2], reproduced in [3] and [6].

Propostion 2.1 *Every Banach space X with a basis $(x_n)_{n\in\mathbb{N}}$ is separable.*

Proof: We want to show, that the countable set

$$A := \left\{ \sum_{n=1}^{m} \alpha_n x_n : \alpha_1, \ldots, \alpha_n \in \tilde{\mathbb{Q}}, m \in N \right\},$$

where $\tilde{\mathbb{Q}}$ denotes the rational numbers or respectively the complex numbers with rational real and imaginary part, is dense in X. Since an arbitrary element of X having the expansion $\sum_{n=1}^{\infty} \alpha_n x_n$ in terms of $(x_n)_{n\in\mathbb{N}}$ can be written as $\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n$, it follows that $X = \overline{\text{span}\left(\{x_n : n \in \mathbb{N}\}\right)}$. Therefore, it suffices to show that A is a dense subset of $\text{span}(\{x_n : n \in \mathbb{N}\})$.

Fix $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \{x_n : n \in \mathbb{N}\}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$. Due to the fact, that $\tilde{\mathbb{Q}}$ is dense in $\mathbb{F}$, there is a sequence $(\alpha_{j,i})_{i\in\mathbb{N}} \in \tilde{\mathbb{Q}}^n$ converging towards α_j for each $j = 1, \ldots, k$. From the continuity of the vector space operations it follows, that

$$\lim_{i\to\infty} \sum_{j=1}^{k} \alpha_{j,i} x_j = \sum_{j=1}^{k} \alpha_j x_j,$$

i.e. an arbitrary element of $\text{span}(\{x_n : n \in \mathbb{N}\})$ can be written as the limit of a sequence in A, which completes the proof of this proposition.

□

2.2 A Counterexample to the Basis Problem

In our counterexample we will prove the existence of a closed subspace of l_p, $2 < p < \infty$, lacking a certain property. Therefore, it is essential, that every Banach space with a basis has this property.

Definition 2.2 A Banach space X is said to have the *approximation property* if for each compact set $C \subset X$ and each $\epsilon > 0$ there exists a bounded linear Operator $A_{C,\epsilon}$ from X into X having finite rank such that $\|A_{C,\epsilon}x - x\|_X < \epsilon$ for each x in C.

Theorem 2.3 *Let $(X, \|.\|_X)$ be a Banach space with basis $(x_n)_{n\in\mathbb{N}}$. Then X has the approximation property.*

Proof: Let C be a compact subset of X and $\epsilon > 0$. By the preceding lemma it is sufficient to show that there is an N in $\mathbb{N}$ such that $\|P_N x - x\|_X < \epsilon$ for each x in C.

It follows readily from the uniform boundedness principle[1] that $\Pi := \sup_{n\in\mathbb{N}} \|P_n\|$ is finite. Due to the compactness of C we can pick finitely many $y_1, \dots, y_l$ in C such that $\min_{i=1,\dots,l} \|x - y_i\|_X \leq \frac{\epsilon}{2(1+\Pi)}$ for each x in C.

Let x_0 be in C. Then there is $j \in \mathbb{N}$ such that $\|x_0 - y_j\|_X \leq \frac{\epsilon}{2(1+C)}$ and, since it follows from definition 0.1 that $\lim_{n\to\infty} \|y_j - P_n y_j\|_X = 0$, there is $N_\epsilon \in \mathbb{N}$ such that $\|y_j - P_n y_j\|_X \leq \frac{\epsilon}{2}$ for each $n \geq N_\epsilon$. We conclude

$$\begin{aligned} \|P_n x_0 - x_0\|_X &= \|P_n x_0 - x_0 \pm y_j \pm P_n y_j\|_X \\ &\leq \|y_j - x_0\|_X + \underbrace{\|P_n y_j - P_n x_0\|_X}_{\leq \|P_n\| \|y_j - x_0\|_X \leq \Pi \|y_j - x_0\|_X} + \underbrace{\|y_j - P_n y_j\|_X}_{\frac{\epsilon}{2}} \\ &\leq (1+\Pi)\|y_j - x_0\|_X + \frac{\epsilon}{2} \leq (1+\Pi)\frac{\epsilon}{2(1+\Pi)} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

In the following lemma we give an equivalent condition for a Banach space to have the approximation property. For a proof we refer e.g. to [3] theorem 1.e.4.

Lemma 2.4 *Suppose that $(X, \|.\|_X)$ is Banach space. Then the following are equivalent.*

(i) X has the approximation property.

(ii) $\sum_{n=1}^{\infty} x_n^*(x_n) = 0$ for all sequences $(x_n)_{n\in\mathbb{N}}$ in X and $(x_n^*)_{n\in\mathbb{N}}$ in X^* satisfying $\sum_{n=1}^{\infty} \|x_n\|_X \|x_n^*\| < \infty$ and $\sum_{n=1}^{\infty} x_n^*(x) x_n = 0$ for each x in X, when X' denotes the continuous dual space of X.

In our proof of the existence of a subspace of l_p, $2 < p < \infty$, lacking the approximation property, we will use an infinite matrix $A = (a_{i,j})_{i,j\in\mathbb{N}}$ of a certain type. For the construction of A we need two lemmata. Thus we want to recall some simple facts from probability theory and from group theory.

Remark 2.5

(i) If η is a discrete random variable, i.e. a measurable function from a probability space to a discrete subset of the real numbers[2], taking the values η_n with probabilities $\mathbb{P}(\eta = \eta_n)$, then the expected value of η is given by $\mathbb{E}(\eta) = \sum_n \eta_n \mathbb{P}(\eta = \eta_n)$.

[1] see e.g. [1] theorem 14.1

[2] or more general, of a measurable space

(ii) The expected value is linear and monotonic and if $\eta_1, \ldots, \eta_n$ are independent random variables we have $\mathbb{E}\left(\prod_{i=1}^n \eta_i\right) = \prod_{i=1}^n \mathbb{E}(\eta_i)$.

(iii) If γ is a random variable and $\lambda > 0$ we have $\mathbb{P}(\gamma > 0) = \mathbb{E}(\mathbf{1}_{(0,\infty)}(\gamma))$, where $\mathbf{1}_{(0,\infty)}$ denotes the characteristic funcion of $(0, \infty)$. Since $\mathbf{1}_{(0,\infty)}(\gamma) \leq e^{\lambda\gamma}$ for all γ and $\lambda > 0$, we obtain $\mathbb{P}(\gamma > 0) \leq \mathbb{E}(e^{\lambda\gamma})$.

(iv) Let $(\alpha_i)_{i=1}^N$ be a sequence of real numbers. Suppose, that the sequence $(p_i)_{i=1}^N \subset [0, 1]$ satisfies $\sum_{i=1}^N p_i = 1$. Then there exists a sequence of independent random variables $(\rho_n)_{n\in\mathbb{N}}$, such that each ρ_n takes the value α_i with probability p_i, for each $i = 1, \ldots, N$.

Let be G be a finite abelian Group of order k, i.e. an abelian group having k elements, then

(v) G has exactly k characters, i.e. homomorphism from G into the multiplicative group $(\{z \in \mathbb{C} : |z| = 1\}, \cdot)$,

(vi) if ω is a character of G we have $\overline{\omega(g)} = \omega(g^{-1})$ for all $g \in G$ and $\omega(e) = 1$ for the identity element e of G and

(vii) any two different characters ω_1 and ω_2 of G are orthogonal, i.e. $\sum_{g\in G} \omega_1(g)\omega_2(g^{-1}) = \sum_{g\in G} \omega_1(g)\overline{\omega_2(g)} = 0$.

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Lemma 2.6 *Let $(\rho_n)_{n=1}^N$ and $(\alpha_n)_{n=1}^N$ be finite sequences of independent random variables and complex numbers, respectively. If*

(i) each ρ_n takes the value 2 with probability $\frac{1}{3}$ and -1 with probability $\frac{2}{3}$ or

(ii) each ρ_n takes the values 1 and -1 with probability $\frac{1}{2}$,

then there exists an absolute constant L such that

$$\mathbb{P}\left\{\left|\sum_{n=1}^N \alpha_n\rho_n\right| > L\left(\log(N)\sum_{n=1}^N |\alpha_n|^2\right)^{\frac{1}{2}}\right\} < \frac{L}{N^3}. \tag{2.1}$$

One possible choice of L is $L = 3\sqrt{3}$.

Proof: First we consider real α_n's. If $\alpha_n = 0$ for all $n = 1, \ldots, N$, the assertion of this lemma is trivial. Thus, let $(\alpha_n)_{n=1}^N$ be so that $\sum_{n=1}^N |\alpha_n|^2 \neq 0$. Furthermore we can assume without loss of generality that $\sum_{n=1}^N |\alpha_n|^2 = 1$, because otherwise we have

$$\mathbb{P}\left\{\left|\sum_{n=1}^N \alpha_n\rho_n\right| > L\left(\log(N)\sum_{n=1}^N |\alpha_n|^2\right)^{\frac{1}{2}}\right\} = \mathbb{P}\left\{\left|\sum_{n=1}^N \frac{\alpha_n}{\sqrt{\sum_{j=1}^N |\alpha_j|^2}}\rho_n\right| > L\left(\log(N)\cdot 1\right)^{\frac{1}{2}}\right\}.$$

$\rho_1, \dots, \rho_N$ are independent and $e^{|t|} \le e^{|t|} + e^{-|t|} = e^t + e^{-t}$ for all $t \in \mathbb{R}$. Thus ,taking remark 2.5 into account we obtain for any $\lambda > 0$

$$\begin{aligned} \mathbb{E}\left(e^{\lambda|\sum_{n=1}^N \alpha_n\rho_n|}\right) &\le \mathbb{E}\left(e^{\lambda\sum_{n=1}^N \alpha_n\rho_n}\right) + \mathbb{E}\left(e^{-\lambda\sum_{n=1}^N \alpha_n\rho_n}\right) \\ = \mathbb{E}\left(\prod_{n=1}^N e^{\lambda\alpha_n\rho_n}\right) + \mathbb{E}\left(\prod_{n=1}^N e^{-\lambda\alpha_n\rho_n}\right) &= \prod_{n=1}^N \mathbb{E}(e^{\lambda\alpha_n\rho_n}) + \prod_{n=1}^N \mathbb{E}(e^{-\lambda\alpha_n\rho_n}) \end{aligned} \tag{2.2}$$

Suppose first that (i) holds. Then (2.2) yields

$$\begin{aligned} \mathbb{E}\left(e^{\lambda|\sum_{n=1}^N \alpha_n\rho_n|}\right) &\le \prod_{n=1}^N \mathbb{E}(e^{\lambda\alpha_n\rho_n}) + \prod_{n=1}^N \mathbb{E}(e^{-\lambda\alpha_n\rho_n}) \\ &= \prod_{n=1}^N \left(\frac{1}{3}e^{2\lambda\alpha_n} + \frac{2}{3}e^{-\lambda\alpha_n}\right) + \prod_{n=1}^N \left(\frac{1}{3}e^{-2\lambda\alpha_n} + \frac{2}{3}e^{\lambda\alpha_n}\right). \end{aligned} \tag{2.3}$$

In order to see that

$$\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \le e^{2t^2} \quad \text{for all } t \in \mathbb{R}, \tag{2.4}$$

consider

$$\begin{aligned} e^{2t} + 2e^{-t} &= \sum_{n=0}^\infty \frac{(2t)^n + 2(-t)^n}{n!} = \sum_{n=0}^\infty \left(\frac{(2t)^{2n} + 2(-t)^{2n}}{(2n)!} + \frac{(2t)^{2n+1} + 2(-t)^{2n+1}}{(2n+1)!}\right) \\ &= 3 + 3t^2 + t^3 + \sum_{n=2}^\infty \left(\frac{2^{2n}+2}{(2n)!} + t\frac{2^{2n+1}-2}{(2n+1)!}\right). \end{aligned} \tag{2.5}$$

It is instantly verified, that (2.4) holds for $t \ge 1$. If $t < 1$ and $n \ge 2$ then

$$\begin{aligned} \frac{2^{2n}+2}{(2n)!} + t\frac{2^{2n+1}-2}{(2n+1)!} &< \frac{2^{2n+1}}{(2n)!} + \frac{2^{2n+1}}{(2n+1)!} = \frac{2^{2n+1}(2n+2)}{(2n+1)!} \\ < \frac{3 \cdot 2^{2n} \cdot 2(n+1)}{n!(n+1)(n+2)\dots(2n+1)} &= \frac{3 \cdot 2^{2n}}{n!} \cdot \frac{2}{(n+2)(n+3)\dots(2n+1)} < \frac{3 \cdot 2^{2n}}{n!}. \end{aligned}$$

Now (2.5) yields for $t < 1$

$$e^{2t} + 2e^{-t} \le 3 + 6t^2 + \sum_{n=2}^\infty \frac{3 \cdot 2^n}{n!} t^{2n} = 3e^{2t^2}.$$

Thus, (2.4) is proven. Inserting (2.4) back in (2.3) for $t = \lambda\alpha_n$ and $t = -\lambda\alpha_n$, respectively, for $n = 1, \dots, N$ we obtain

$$\mathbb{E}\left(e^{\lambda|\sum_{n=1}^N \alpha_n\rho_n|}\right) \le 2\prod_{n=1}^N e^{2(\lambda\alpha_n)^2} = 2e^{2\lambda^2\sum_{n=1}^N |\alpha_n|^2} = 2e^{2\lambda^2}$$

If (ii) holds, (2.2) shows that

$$\mathbb{E}\left(e^{\lambda\left|\sum_{n=1}^{N}\alpha_n\rho_n\right|}\right) \;\leq\; 2\prod_{n=1}^{N}\left(\frac{e^{-\lambda\alpha_n}+e^{\lambda\alpha_n}}{2}\right) \tag{2.6}$$

and because of

$$\frac{e^t+e^{-t}}{2} \;=\; \cosh t \;=\; \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!} \;\leq\; \sum_{n=0}^{\infty}\frac{t^{2n}}{n!} \;=\; e^{t^2}, \quad \text{for all } t\in\mathbb{R},$$

furthermore

$$\mathbb{E}\left(e^{\lambda\left|\sum_{n=1}^{N}\alpha_n\rho_n\right|}\right) \;\leq\; 2\prod_{n=1}^{N}e^{(\lambda\alpha_n)^2} \;=\; 2e^{\lambda^2\sum_{n=1}^{N}|\alpha_n|^2} \;=\; 2e^{\lambda^2} \;\leq\; 2e^{2\lambda^2}.$$

So in either case we have

$$\mathbb{E}\left(e^{\lambda\left|\sum_{n=1}^{N}\alpha_n\rho_n\right|}\right) \;\leq 2e^{2\lambda^2}.$$

Consequently

$$\mathbb{E}\left(e^{\lambda\left|\sum_{n=1}^{N}\alpha_n\rho_n\right|-2\lambda^2-3\log N}\right) = e^{-2\lambda^2-3\log N}\mathbb{E}\left(e^{\lambda\left|\sum_{n=1}^{N}\alpha_n\rho_n\right|}\right) \leq 2e^{-3\log N} = \frac{2}{N^3}.$$

Now putting $\lambda = \sqrt{3\log N}$ gives

$$\mathbb{E}\left(e^{\sqrt{3\log N}\left|\sum_{n=1}^{N}\alpha_n\rho_n\right|-9\log N}\right) \leq \frac{2}{N^3}.$$

Applying remark 2.5 (iii) to $\gamma = \left|\sum_{n=1}^{N}\alpha_n\rho_n\right| - 3\sqrt{3}(\log N)^{\frac{1}{2}}$ and $\lambda = \sqrt{3\log N}$ we obtain

$$\mathbb{P}\left\{\left|\sum_{n=1}^{N}\alpha_n\rho_n\right| - 3\sqrt{3}(\log N)^{\frac{1}{2}} > 0\right\} \leq \mathbb{E}\left(e^{\sqrt{3\log N}\left|\sum_{n=1}^{N}\alpha_n\rho_n\right|-9\log N}\right) \leq \frac{2}{N^3} \leq \frac{3\sqrt{3}}{N^3}, \tag{2.7}$$

i.e. (2.1) is proven for real α_n's. In order to see that (2.1) still holds for complex $\alpha_n = u_n + iv_n$, $n = 1,\ldots,N$, observe that $|\alpha_n|^2 = |u_n|^2 + |v_n|^2$, $n = 1,\ldots,N$, and

$\left|\sum_{n=1}^N \alpha_n\rho_n\right|^2 = \left|\sum_{n=1}^N u_n\rho_n\right|^2 + \left|\sum_{n=1}^N v_n\rho_n\right|^2$. By (2.7) we have

$$
\begin{aligned}
&\mathbb{P}\left\{\left|\sum_{n=1}^N \alpha_n\rho_n\right| > 3\sqrt{3}\left(\log(N)\sum_{n=1}^N |\alpha_n|^2\right)^{\frac{1}{2}}\right\} \\
= \;& \mathbb{P}\left\{\left|\sum_{n=1}^N \alpha_n\rho_n\right|^2 > 27\log(N)\sum_{n=1}^N |\alpha_n|^2\right\} \\
\le \;& \mathbb{P}\left\{\left(\left|\sum_{n=1}^N u_n\rho_n\right|^2 > 27\log(N)\sum_{n=1}^N |u_n|^2\right) \vee \left(\left|\sum_{n=1}^N v_n\rho_n\right|^2 > 27\log(N)\sum_{n=1}^N |v_n|^2\right)\right\} \\
\le \;& \mathbb{P}\left\{\left|\sum_{n=1}^N u_n\rho_n\right|^2 > 27\log(N)\sum_{n=1}^N |u_n|^2\right\} + \mathbb{P}\left\{\left|\sum_{n=1}^N v_n\rho_n\right|^2 > 27\log(N)\sum_{n=1}^N |v_n|^2\right\} \\
\le \;& \frac{2}{N^3} + \frac{2}{N^3} = \frac{4}{N^3} \le \frac{3\sqrt{3}}{N^3},
\end{aligned}
$$

which completes the proof of this lemma.

□

Corollary 2.7 *Let G_k and G_{k-1} be abelian groups*[3] *of order $3{\cdot}2^k$ and $3{\cdot}2^{k-1}$, respectively. Denote the characters of G_k and G_{k-1} by ω_n, $n = 1,\dots,3{\cdot}2^k$, and γ_n, $n = 1,\dots,3{\cdot}2^{k-1}$. Let $(j_n)_{n=1}^N$ be in $\{1,\dots,3{\cdot}2^k\}^N$ and let $(l_n)_{n=1}^N$ be in $\{1,\dots,3{\cdot}2^{k-1}\}^N$, for an $N \ge 3{\cdot}2^k$. Then there exist sequences $(\rho_n^1)_{n=1}^N \in \{-1,2\}^N$ and $(\rho_n^2)_{n=1}^N \in \{-1,1\}^N$ such that for some absolute constant M and for all $g \in G_k$ and for all $h \in G_{k-1}$*

$$\left|\sum_{n=1}^N \rho_n^i\overline{\omega_{j_n}(g)}\gamma_{l_n}(h)\right| \le Mk^{\frac{1}{2}}2^{\frac{k-1}{2}}, \quad i \in \{1,2\}.$$

Proof: Let the $\tilde\rho_n^1$'s be random variables[4] as in lemma 2.6 (*i*) and the $\tilde\rho_n^2$ as in lemma 2.6 (*ii*). Recall that $|\omega_n(g)| = |\gamma_n(h)| = 1$ for all n, $g \in G_k$, $h \in G_{k-1}$.

Applying lemma 2.6 (*i*) and (*ii*) to $(\alpha_n)_{n=1}^N = (\overline{\omega_{j_n}(g)}\gamma_{l_n}(h))_{n=1}^N$, respectively, yields

$$\mathbb{P}\left\{\left|\sum_{n=1}^N \overline{\omega_{j_n}(g)}\gamma_{l_n}(h)\tilde\rho_n^i\right| > 3\sqrt{3}\,(\log(N)N)^{\frac{1}{2}}\right\} < \frac{3\sqrt{3}}{N^3}, \quad \text{for all } g \in G_k,\ h \in G_{k-1},\ i \in \{1,2\}.$$

[3] for a given $m \in \mathbb{N}$ one example of an abelian group of order m is $\mathbb{Z}/m\mathbb{Z}$, the factor group of $\mathbb{Z}$ over $m\mathbb{Z}$

[4] the existence of these random variables follows from remark 2.5 [iv]

We conclude

$$\mathbb{P}\left\{\exists (g,h)\in G_k\times G_{k-1}\ :\ \left|\sum_{n=1}^{N}\overline{\omega_{j_n}(g)}\gamma_{l_n}(h)\tilde{\rho}_n^i\right| > 3\sqrt{3}\,(\log(N)N)^{\frac{1}{2}}\right\}$$

$$\leq \sum_{(g,h)\in G_k\times G_{k-1}} \mathbb{P}\left\{\left|\sum_{n=1}^{N}\overline{\omega_{j_n}(g)}\gamma_{l_n}(h)\tilde{\rho}_n^i\right| > 3\sqrt{3}\,(\log(N)N)^{\frac{1}{2}}\right\} < \frac{3\cdot 2^k 3\cdot 2^{k-1}3\sqrt{3}}{N^3}.$$

Since $N\geq 2^k$, we obtain for k large enough ($k>k_0$)

$$\mathbb{P}\left\{\exists (g,h)\in G_k\times G_{k-1}\ :\ \left|\sum_{n=1}^{N}\overline{\omega_{j_n}(g)}\gamma_{l_n}(h)\tilde{\rho}_n^i\right| > 3\sqrt{3}\,(\log(N)N)^{\frac{1}{2}}\right\} < 1.$$

Consequently

$$\mathbb{P}\left\{\forall (g,h)\in G_k\times G_{k-1}\ :\ \left|\sum_{n=1}^{N}\overline{\omega_{j_n}(g)}\gamma_{l_n}(h)\tilde{\rho}_n^i\right| \leq 3\sqrt{3}\,(\log(N)N)^{\frac{1}{2}}\right\} > 0.$$

Thus there exist sequences $(\rho_n^1)_{n=1}^{3\cdot 2^k}\in\{-1,2\}^N$ and $(\rho_n^2)_{n=1}^{3\cdot 2^k}\in\{-1,1\}^N$ for k large enough such that for all $g\in G_k$ and for all $h\in G_{k-1}$

$$\left|\sum_{n=1}^{N}\overline{\omega_{j_n}(g)}\gamma_{l_n}(h)\rho_n^i\right| \leq 3\sqrt{3}\,(\log(N)N)^{\frac{1}{2}} \leq 3\sqrt{3}(\log(3\cdot 2^{k-1})3\cdot 2^{k-1})^{\frac{1}{2}} \leq Mk^{\frac{1}{2}}2^{\frac{k-1}{2}}.$$

Since for each $k\in\mathbb{N}\cup\{0\}$ the product $G_k\times G_{k-1}$ is a finite set, the same inequalities hold for $k=0,\ldots,k_0$, by increasing M if necessary, which completes the proof of this corollary.

□

Lemma 2.8 *Let G_k be an abelian group of order $3\cdot 2^k$, for each k in $\mathbb{N}\cup\{0\}$. Then there exists a partition of the set of all characters $H_k=\{\omega_n\}_{n=1}^{3\cdot 2^k}$ of G_k into two disjoint sets $H_k^+=\{\sigma_n\}_{n=1}^{2^k}$ and $H_k^-=\{\tau_n\}_{n=1}^{2^{k+1}}$ with cardinalities satisfying $|H_k^+|=2^k$ and $|H_k^-|=2^{k+1}$ such that for some absolute constant K and for each $g\in G_k$*

$$\left|2\sum_{n=1}^{2^k}\sigma_n(g)-\sum_{n=1}^{2^{k+1}}\tau_n(g)\right|\leq K(k+1)^{\frac{1}{2}}2^{\frac{k}{2}}. \tag{2.8}$$

Proof: It suffices to show that there exists a sequence of numbers $(\rho_n)_{n=1}^{3\cdot 2^k}\in\{-1,2\}^{3\cdot 2^k}$ such that

$$\left|\sum_{n=1}^{3\cdot 2^k}\rho_n\omega_n(g)\right|\leq K(n+1)^{\frac{1}{2}}2^{\frac{n}{2}}\quad \text{for all } g\in G_k \tag{2.9}$$

and

$$\sum_{n=1}^{3\cdot 2^k} \rho_n = 0, \tag{2.10}$$

since then (2.9) proofs (2.8), while (2.10) ensures that the requirements concerning the cardinalities of our subsets H_k^+ and H_k^- are satisfied.

Let G_{k+1} be any abelian group of order $3 \cdot 2^{k+1}$ and let I be the trivial character[5] of G_{k+1}. Applying corollary 2.7 to G_k, G_{k+1}, $\{\omega_n\}_{n=1}^{3\cdot 2^k}$ and $\{I\}_{n=1}^{3\cdot 2^k}$ shows, that there exists a sequence $(\rho_n)_{n=1}^{3\cdot 2^k} \in \{-1,2\}^{3\cdot 2^k}$ such that,

$$\left|\sum_{n=1}^{3\cdot 2^k} \rho_n \omega_n(g)\right| \;\leq\; K 2^{\frac{k}{2}} (k+1)^{\frac{1}{2}},$$

for some absolute constant K, i.e. (2.9) holds for $(\rho_n)_{n=1}^{3\cdot 2^k}$.

Put $S_j := \left\{\rho_n : n \in \{1,\ldots,3\cdot 2^k\}, \rho_n = j\right\}$ for $j \in \{-1,2\}$. For the cardinalities of this to sets we obviously have $|S_{-1}| + |S_2| = 3\cdot 2^k$. Therefore

$$\sum_{n=1}^{3\cdot 2^k} \rho_n \;=\; 2|S_2| - |S_{-1}| \;=\; 2|S_2| - 3\cdot 2^k + |S_2| \;=\; 3(|S_2| - 2^k). \tag{2.11}$$

If $|S_2| = 2^k$ our sequence $(\rho_n)_{n=1}^{3\cdot 2^k}$ satisfies (2.10) in addition to (2.9) and we are done. Otherwise we have to change an appropriate number of the ρ_n's

Suppose first, that $|S_2| < 2^k$ and let $I_1 \subset \{1,\ldots,3\cdot 2k\}$ be such that $|I_1| = 2^k - |S_2|$ and $\rho_n = -1$ for all $n \in I_1$. We define a sequence $\{\rho_{n,1}\}_{n=1}^{3\cdot 2^k}$ by

$$\rho_{n,1} = \begin{cases} 2, & \text{for } n \in I_1, \\ \rho_n, & \text{else.} \end{cases}$$

If $|S_2| > 2^k$, we fix $I_2 \subset \{1,\ldots,3\cdot 2k\}$ such that $|I_2| = |S_2| - 2^k$ and $\rho_n = 2$ for all $n \in I_2$. The sequence $\{\rho_{n,2}\}_{n=1}^{3\cdot 2^k}$ is defined by

$$\rho_{n,2} = \begin{cases} -1, & \text{for } n \in I_2, \\ \rho_n, & \text{else.} \end{cases}$$

In both cases (2.9) and (2.11) give

$$\left|\sum_{n=1}^{3\cdot 2^k} \rho_n \omega_n(g) - \sum_{n=1}^{3\cdot 2^k} \rho_{n,j} \omega_n(g)\right| \;=\; \left|\sum_{i\in I_j} 3\cdot \omega_i(g)\right| \leq 3|I_j| \;=\; 3\left||S_2| - 2^k\right| \;=\; \left|\sum_{n=1}^{3\cdot 2^k} \rho_n\right|$$
$$=\; \left|\sum_{n=1}^{3\cdot 2^k} \rho_n \omega_n(e)\right| \;\leq\; K(n+1)^{\frac{1}{2}} 2^{\frac{n}{2}}$$

[5]i.e. the map $h \mapsto 1,\ h \in G_{k+1}$

for all $g \in G_k$, $j \in \{1,2\}$ and the identity element e of G_k. Using this, the triangle inequality and (2.9) we obtain

$$\left|\sum_{n=1}^{3\cdot 2^k} \rho_{n,j}\right| \leq 2K(k+1)^{\frac{1}{2}} 2^{\frac{n}{2}}, \quad \text{for all } g \in G_k,\ j \in \{1,2\},$$

and thus (2.9) holds for K and $\{\rho_n\}_{n=1}^{3\cdot 2^k}$ replaced by $2K$ and $\{\rho_{n,j}\}_{n=1}^{3\cdot 2^k}$ ($j \in \{1,2\}$), respectively. In addition, by (2.11), we have

$$\sum_{n=1}^{3\cdot 2^k} \rho_n - \sum_{n=1}^{3\cdot 2^k} \rho_{n,j} = \sum_{i \in I_j} 3 = 3|I_j| = 3\left||S_2| - 2^k\right| = \sum_{n=1}^{3\cdot 2^k} \rho_n \quad j \in \{1,2\}$$

and consequently $\sum_{n=1}^{3\cdot 2^k} \rho_{n,j} = 0$ for $j \in \{1,2\}$, i.e. we have (2.10) for $\{\rho_n\}_{n=1}^{3\cdot 2^k}$ replaced by $\{\rho_{n,j}\}_{n=1}^{3\cdot 2^k}$.

□

Now we are able to pass the construction of the infinite matrix A needed for our counterexample.

Lemma 2.9 *There exists an infinite matrix $A = (a_{i,j})_{i,j\in\mathbb{N}}$ of complex numbers satisfying $A^2 = 0$, $\mathrm{tr}A = \sum_{i=1}^{\infty} a_{i,i} \neq 0$ and $\sum_{i=1}^{\infty} (\max_{j\in\mathbb{N}} |a_{i,j}|)^r < \infty$ for each $r > \frac{2}{3}$. Furthermore each row and each column contains only finitely many nonzero entries.*

Proof: For each k in $\mathbb{N} \cup \{0\}$ let $G_k = (\{g_1, \ldots, g_{3\cdot 2^k}\}, *)$ be an abelian group of order $3 \cdot 2^k$. Furthermore let $H_k^+ = \{\sigma_{n,k}\}_{n=1}^{2^k}$ and $H_k^- = \{\tau_{n,k}\}_{n=1}^{2^{k+1}}$ be as in the preceding lemma.

We define complex matrices

$$P_k = \left(3^{-\frac{1}{2}} 2^{-\frac{2k+1}{2}} \tau_{i,k}(g_j)\right)_{i=1,\ldots,2^{k+1};j=1,\ldots,3\cdot 2^k},\ Q_k = \left(3^{-\frac{1}{2}} 2^{-k} \rho_{i,k} \sigma_{i,k}(g_j)\right)_{i=1,\ldots,2^k;j=1,\ldots,3\cdot 2^k},$$

for each k in $\mathbb{N} \cup \{0\}$, where $(\rho_{i,k})_{i=1}^{2^k} \in \{-1,1\}^{2^k}$ will be determined below. We have

$$P_k P_k^* = \left(3^{-1} 2^{-2k-1} \sum_{l=1}^{3\cdot 2^k} \tau_{i,k}(g_l)\overline{\tau_{j,k}(g_l)}\right)_{i,j=1,\ldots 2^{k+1}}, \tag{2.12}$$

where P_k^* denotes the conjugate transpose of P_k. Taking remark 2.5 into account we obtain

$$\sum_{l=1}^{3\cdot 2^k} \tau_{i,k}(g_l)\overline{\tau_{j,k}(g_l)} = \begin{cases} \sum_{l=1}^{3\cdot 2^k} |\tau_{i,k}(g_l)|^2 = 3\cdot 2^k, & j = i = 1, \ldots, 2^k \\ 0, & i,j \in \{1,\ldots,2^k\},\ i \neq j. \end{cases}$$

Hence, we have

$$P_k P_k^* = 2^{-k-1} I_{2^{k+1}}, \quad k \in \mathbb{N} \cup \{0\}, \tag{2.13}$$

where I_m denotes the identity matrix of order m, $m \in \mathbb{N}$. Analogously it follows that for each possible choice of $(\rho_{j,k})_{j=1}^{2^k}$

$$Q_k Q_k^* = 2^{-k} I_{2^k} \quad \text{and} \quad P_k Q_k^* = 0 = Q_k P_k^*, \quad k \in \mathbb{N} \cup \{0\}. \tag{2.14}$$

Put

$$A = (a_{i,j})_{i,j=1}^{\infty} \begin{pmatrix} P_0^* P_0 & P_0^* Q_1 & 0 & 0 & 0 & \cdots \\ -Q_1^* P_0 & (P_1^* P_1 - Q_1^* Q_1) & P_1^* Q_2 & 0 & 0 & \cdots \\ 0 & -Q_2^* P_1 & (P_2^* P_2 - Q_2^* Q_2) & P_2^* Q_3 & 0 & \cdots \\ 0 & 0 & -Q_3^* P_2 & (P_3^* P_3 - Q_3^* Q_3) & P_3^* Q_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

For the trace of A we have $\mathrm{tr} A = \mathrm{tr}(P_0^* P_0) + \sum_{k=1}^{\infty} \mathrm{tr}(P_k^* P_k - Q_k^* Q_k)$. Taking 2.13 and 2.14 into account for $k \geq 1$ the trace of $P_k^* P_k - Q_k^* Q_k$ is given by

$$\mathrm{tr}(P_k^* P_k - Q_k^* Q_k) \;=\; \mathrm{tr}(P_k P_k^*) - \mathrm{tr}(Q_k Q_k^*) \;=\; \mathrm{tr}(2^{-k-1} I_{2^{k+1}}) - \mathrm{tr}(2^{-k} I_{2^k}) \;=\; 0$$

Therefore $\mathrm{tr} A = \mathrm{tr}(P_0^* P_0) = \mathrm{tr}(P_0 P_0^*) = \mathrm{tr}(2^{-1} I_2) = 1 \neq 0$. An elementary computation[6] also shows that $A^2 = 0$.

So it remains to be shown, that, for a suitable choice of $(\rho_{j,k})_{j=1}^{2^k}$, $\sum_{i=1}^{\infty} (\max_{j \in \mathbb{N}} |a_{i,j}|)^r < \infty$ for each $r > \frac{2}{3}$.

For the first row of blocks it is obvious, that $\sum_{l=1}^{3} \max_{j \in \mathbb{N}} |a_{l,j}|^r = \sum_{l=1}^{3} \max_{j=1,\ldots,9} |a_{l,j}|^r < \infty$. For each $k \geq 1$ we will show next, that the absolute value of each element in the $(k+1)$'th row of blocks of A, i.e. each element of the matrices $-Q_k^* P_{k-1}$, $P_k^* P_k - Q_k^* Q_k$ and $P_k^* Q_{k+1}$, is less or equal to $C(k+1)^{\frac{1}{2}} 2^{-\frac{3k}{2}}$, for some absolute constant C. This will imply

$$\sum_{i=1}^{\infty} \left(\max_{j \in \mathbb{N}} |a_{i,j}| \right)^r \leq \sum_{l=1}^{3} \max_{j=1,\ldots,9} |a_{i,j}|^r + \sum_{k=1}^{\infty} 3 \cdot 2^k C^r (k+1)^{\frac{r}{2}} 2^{-\frac{3k}{2}} < \infty. \tag{2.15}$$

Since $P_{k-1}^* Q_k = (Q_k^* P_{k-1})^*$, it suffices to consider the matrices $Q_k^* P_{k-1}$ and $(P_k^* P_k - Q_k^* Q_k)$.

For the latter we have

$$\begin{aligned} P_k^* P_k - Q_k^* Q_k &= \left(3^{-1} 2^{-2k-1} \sum_{l=1}^{2^{k+1}} \overline{\tau_{l,k}(g_i)} \tau_{l,k}(g_j) - 3^{-1} 2^{-2k} \sum_{l=1}^{2^k} \rho_{l,k}^2 \overline{\sigma_{l,k}(g_i)} \sigma_{l,k}(g_j) \right)_{i,j=1}^{3 \cdot 2^k} \\ &= \left(3^{-1} 2^{-2k-1} \left(\sum_{l=1}^{2^{k+1}} \tau_{l,k}(g_i^{-1} * g_j) - 2 \sum_{l=1}^{2^k} \sigma_{l,k}(g_i^{-1} * g_j) \right) \right)_{i,j=1}^{3 \cdot 2^k} . \end{aligned}$$

[6] A^2 is a block matrix with 'nonzero' blocks only in the main diagonal (of blocks) and the two diagonals (of blocks) above and below this. Using the properties (2.13) and (2.14) it is easy to show that the 'nonzero' blocks in the first three rows (of blocks) must be zero too. Now (2.13) and (2.14) hold for all $k \in \mathbb{N} \cup \{0\}$. For $k \geq 4$ the 'nonzero' blocks of the k'th row basically only differ from those of the third row in their dimension and their position. Thus it is straight forward to infer, that all elements of A^2 must be zero.

Thus lemma[7] 2.8 shows, that the absolute value of each element of $(P_k^* P_k - Q_k^* Q_k)$ is less or equal to

$$3^{-1}2^{-2k-1}K(k+1)^{\frac{1}{2}}2^{\frac{k}{2}} = \frac{K}{6}(k+1)^{\frac{1}{2}}2^{-\frac{3k}{2}}.$$

The matrix $Q_k^* P_{k-1}$ is given by

$$Q_k^* P_{k-1} = \left(3^{-1}2^{\frac{1}{2}-2k}\sum_{l=1}^{2^k}\rho_{l,k}\overline{\sigma_{l,k}(g_i)}\tau_{l,k-1}(g_j)\right)_{i,j=1}^{3\cdot 2^k}.$$

Now corollary 2.7 shows, that we can chose the $\rho_{i,k}$'s so that the absolute value of each element of $Q_k^* P_{k-1}$ is less or equal to

$$3^{-1}2^{\frac{1}{2}-2k}Mk^{\frac{1}{2}}2^{\frac{k-1}{2}} \leq \frac{M}{3}(k+1)^{\frac{1}{2}}2^{-\frac{3k}{2}}.$$

□

Theorem 2.10 *For $2 < p < \infty$ the space l_p contains a closed subspace lacking the approximation property.*

Proof: Let $A = (a_{i,j})_{i,j=1}^{\infty}$ be a matrix satisfying the requirements of the preceding lemma. For $i \in \mathbb{N}$ put $A_i = \max_{j\in\mathbb{N}} |a_{i,j}|$ and $b_{i,j} = a_{i,j}\left(\frac{A_j}{A_i}\right)^{\frac{1}{p+1}}$ for $A_i \neq 0$ and $b_{i,j} = 0$ else. Furthermore put $y_i = (b_{i,n})_{n\in\mathbb{N}}$. For the matrix $B = (b_{i,j})_{i,j=1}^{\infty}$ we have $B^2 = A^2 = 0$ and $\mathrm{tr}B = \mathrm{tr}A \neq 0$. Each row and each column of A contains only finitely many nonzero entries and so must B. Furthermore

$$\|y_i\|_{l^p} = \left(\sum_{j=1}^{\infty}|b_{i,j}|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{\infty}\left(\frac{A_j}{A_i}\right)^{\frac{p}{p+1}}\max_{l\in\mathbb{N}}|a_{i,l}|^p\right)^{\frac{1}{p}} = A_i^{\frac{p}{p+1}}\left(\sum_{j=1}^{\infty}A_j^{\frac{p}{p+1}}\right)^{\frac{1}{p}}. \quad (2.16)$$

Since $\frac{p}{p+1} > \frac{2}{3}$ the preceding lemma ensures, that $y_i \in l^p$ for each $i \in \mathbb{N}$.

We want to use lemma 2.4 to show, that $\overline{\mathrm{span}}(\{y_i\}_{i=1}^{\infty})$ fails to have the approximation property. For this purpose denote the standard unit vector space basis of the continuos dual space $l^{p'}(\cong l^q)$ of l^p by $\{e_n\}_{n\in\mathbb{N}}$. Then (2.16) and lemma 2.9 show, that $\sum_{i=1}^{\infty}\|y_i\|_{l^p}\|e_i\|_{l^q} = \sum_{i=1}^{\infty}\|y_i\|_{l^p} < \infty$. Let y be in $\overline{\mathrm{span}}(\{y_i\}_{i=1}^{\infty})$. Since $\mathrm{span}(\{y_i\}_{i=1}^{\infty})$ is a dense subspace of $\overline{\mathrm{span}}(\{y_i\}_{i=1}^{\infty})$, y can be written as $y = \lim_{N\to\infty} y_N$, where $y_N = \sum_{k=1}^{m(N)}\alpha_{k,N}y_k$, $\alpha_{k,N} \in \mathbb{F}$ for all k and all N in $\mathbb{N}$. For each $N \in \mathbb{N}$ considering $\sum_{i=1}^{\infty} y_i e_i(y_N)$ component wise yields

$$\sum_{i=1}^{\infty} y_i e_i(y_N) = \sum_{i=1}^{\infty}\left(y_i\sum_{k=1}^{m(N)}\alpha_{k,N}b_{k,i}\right) = \left(\sum_{i=1}^{\infty}\left(b_{i,n}\sum_{k=1}^{m(N)}\alpha_{k,N}b_{k,i}\right)\right)_{n\in\mathbb{N}}.$$

[7] recall, that at the beginning of this proof the $\sigma_{n,k}$'s and the $\tau_{n,k}$'s were chosen like in lemma 2.8

Due to the structure[8] of the matrix B these series are finite. Therefore,

$$\sum_{i=1}^{\infty} y_i e_i(y_N) \;=\; \left(\sum_{k=1}^{m(N)} \left(\alpha_{k,N} \underbrace{\sum_{i=1}^{\infty} b_{i,n} b_{k,i}}_{=0} \right) \right)_{n\in\mathbb{N}} \;=\; 0.$$

Now the map $x \mapsto \sum_{i=1}^{\infty} y_i e_i(x)$ is continuous. Thus we obtain

$$\sum_{i=1}^{\infty} y_i e_i(y) \;=\; \sum_{i=1}^{\infty} y_i e_i(\lim_{N\to\infty} y_N) \;=\; \lim_{N\to\infty} \left(\sum_{i=1}^{\infty} y_i e_i(y_N) \right) \;=\; 0.$$

But $\sum_{i=1}^{\infty} e_i(y_i) = \mathrm{tr} B \neq 0$. By lemma 2.4 $\overline{\mathrm{span}}(\{y_i\}_{i=1}^{\infty})$ fails to have the approximation property.

□

Taking theorem 2.3 into account, we conclude

Corollary 2.11 *For $2 < p < \infty$ the space l_p contains a closed subspace that does not have a basis.*

□

[8] recall that each row and each column of B contains only finitely many nonzero entries

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