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BACHELOR THESIS

Introduction to absolute neighborhood retract Theory

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by

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Affidavit

I declare in lieu of oath, that I wrote this thesis and performed the associated research myself, using only literature cited in this volume. If text passages from sources are used literally, they are marked as such.

Vienna, Datum

Signature

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1 Introduction

As stated in the title, the aim of this bachelor thesis is to introduce absolute neighborhood retracts, find some examples of them and find out some of their properties. Basic knowledge about topological spaces is required in order to understand this bachelor thesis. All important theorems, which the author does not count as basic knowledge in topology, are either proofed in this bachelor thesis or at least stated with a reference to a place where the proof can be found. What is assumed to be known is strongly related to what is taught during the bachelor study at TU Wien. The introduction to absolute neighborhood retract theory should also enable the reader to easily continue studying absolute neighborhood retracts in [Hu65] or [Mil01]. The book [Hu65], while being rather old, comprises a very comprehensive study of absolute neighborhood retracts. The book [Mil01] introduces absolute neighborhood retracts rather as a tool that is used to proof other theorems.

The bachelor thesis starts with a chapter about topological spaces, where a lot of preliminary work for the following chapters is done. Chapter two is concerned with absolute neighborhood extensors because they are strongly related to absolute neighborhood retracts. The third chapter immediately starts with an investigation of this relation, which enables us to state many results from the second chapter for absolute neighborhood retracts. After this, the relation between special homotopies and absolute neighborhood retracts is investigated.

2 Topological spaces

Topological spaces are the most important mathematical structures in this bachelor thesis. Many properties of topological spaces, which are assumed to be known when reading this paper, can be found in [Kal14]. Throughout this paper, we employ the notation $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{Z}^+ := \{1, 2, 3, \dots\}$ and $\mathbb{R}^+ := (0, +\infty)$.

2.1 Basic properties

Lemma 2.1.1. If A is a closed subset of a topological space $(X, \mathcal{T})$, U is an open subset of $(X, \mathcal{T})$ and V is an open subset of $(A, \mathcal{T}|_A)$ such that $V \subseteq U$, then $V \cup (U \setminus A)$ is open in $(X, \mathcal{T})$.

Proof. By definition of the subspace topology, there exists an open subset W of $(X, \mathcal{T})$ such that $W \cap A = V$. Hence,

$$\begin{aligned} V \cup (U \setminus A) &= (W \cap A) \cup (U \setminus A) = (W \cap A \cap U) \cup (U \setminus A) \\ &= (W \cup (U \setminus A)) \cap (A \cup (U \setminus A)) \cap (U \cup (U \setminus A)) \\ &= (W \cup (U \setminus A)) \cap (U \cup A) \cap U = (W \cup (U \setminus A)) \cap U. \end{aligned}$$

Since W, as well as $U \setminus A$ and U are open in $(X, \mathcal{T})$, so is $(W \cup (U \setminus A))$ and therefore, $V \cup (U \setminus A)$. □

Definition 2.1.2. Let X be a set and $\mathrm{d} : X \times X \to \mathbb{R}$ a metric on X. Given $x \in X$ and $\varepsilon \in \mathbb{R}^+$, we define the open ball

$$\mathrm{B}_{\mathrm{d}}(x, \varepsilon) := \{y \in X \mid \mathrm{d}(x, y) < \varepsilon\}.$$

Furthermore, we define the distance between two non-empty sets $A, B \subseteq X$ by

$$\operatorname{dist}(A, B) := \inf \{\mathrm{d}(a, b) \mid a \in A, b \in B\}$$

and $\operatorname{dist}(x, A) := \operatorname{dist}(\{x\}, A)$.

Lemma 2.1.3. If $(X, \mathcal{T})$ is a metrizable space and $\lambda \in \mathbb{R}^+$, then there exists a metric $\mathrm{d} : X \times X \to [0, \lambda]$ that induces $\mathcal{T}$.

Proof. Let $(X, \mathcal{T})$ be a metrizable space, $\lambda \in \mathbb{R}^+$ and $\tilde{\mathrm{d}}$ a metric that induces $\mathcal{T}$. We define a function $\mathrm{d} : X \times X \to [0, \lambda]$ by $\mathrm{d}(x, y) := \min \left\{\tilde{\mathrm{d}}(x, y), \lambda\right\}$. For arbitrary $x, y, z \in X$ clearly $\mathrm{d}(x, y) \geq 0$. Furthermore,

$$\mathrm{d}(x, y) = 0 \Leftrightarrow \tilde{\mathrm{d}}(x, y) = 0 \Leftrightarrow x = y.$$

The symmetry property of $\tilde{\mathrm{d}}$ is obviously transferred to d and

$$\begin{aligned}\mathrm{d}(x,z) &= \min\left\{\tilde{\mathrm{d}}(x,z),\lambda\right\} \le \min\left\{\tilde{\mathrm{d}}(x,y)+\tilde{\mathrm{d}}(y,z),\lambda\right\} \\ &\le \min\left\{\tilde{\mathrm{d}}(x,y),\lambda\right\} + \min\left\{\tilde{\mathrm{d}}(y,z),\lambda\right\} = \mathrm{d}(x,y)+\mathrm{d}(y,z).\end{aligned}$$

Therefore, d is a metric.

It remains to show that d induces $\mathcal{T}$. In order to do this, consider an arbitrary $x \in X$ and $\varepsilon \in \mathbb{R}^+$. Defining $\rho := \min\{\varepsilon, \lambda\}$, for arbitrary $y \in \mathrm{B}_{\mathrm{d}}(x,\rho)$ we have

$$\tilde{\mathrm{d}}(x,y) \le \min\left\{\tilde{\mathrm{d}}(x,y),\lambda\right\} = \mathrm{d}(x,y) < \varepsilon.$$

Hence, $\mathrm{B}_{\mathrm{d}}(x,\rho) \subseteq \mathrm{B}_{\tilde{\mathrm{d}}}(x,\varepsilon)$. For $z \in \mathrm{B}_{\tilde{\mathrm{d}}}(x,\rho)$ we obtain

$$\mathrm{d}(x,z) = \min\left\{\tilde{\mathrm{d}}(x,z),\lambda\right\} = \tilde{\mathrm{d}}(x,z) < \varepsilon,$$

which shows $\mathrm{B}_{\tilde{\mathrm{d}}}(x,\rho) \subseteq \mathrm{B}_{\mathrm{d}}(x,\varepsilon)$. Therefore, $\tilde{\mathrm{d}}$ and d induce the same topology $\mathcal{T}$. □

Definition 2.1.4. Let $(X,\mathcal{T})$ be a topological space.

Two subsets $A, B \subseteq X$ are said to be *separated* in $(X,\mathcal{T})$, if each is disjoint from the others closure in $(X,\mathcal{T})$. Two points $x, y \in X$ are said to be separated in $(X,\mathcal{T})$, if the sets $\{x\}$ and $\{y\}$ can be separated in $(X,\mathcal{T})$.

A subset U of X is called a *neighborhood* of a subset A of X in $(X,\mathcal{T})$ if there exists a set $O \in \mathcal{T}$ with $A \subseteq O \subseteq U$. The neighborhoods of a point $x \in X$ in $(X,\mathcal{T})$ are the neighborhoods of the set $\{x\}$ in $(X,\mathcal{T})$.

Two subsets A and B of X are said to be *separated by neighborhoods* in $(X,\mathcal{T})$, if there are disjoint neighbourhoods of the two sets.

Definition 2.1.5. We define the following separation axioms for a topological space $(X,\mathcal{T})$.

(T_1) Any two distinct points can be separated in $(X,\mathcal{T})$.

(T_2) Any two distinct points can be separated by neighbourhoods in $(X,\mathcal{T})$.

(T_3) Any closed subset A of $(X,\mathcal{T})$ and any point $x \in X \setminus A$ can be separated by neighbourhoods in $(X,\mathcal{T})$.

(T_4) Any two disjoint closed subsets of $(X,\mathcal{T})$ can be separated by neighbourhoods.

Remark 2.1.6. A topological space that satisfies (T_2) is also called a *Hausdorff* space.

Lemma 2.1.7. Let A and X' be closed subsets of a topological space $(X,\mathcal{T})$ that satisfies (T_4) and $A' := X' \cap A$. If B' is a closed neighborhood of A' in $(X', \mathcal{T}|_{X'})$, then there exists a closed neighborhood B of A in $(X,\mathcal{T})$ such that $B' = X' \cap B$.

Proof. Since B' is a neighborhood of A' in $(X', \mathcal{T}|_{X'})$, there exists an open subset O' of $(X', \mathcal{T}|_{X'})$ such that $A' \subseteq O' \subseteq B'$. There exists an open subset O of $(X,\mathcal{T})$ such that $O \cap X' = O'$. We obtain $X' \setminus O' = X' \setminus (O \cap X') = X' \setminus O$. Therefore, $X' \setminus O'$ is closed in $(X, \mathcal{T})$. Since $X' \setminus O'$ and A are two disjoint and closed subsets of $(X, \mathcal{T})$, a space that satisfies (T_4), there exists a closed neighborhood C of A in $(X, \mathcal{T})$ such that $C \cap (X' \setminus O') = \emptyset$. Hence, $X' \cap C \subseteq O' \subseteq B'$.

Since B' and C are closed subsets of $(X, \mathcal{T})$, so is $B := B' \cup C$. Since C is a neighborhood of A in $(X, \mathcal{T})$, so is B. We have

$$X' \cap B = X' \cap \big(B' \cup C\big) = (X' \cap B') \cup \big(X' \cap C\big) = B' \cup \big(X' \cap C\big) = B'.$$

□

Definition 2.1.8. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The tuple $\big(L, +, (\omega_\lambda)_{\lambda \in \mathbb{K}}, \mathcal{O}\big)$ is said to be a *topological vector space* over $\mathbb{K}$, if and only if $\big(L, +, (\omega_\lambda)_{\lambda \in \mathbb{K}}\big)$ is a vector space over $\mathbb{K}$ with the scalar multiplication $\cdot : \mathbb{K} \times X \to X$ defined by $\cdot(\lambda, x) = \omega_\lambda(x)$, and $(L, \mathcal{O})$ is a topological space such that $+$ and $\cdot$ are continuous functions, when $\mathbb{K}$ is furnished with the standard toplogy and products of sets with the product toplogy.

The topological vector space is called *locally convex*, if and only if for every neighborhood U of 0 in $(L, \mathcal{O})$ there exists a convex neighborhood V of 0 in $(L, \mathcal{O})$ such that $V \subseteq U$.

Remark 2.1.9. Whenever we deal with a subset A of some $\mathbb{R}^n$ or a set with an obvious bijection to a subset of some $\mathbb{R}^n$, then we denote with $(A, \mathcal{E}(A))$ the topological space endowed with the topology induced by the euclidean norm $\|\cdot\|_2 : A \to [0, \infty)$ defined by

$$\|x\|_2 := \left(\sum_{i=1}^{n} x_i^* x_i\right)^{\frac{1}{2}}.$$

2.2 The category of topological spaces

We want to use the notion of a *class* in this bachelor thesis. Since the whole bachelor thesis is based on the ZFC-axioms, we can not define classes as objects, because they do not exist in this axiomatic system. We can think of a class as a property, written down as a mathematical formula. An object, which is always a set in our axiomatic system, is said to be in the class $\mathcal{C}$, if and only if it has the property that characterizes $\mathcal{C}$. We will for example talk about the class of all metrizable spaces.

Definition 2.2.1. A *category* $\mathfrak{C}$ is given by the following.

1. A class $\mathrm{Ob}\,(\mathfrak{C})$ of *objects*.
2. For any two objects A, B of $\mathfrak{C}$ there is a set $\mathrm{Hom}_{\mathfrak{C}}\,(A, B)$ of *morphisms*. All these sets have to be pairwise disjoint. For every object A there is the *identity* morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathfrak{C}}\,(A, A)$.
3. For all A, B, C in $\mathrm{Ob}\,(\mathfrak{C})$ there is a function

$$\mathrm{Hom}_{\mathfrak{C}}\,(B, C) \times \mathrm{Hom}_{\mathfrak{C}}\,(A, B) \to \mathrm{Hom}_{\mathfrak{C}}\,(A, C)$$

given by $(g, f) \mapsto g \circ f$ which is called *composition*. For all A, B, C, D in $\mathrm{Ob}(\mathfrak{C})$ and for all $f \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathfrak{C}}(B, C)$ and $h \in \mathrm{Hom}_{\mathfrak{C}}(C, D)$ we have $h \circ (g \circ f) = (h \circ g) \circ f$ and $\mathrm{id}_B \circ f = f$ as well as $g \circ \mathrm{id}_B = g$.

Definition 2.2.2. A *full subcategory* of a category $\mathfrak{C}$ consists of a subclass of objects of $\mathfrak{C}$ together with all the morphisms between sets of this subclass and the composition of $\mathfrak{C}$.

Example 2.2.3. Examples for categories are the following.

1. The category $\mathfrak{Set}$ of all sets with the functions between sets as the morphisms and the usual composition of functions as the composition.

2. The categroy $\mathfrak{Top}$ of all topological spaces with the continuous functions as morphisms and the usual composition of functions as the composition.

3. The category $\mathfrak{TVS}_{\mathbb{K}}$ of all topological vector spaces over a given topological field $\mathbb{K}$ with the continuous and $\mathbb{K}$-linear functions as morphisms and the usual composition of functions as the composition.

4. The full subcategory $\mathfrak{LCTVS}_{\mathbb{K}}$ of $\mathfrak{TVS}_{\mathbb{K}}$ containing all locally convex topological vector spaces over a given topological field $\mathbb{K}$.

Definition 2.2.4. A full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is said to be *weakly hereditary*, if and only if for all objects $(X, \mathcal{T})$ of $\mathfrak{C}$ and all closed subspaces A of $(X, \mathcal{T})$, the topological space $(A, \mathcal{T}|_A)$ is also an object of $\mathfrak{C}$.

Example 2.2.5. Examples for weakly hereditary categories of topological spaces are the following.

1. The full subcategory $\mathfrak{Haus}$ of $\mathfrak{Top}$ containing all topological spaces that satisfy the axiom (T_2).

2. The full subcategory $\mathfrak{T}4$ of $\mathfrak{Top}$ containing all topological spaces that satisfy the axiom (T_4).

3. The full subcategory $\mathfrak{Met}$ of $\mathfrak{Top}$ containing all metrizable spaces.

4. The full subcategory $\mathfrak{sepMet}$ of $\mathfrak{Top}$ containing all separable and metrizable spaces.

Some properties of categories can be found in [Rei20].

Remark 2.2.6. Instead of introducing categories, it would be possible to solely work with classes, but this would not make much difference for our purposes. Since we will only deal with the category of topological spaces in this bachelor thesis, we are going to write $\operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$ for the set $\operatorname{Hom}_{\mathfrak{Top}}((X, \mathcal{T}), (Y, \mathcal{O}))$ of all continuous function from $(X, \mathcal{T})$ to $(Y, \mathcal{O})$.

Lemma 2.2.7. If $(X, \mathcal{T})$ be a topological space and $(K, \mathcal{O})$ a compact topological space, then the projection $\pi_1 \in \operatorname{Hom}((X \times K, \mathcal{T} \times O), (X, \mathcal{T}))$ defined by $\pi_1((x, k)) := x$ is a closed function.

Proof. Let A be a closed subset of $(X \times K, \mathcal{T} \times O)$ and $(\pi_1((x_i, k_i)))_{i \in I}$ a net in $\pi_1[A]$ that converges in $(X, \mathcal{T})$ to some $x \in X$. Since K is compact, there exists a subnet $\left(k_{i_j}\right)_{j \in J}$ of $(k_i)_{i \in I}$ that converges in $(K, \mathcal{O})$ to some $k \in K$. Given any neighborhoods U of x in $(X, \mathcal{T})$ and V of k in $(K, \mathcal{O})$ there exists $j_0 \in J$ such that for all $j \geq j_0$ we have $\left(x_{i_j}, k_{i_j}\right) \in U \times V$. Hence, $\left(x_{i_j}, k_{i_j}\right)_{j \in J}$ converges in $(X \times K, \mathcal{T} \times O)$ to (x, k). Since A is closed, we have $(x, k) \in A$ and $x = \pi_1((x, k)) \in \pi_1[A]$. This shows that $\pi_1[A]$ is closed in $(X, \mathcal{T})$. $\square$

The following Lemma 2.2.8 is a well known result in topology, which can for example be found in [Kal14, p. 445].

Lemma 2.2.8 (Urysohn's Lemma)**.** If A and B are two closed and disjoint subsets of a topological space $(X, \mathcal{T})$ that satisfies (T_4), then there exists an

$$f \in \operatorname{Hom}((X, \mathcal{T}), ([0,1], \mathcal{E}([0,1])))$$

such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.

2.3 Covers of topological spaces

Definition 2.3.1. Let $(X, \mathcal{T})$ be a topological space. A set $\mathcal{U}$ of subsets of X is called *locally finite* if for every $x \in X$ there exists a neighborhood W of x in $(X, \mathcal{T})$ such that the set $\{U \in \mathcal{U} \mid U \cap W \neq \emptyset\}$ is finite. Another set $\mathcal{V}$ of subsets of X is called a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$. The set $\mathcal{U}$ is called a *cover* of X if $\bigcup \mathcal{U} = X$. The set $\mathcal{U}$ is said to be an *open (closed) cover* of $(X, \mathcal{T})$ if $\mathcal{U}$ is a cover of X and $\mathcal{U} \subseteq \mathcal{T}(\{X \setminus U \mid U \in \mathcal{U}\} \subseteq \mathcal{T})$.

Lemma 2.3.2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{O})$ be two topological spaces and $\mathcal{A}$ a locally finite, closed cover of $(X, \mathcal{T})$. If for all $A \in \mathcal{A}$ there exists a function $f_A \in \operatorname{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$ such that for all $A, B \in \mathcal{A}$ the equality $f_A|_{A \cap B} = f_B|_{A \cap B}$ is satisfied, then the function $f : X \to Y$ defined by $f(x) := f_A(x)$, $x \in A$, satisfies $f \in \operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$.

Proof. Let $x \in X$ be a given point. Since $\mathcal{A}$ is a locally finite cover of $(X, \mathcal{T})$, there exists an open neighborhood U_x of x in $(X, \mathcal{T})$, such that $\mathcal{A}_x := \{A \in \mathcal{A} \mid U_x \cap A \neq \emptyset\}$ is finite. Consequently, the set $\mathcal{A}'_x := \{A \in \mathcal{A} \mid x \in A\}$ is finite as well. Consider an arbitrary neighborhood W of $f(x)$ in $(Y, \mathcal{O})$. For every $A \in \mathcal{A}'_x$ there exists an open neighborhood $\tilde{V}_A$ of x in $(A, \mathcal{T}|_A)$ such that $f_A\left[\tilde{V}_A\right] \subseteq W$. Clearly, $\tilde{V}_A = V_A \cap A$ for some $V_A \in \mathcal{T}$. Consequently

$$V := \left(U_x \cap \left(\bigcap \{V_A \mid A \in \mathcal{A}'_x\}\right)\right) \setminus \left(\bigcup (\mathcal{A}_x \setminus \mathcal{A}'_x)\right)$$

is an open neighborhood of x in $(X, \mathcal{T})$.

Let $z \in V$ be a given point. Since $\mathcal{A}$ is a cover of X, there exists $A \in \mathcal{A}$ such that $z \in A$. Hence, $z \in U_x \cap A$ and consequently, we have $A \in \mathcal{A}_x$. Since $z \notin \bigcup (\mathcal{A}_x \setminus \mathcal{A}'_x)$, we even obtain $A \in \mathcal{A}'_x$. Hence, by definition of V we have $z \in V_A$ and in turn $z \in V_A \cap A = \tilde{V}_A$. Consequently, $f(z) \in W$. □

Remark 2.3.3. We denote by $\operatorname{cl}_{\mathcal{T}}(A)$ the closure of a subset A of a topological space $(X, \mathcal{T})$. For a subset $\mathcal{A}$ of $\mathfrak{P}(X)$ we define $\operatorname{cl}_{\mathcal{T}}[\mathcal{A}] := \{\operatorname{cl}_{\mathcal{T}}(A) \mid A \in \mathcal{A}\}$.

Lemma 2.3.4. Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{A} \subseteq \mathfrak{P}(X)$ is locally finite, then

$$\operatorname{cl}_{\mathcal{T}}\left(\bigcup \mathcal{A}\right) = \bigcup \operatorname{cl}_{\mathcal{T}}[\mathcal{A}].$$

Proof. For any $A \in \mathcal{A}$ we have $A \subseteq \bigcup \mathcal{A}$ and hence, $\mathrm{cl}_{\mathcal{T}}(A) \subseteq \mathrm{cl}_{\mathcal{T}}(\bigcup \mathcal{A})$. Thus, $\bigcup \mathrm{cl}_{\mathcal{T}}[\mathcal{A}] \subseteq \mathrm{cl}_{\mathcal{T}}(\bigcup \mathcal{A})$.

For the converse inclusion we take $x \in \mathrm{cl}_{\mathcal{T}}(\bigcup \mathcal{A})$. There exists a neighborhood U of x in $(X, \mathcal{T})$ such that the set $\mathcal{A}_x := \{A \in \mathcal{A} \mid A \cap U \neq \emptyset\}$ is finite. Since $U \cap (\bigcup (\mathcal{A} \setminus \mathcal{A}_x)) = \emptyset$, we have $x \notin \mathrm{cl}_{\mathcal{T}}(\bigcup (\mathcal{A} \setminus \mathcal{A}_x))$. From

$$x \in \mathrm{cl}_{\mathcal{T}}\Big(\bigcup \mathcal{A}\Big) = \mathrm{cl}_{\mathcal{T}}\Big(\bigcup (\mathcal{A} \setminus \mathcal{A}_x)\Big) \cup \mathrm{cl}_{\mathcal{T}}\Big(\bigcup \mathcal{A}_x\Big)$$

we conclude

$$x \in \mathrm{cl}_{\mathcal{T}}\Big(\bigcup \mathcal{A}_x\Big) = \bigcup \mathrm{cl}_{\mathcal{T}}[\mathcal{A}_x] \subseteq \bigcup \mathrm{cl}_{\mathcal{T}}[\mathcal{A}].$$

□

Lemma 2.3.5. If $\mathcal{A}$ is a locally finite, closed cover of a topological space $(X, \mathcal{T})$, then every point $x \in X$ has an open neighborhood U in $(X, \mathcal{T})$ such that $\{A \in \mathcal{A} \mid A \cap U \neq \emptyset\}$ is finite and coincides with $\{A \in \mathcal{A} \mid x \in A\}$.

Proof. Let $x \in X$ be a given point. Since $\mathcal{A}$ is locally finite, there exists an open neighborhood V of x in $(X, \mathcal{T})$ such that the set $\mathcal{A}_V := \{A \in \mathcal{A} \mid A \cap V \neq \emptyset\}$ is finite. Let $\mathcal{A}_x := \{A \in \mathcal{A} \mid x \in A\}$ and $\mathcal{A}' := \mathcal{A}_V \setminus \mathcal{A}_x$. Since all sets contained in the finite set $\mathcal{A}_V$ are closed in $(X, \mathcal{T})$, so are all sets contained in $\mathcal{A}'$ and therefore also $\bigcup \mathcal{A}'$. Hence, the set $U := V \setminus (\bigcup \mathcal{A}')$ is an open neighborhood of x in $(X, \mathcal{T})$.

If $A \in \mathcal{A}$ is a given set and $A \cap U \neq \emptyset$, then $A \cap V \neq \emptyset$ implies $A \in \mathcal{A}_V$. Furthermore, $A \notin \mathcal{A}'$ and therefore $A \in \mathcal{A}_x$, which implies $x \in A$. □

Lemma 2.3.6. Let A be a subset of a topological space $(X, \mathcal{T})$ and let $\mathcal{B}$ be a locally finite, closed cover of $(X, \mathcal{T})$. If for all $B \in \mathcal{B}$ the set U_B is a neighborhood of $B \cap A$ in $(B, \mathcal{T}|_B)$, then $U := \bigcup \{U_B \mid B \in \mathcal{B}\}$ is a neighborhood of A in $(X, \mathcal{T})$.

Proof. Let x be a given element of A. By Lemma 2.3.5 there exists an open neighborhood N_x of x in $(X, \mathcal{T})$ such that the set $\mathcal{B}_x := \{B \in \mathcal{B} \mid N_x \cap B \neq \emptyset\} = \{B \in \mathcal{B} \mid x \in B\}$ is finite. For a given $B \in \mathcal{B}_x$ we have $x \in B$, which implies $x \in B \cap A \subseteq U_B$. Hence, U_B is a neighborhood of x in $(B, \mathcal{T}|_B)$. Therefore, there exists an open subset W_B of $(X, \mathcal{T})$ such that $x \in W_B \cap B \subseteq U_B$.

Since $\mathcal{B}_x$ is finite, $W_x := N_x \cap (\bigcap \{W_B \mid B \in \mathcal{B}_x\})$ is open in $(X, \mathcal{T})$. Let $z \in W_x$ be given. Since $\mathcal{B}$ is a cover of $(X, \mathcal{T})$, there exists a $\tilde{B} \in \mathcal{B}$ such that $z \in \tilde{B}$. The fact that $z \in N_x$, implies $\tilde{B} \cap N_x \neq \emptyset$. Hence, $\tilde{B} \in \mathcal{B}_x$ and therefore $z \in \tilde{B} \cap W_{\tilde{B}} \subseteq U_{\tilde{B}} \subseteq U$. We conclude that $W_x \subseteq U$.

The set $W := \bigcup \{W_x \mid x \in A\}$ is clearly open in $(X, \mathcal{T})$ satisfying $A \subseteq W \subseteq U$. Therefore U is a neighborhood of A in $(X, \mathcal{T})$. □

Definition 2.3.7. Let $<$ be a well order on a set S and $0 := \min S$. An element $b \in S$ is called *successor* of $a \in S$, if $a < b$ and there exists no $b' \in S$, such that $a < b' < b$. All elements of S which are neither 0 nor the successor of another element of S are called *limit points*.

The proof of the following theorem is from [Bra03] and uses transfinite induction. We will make use of the well-ordering theorem, a proof of which can be found in [GGH20, p. 126].

Lemma 2.3.8. Let $\mathcal{U}$ be a locally finite, open cover of a topological space $(X, \mathcal{T})$ that satisfies (T_4). Then there exists an open cover $\mathcal{V} = \{V_U \mid U \in \mathcal{U}\}$ of $(X, \mathcal{T})$ such for all $U \in \mathcal{U}$ the inclusion $\mathrm{cl}_{\mathcal{T}}(V_U) \subseteq U$ holds true.

Proof. By the well-ordering theorem, there exists a well order $<$ on the set $\mathcal{U}$. Define $0 := \min \mathcal{U}$ and for every $W \in \mathcal{U}$ the set $\mathcal{U}_W := \{U \in \mathcal{U} \mid U > W\}$. We claim that for all $W \in \mathcal{U}$ there exists a function $f_W : \mathcal{U} \setminus \mathcal{U}_W \to \mathcal{T}$, such that the set $f_W[\mathcal{U} \setminus \mathcal{U}_W] \cup \mathcal{U}_W$ is a cover of X and for all $\hat{W} \leq W$ the function f_W is an extension of $f_{\hat{W}}$ and the inclusion $\mathrm{cl}_{\mathcal{T}}\Big(f_W\Big(\hat{W}\Big)\Big) \subseteq \hat{W}$ holds true. The definition $V_U := f_U(U)$ for all $U \in \mathcal{U}$ will then finish proof.

We proof our claim with transfinite induction and start with the base case defining the open subset $A := X \setminus (\bigcup \mathcal{U}_0)$ of $(X, \mathcal{T})$ that clearly satisfies $A \subseteq 0$. Since $(X, \mathcal{T})$ satisfies (T_4) there exists $V_0 \in \mathcal{T}$ such that $A \subseteq V_0 \subseteq \mathrm{cl}_{\mathcal{T}}(V_0) \subseteq 0$. Defining $f_0(0) := V_0$ finishes the the treatment of the base case.

We need to proof that our claim is true for a successor W' of an element $W \in \mathcal{U}$ for which the claim is true. In order to do this, define $\mathcal{V}_{W'} := f_W[\mathcal{U} \setminus \mathcal{U}_W]$ and $A := X \setminus (\bigcup (\mathcal{V}_{W'} \cup \mathcal{U}_{W'}))$. The set A is clearly closed in $(X, \mathcal{T})$ and $\bigcup (\mathcal{V}_{W'} \cup W' \cup \mathcal{U}_{W'}) = \bigcup (\mathcal{V}_{W'} \cup \mathcal{U}_W) = X$ implies $A \subseteq W'$. Since $(X, \mathcal{T})$ satisfies (T_4) there exists an open subset $V_{W'}$ of $(X, \mathcal{T})$ such that $A \subseteq V_{W'} \subseteq \mathrm{cl}_{\mathcal{T}}(V_{W'}) \subseteq W$. Defining $f_{W'} := f_W \cup \{(W', V_{W'})\}$ the claim is true for W'.

It remains to proof the claim for a limit point W under the assumption that for all $\hat{W} < W$ the claim is true. Define $\mathcal{V}_W := \Big\{f_{\hat{W}}\Big(\hat{W}\Big) \mid \hat{W} < W\Big\}$ and the closed subset $A := X \setminus (\bigcup (\mathcal{V}_W \cup \mathcal{U}_W))$ of $(X, \mathcal{T})$. In the case $A \neq \emptyset$ let $x \in A$ be given. There exists $\hat{W} \in \mathcal{U}$ such that $x \in \hat{W}$ and the definition of A guarantees $\hat{W} \leq W$. Supposing $\hat{W} < W$ and recalling that $\mathcal{U}_x := \{U \in \mathcal{U} \mid x \in U\}$ is finite, we find $\tilde{W}$ such that $U < \tilde{W} < W$ for all $U \in \mathcal{U}_x$. By assumption $f_{\tilde{W}}[\mathcal{U} \setminus \mathcal{U}_{\tilde{W}}] \cup \mathcal{U}_{\tilde{W}}$ covers X and $\mathcal{U}_{\tilde{W}} \cap \mathcal{U}_x = \emptyset$ implies $x \in f_{\tilde{W}}[\mathcal{U} \setminus \mathcal{U}_{\tilde{W}}]$. This clearly contradicts $x \notin \bigcup \mathcal{V}_W \supseteq f_{\tilde{W}}[\mathcal{U} \setminus \mathcal{U}_{\tilde{W}}]$ showing $x \in W$. Hence, in any case $A \subseteq W$. Just like in the previous steps we find $V_W \in \mathcal{T}$ such that $A \subseteq V_W \subseteq \mathrm{cl}_{\mathcal{T}}(V_W) \subseteq W$. With the definition $f_W := \{(W, V_W)\} \cup \Big(\bigcup \Big\{f_{\hat{W}} \mid \hat{W} < W\Big\}\Big)$ we are finished. □

Lemma 2.3.9. Let A be a closed subspace of a topological space $(X, \mathcal{T})$ that satisfies (T_4). If $\mathcal{U}$ is a locally finite, open cover of $(A, \mathcal{T}|_A)$ and $\mathcal{V} := \{V_U \mid U \in \mathcal{U}\}$ is a locally finite, open cover of $(X, \mathcal{T})$ such that for all $U \in \mathcal{U}$ the inclusion $V_U \cap A \subseteq U$ is satisfied, then there exists a closed neighborhood B of A in $(X, \mathcal{T})$ and a locally finite, closed cover $\mathcal{W} = \{W_U \mid U \in \mathcal{U}\}$ of $(B, \mathcal{T}|_B)$ such that for all $U \in \mathcal{U}$ the inclusion $W_U \cap A \subseteq U$ holds true and for all finite subsets $\mathcal{U}' \neq \emptyset$ of $\mathcal{U}$ we have

$$\bigcap \mathcal{U}' = \emptyset \Rightarrow \bigcap \{W_U \mid U \in \mathcal{U}'\} = \emptyset.$$

Proof. Since $\mathcal{V}$ is a locally finite cover of a topological space satisfying (T_4), by Lemma 2.3.8 there exists an open, locally finite cover $\mathcal{V}' = \{V'_U \mid U \in \mathcal{U}\}$ of $(X, \mathcal{T})$ such that for all $U \in \mathcal{U}$ the inclusion $\mathrm{cl}_{\mathcal{T}}(V'_U) \subseteq V_U$ holds true. Clearly, $\mathrm{cl}_{\mathcal{T}}[\mathcal{V}']$ is a closed, locally finite cover of $(X, \mathcal{T})$. Hence, by Lemma 2.3.5, for every $x \in X$ we find a neighborhood N_x of x in $(X, \mathcal{T})$, such that for all $U \in \mathcal{U}$ with $\mathrm{cl}_{\mathcal{T}}(V'_U) \cap N_x \neq \emptyset$ we have $x \in \mathrm{cl}_{\mathcal{T}}(V'_U)$.

For every $x \in X$ we define $\mathcal{U}_x := \{U \in \mathcal{U} \mid x \in \mathrm{cl}_{\mathcal{T}}(V'_U)\}$ and claim that the set $O := \{x \in X \mid \bigcap \mathcal{U}_x \neq \emptyset\}$ is open in $(X, \mathcal{T})$. In order to show this, let $x \in O$ and consider $z \in N_x$. For any given $U \in \mathcal{U}_z$ we have $z \in N_x \cap \mathrm{cl}_{\mathcal{T}}(V'_U)$, hence $\mathrm{cl}_{\mathcal{T}}(V'_U) \cap N_x \neq \emptyset$. This implies $x \in \mathrm{cl}_{\mathcal{T}}(V'_U)$, which means $U \in \mathcal{U}_x$ showing $\mathcal{U}_z \subseteq \mathcal{U}_x$. Therefore, $\bigcap \mathcal{U}_x \neq \emptyset$ yields $\bigcap \mathcal{U}_z \neq \emptyset$ and in turn $z \in O$. Since z was arbitrary in N_x, we have $N_x \subseteq O$ showing that O is open in $(X, \mathcal{T})$.

We also claim that $A \subseteq O$. Given any $x \in A$ for every $U \in \mathcal{U}_x$ we have $U \supseteq V_U \cap A \supseteq \mathrm{cl}_{\mathcal{T}}(V'_U) \cap A$ and $x \in \mathrm{cl}_{\mathcal{T}}(V'_U)$. Hence,

$$\bigcap \mathcal{U}_x \supseteq \bigcap \{V_U \cap A \mid U \in \mathcal{U}_x\} \supseteq \bigcap \big\{\mathrm{cl}_{\mathcal{T}}\big(V'_U\big) \cap A \mid U \in \mathcal{U}_x\big\} \supseteq \{x\} \neq \emptyset$$

verifying $x \in O$ and in turn $A \subseteq O$.

Since $(X, \mathcal{T})$ satisfies (T_4), there exists a closed neighborhood B of A in $(X, \mathcal{T})$ such that $A \subseteq B \subseteq O$. For every $U \in \mathcal{U}$ we define $W_U := B \cap \mathrm{cl}_{\mathcal{T}}(V'_U)$ and $\mathcal{W} := \{W_U \mid U \in \mathcal{U}\}$. Since $\overline{\mathcal{V}} = \{\mathrm{cl}_{\mathcal{T}}(V'_U) \mid U \in \mathcal{U}\}$ is a locally finite, closed cover of $(X, \mathcal{T})$, $\mathcal{W}$ is a locally finite, closed cover of $(B, \mathcal{T}|_B)$. Furthermore, $W_U \cap A \subseteq V_U \cap A \subseteq U$ for every $U \in \mathcal{U}$.

It remains to show the second property of $\mathcal{W}$. In order to do this, consider a finite, non-vanishing $\mathcal{U}' \subseteq \mathcal{U}$ with $\bigcap \{W_U \mid U \in \mathcal{U}'\} \neq \emptyset$. For $x \in \bigcap \{W_U \mid U \in \mathcal{U}'\} \subseteq B$ and a given $U \in \mathcal{U}'$ we have $x \in W_U = B \cap \mathrm{cl}_{\mathcal{T}}(V'_U)$ implying $U \in \mathcal{U}_x$. Hence, $\mathcal{U}' \subseteq \mathcal{U}_x$ and the fact that $x \in B \subseteq O$ finally implies $\emptyset \neq \bigcap \mathcal{U}_x \subseteq \bigcap \mathcal{U}'$. □

2.4 Paracompact and fully T_4 spaces

Definition 2.4.1. Let $X \neq \emptyset$ be a set. For $M \subseteq X$ and $\mathcal{A} \subseteq \mathcal{P}(X)$ we define the *star* of M with respect to $\mathcal{A}$ by

$$\mathrm{St}\,(M, \mathcal{A}) := \bigcup \{A \in \mathcal{A} \mid A \cap M \neq \emptyset\}.$$

A set $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *star refinement* of $\mathcal{A}$ if the set $\{\mathrm{St}\,(B, \mathcal{B}) \mid B \in \mathcal{B}\}$ is a refinement of $\mathcal{A}$.

Definition 2.4.2. A topological space $(X, \mathcal{T})$ is called *paracompact* if every open cover of $(X, \mathcal{T})$ has a locally finite, open refinement that covers $(X, \mathcal{T})$. It is said to be *fully* T_4 if every open cover of $(X, \mathcal{T})$ has an open star refinement that covers $(X, \mathcal{T})$. We denote by $\mathfrak{fT4}$ the full subcategory of $\mathfrak{Top}$ that consists of all fully T_4 spaces.

Lemma 2.4.3. The full subcategory $\mathfrak{fT4}$ of $\mathfrak{Top}$ is a weakly hereditary subcategory of $\mathfrak{T4}$.

Proof. Let $(X, \mathcal{T})$ be a topological space that is fully T_4. First, we want to show that $(X, \mathcal{T})$ satisfies (T_4). In order to do this, consider two closed and disjoint subsets A, B of $(X, \mathcal{T})$. The set $\mathcal{U} := \{X \setminus A, X \setminus B\}$ is clearly an open cover of $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is fully T_4, there exists an open cover $\mathcal{V}$ of $(X, \mathcal{T})$, such that $\mathcal{V}$ is a star refinement of $\mathcal{U}$. The set $W_A := \bigcup \{V \in \mathcal{V} \mid V \cap A \neq \emptyset\}$ is an open neighborhood of A in $(X, \mathcal{T})$ and

$W_B := \bigcup \{V \in \mathcal{V} \mid V \cap B \neq \emptyset\}$ is an open neighborhood of B in $(X, \mathcal{T})$. If $x \in W_A$ is given, then there exists $V \in \mathcal{V}$ such that $x \in V$ and $V \cap A \neq \emptyset$. Since $\mathcal{V}$ is a star refinement of $\mathcal{U}$, we obtain $\mathrm{St}(V, \mathcal{V}) \subseteq X \setminus B$. Therefore, $V \cap B = \emptyset$ for all $V' \in \mathcal{V}$ with $V' \cap V \neq \emptyset$. Consequently, $W_A \cap W_B = \emptyset$.

It remains to show that $\mathfrak{f}\mathfrak{T}4$ is weakly hereditary. Let A be a given closed subset of $(X, \mathcal{T})$ and $\mathcal{U}$ an open cover of $(A, \mathcal{T}|_A)$. Clearly, $\mathcal{U} = \left\{U \cap A \mid U \in \hat{\mathcal{U}}\right\}$ for some $\hat{\mathcal{U}} \subseteq \mathcal{T}$. Obviously, the cover $\tilde{\mathcal{U}} := \hat{\mathcal{U}} \cup \{X \setminus A\}$ is an open cover of $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is fully T_4, there exists an open cover $\tilde{\mathcal{V}}$ of $(X, \mathcal{T})$ such that $\tilde{\mathcal{V}}$ is a star refinement of $\tilde{\mathcal{U}}$. The set $\mathcal{V} := \left\{V \cap A \mid V \in \tilde{\mathcal{V}} \wedge V \cap A \neq \emptyset\right\}$ is clearly an open cover of $(A, \mathcal{T}|_A)$. Consider an arbitrary $V \cap A \in \mathcal{V}$ with $V \in \tilde{\mathcal{V}}$. Since $\tilde{\mathcal{V}}$ is a star refinement of $\tilde{\mathcal{U}}$, there exists $U \in \tilde{\mathcal{U}}$ such that $\mathrm{St}\left(V, \tilde{\mathcal{V}}\right) \subseteq U$. The fact, that $V \not\subseteq X \setminus A$, implies $U \in \hat{\mathcal{U}}$. Finally, $A \supseteq \mathrm{St}(V \cap A, \mathcal{V}) \subseteq \mathrm{St}\left(V, \tilde{\mathcal{V}}\right) \subseteq U$ shows that $\mathcal{V}$ is a star refinement of $\mathcal{U}$. □

The proof of the following Theorem 2.4.4 can be found in [BT19, p. 66].

Theorem 2.4.4. Every metrizable space is paracompact.

Lemma 2.4.5. If A is a closed subspace of a fully T_4 topological space $(X, \mathcal{T})$ and $\mathcal{U}$ is a locally finite, open cover of $(A, \mathcal{T}|_A)$, then there exists a locally finite open cover $\mathcal{V} = \{V_U \mid U \in \mathcal{U}\}$ of $(X, \mathcal{T})$, such that $V_U \cap A = U$ for every $U \in \mathcal{U}$.

Proof. For every $x \in A$ there exists an open subset $\tilde{W}_x$ of $(A, \mathcal{T}|_A)$ that contains x and for which the set $\mathcal{U}_x := \left\{U \in \mathcal{U} \mid U \cap \tilde{W}_x \neq \emptyset\right\}$ is finite. We write $W_x \cap A = \tilde{W}_x$ for some $W_x \in \mathcal{T}$. Since $\hat{\mathcal{W}} := \{W_x \mid x \in A\} \cup \{X \setminus A\}$ is clearly an open cover of the fully T_4 space $(X, \mathcal{T})$, it has a star refinement $\mathcal{W}$ that is an open cover of $(X, \mathcal{T})$.

Fix some $\tilde{U} \in \mathcal{U}$ and define $V_{\tilde{U}} := \tilde{U} \cup (X \setminus A)$, which is clearly open in $(X, \mathcal{T})$. For all $U \in \mathcal{U} \setminus \left\{\tilde{U}\right\}$, the set $V_U := U \cup (\mathrm{St}(U, \mathcal{W}) \setminus A)$ is open in $(X, \mathcal{T})$ by Lemma 2.1.1. Therefore, $\mathcal{V} := \{V_U \mid U \in \mathcal{U}\}$ is an open cover of $(X, \mathcal{T})$. For all $U \in \mathcal{U}$ we clearly have $V_U \cap A = U$.

It remains to show that $\mathcal{V}$ is locally finite. In order to show this, let $x \in X$ be a given point. Since $\mathcal{W}$ is a cover of X, there exists $W \in \mathcal{W}$ such that $x \in W$. For any $U \in \mathcal{U} \setminus \left\{\tilde{U}\right\}$ the inclusion $U \subseteq \mathrm{St}(U, \mathcal{W})$ is satisfied. Therefore, if $V_U \cap W \neq \emptyset$, then $W \cap \mathrm{St}(U, \mathcal{W}) \neq \emptyset$. Equivalently, we can say that there exists $W' \in \mathcal{W}$ such that $W \cap W' \neq \emptyset$ and $W' \cap U \neq \emptyset$, which we can write as $U \cap \mathrm{St}(W, \mathcal{W}) \neq \emptyset$. By definition of $\mathcal{W}$ there exists $\hat{W} \in \hat{\mathcal{W}}$ such that $\mathrm{St}(W, \mathcal{W}) \subseteq \hat{W}$. We obtain $U \cap \hat{W} \supseteq U \cap \mathrm{St}(W, \mathcal{W}) \neq \emptyset$. Since $U \cap (X \setminus A) = \emptyset$, there exists $z \in A$ such that $\hat{W} = W_z$. Hence, $U \in \mathcal{U}_z$ which concludes the proof, because $\left\{V_U \in \mathcal{V} \mid U \in \mathcal{U}_z \cup \left\{\tilde{U}\right\}\right\}$ is clearly finite. □

Lemma 2.4.6. Let $(X, \mathcal{T})$ be a paracompact topological space and A and B two closed subsets of $(X, \mathcal{T})$. If

$$\forall x \in B \; \exists U_x, V_x \in \mathcal{T} : (A \subseteq U_x \wedge x \in V_x \wedge U_x \cap V_x = \emptyset),$$

then A and B are separated by neighbourhoods.

Proof. Let A and B be closed subsets of a paracompact topological space $(X, \mathcal{T})$ and for all $x \in B$ the sets U_x and V_x as in the statement of the Lemma. Since $(X, \mathcal{T})$ is paracompact and $\mathcal{V} := \{V_x \mid x \in B\} \cup (X \setminus B)$ is an open cover of $(X, \mathcal{T})$, there exists a locally finite, open cover $\tilde{\mathcal{W}}$ of $(X, \mathcal{T})$ which is a refinement of $\mathcal{V}$. We define the set $\mathcal{W} := \left\{W \in \tilde{\mathcal{W}} \mid W \nsubseteq X \setminus B\right\}$. For every $W \in \mathcal{W}$ there exists $x \in B$ such that $W \subseteq V_x$. Since $V_x \cap U_x = \emptyset$ and $U_x \in \mathcal{T}$, we obtain $\mathrm{cl}_{\mathcal{T}}(W) \subseteq X \setminus U_x \subseteq X \setminus A$. Therefore $\mathrm{cl}_{\mathcal{T}}(W) \cap A = \emptyset$. Since W was arbitrary, we have $A \cap (\bigcup \mathrm{cl}_{\mathcal{T}}[\mathcal{W}]) = \emptyset$. Defining $V := \bigcup \mathcal{W}$ and applying Lemma 2.3.4 we obtain $A \cap \mathrm{cl}_{\mathcal{T}}(V) = \emptyset$, and hence, $A \subseteq X \setminus \mathrm{cl}_{\mathcal{T}}(V) =: U$. Since $\tilde{\mathcal{W}}$ is a cover of X, we clearly have $B \subseteq V$. Therefore, U and V separate A and B. □

Corollary 2.4.7. Every paracompact Hausdorff space satisfies (T_4).

Proof. We start the proof by showing that $(X, \mathcal{T})$ satisfies (T_3). In order to do this, we consider an arbitrary closed subset B of $(X, \mathcal{T})$ and $z \in X \setminus B$. Since $(X, \mathcal{T})$ is Hausdorff, we find for every $x \in B$ a neighborhood V_x of x and a neighborhood U_x of z such that $V_x \cap U_x = \emptyset$. We apply Lemma 2.4.6 with $A := \{z\}$ and obtain immediately that $(X, \mathcal{T})$ satisfies (T_3).

Let A and B be two given disjoint and closed subsets of $(X, \mathcal{T})$. Since we already know that $(X, \mathcal{T})$ satisfies (T_3), we can easily convince ourselves that all preconditions of Lemma 2.4.6 are met, hence we obtain that A and B are separated by neighborhoods. Therefore, $(X, \mathcal{T})$ satisfies (T_4). □

A proof of the following theorem can be found in [BT19, p. 77].

Theorem 2.4.8 (Stone's coincidence thoerem)**.** If $(X, \mathcal{T})$ is an object of $\mathfrak{Top}$ that fulfills (T_1), then the following statements are equivalent.

1. The topological space $(X, \mathcal{T})$ satisfies (T_2) and is paracompact.
2. The topological space $(X, \mathcal{T})$ is fully T_4.

Our next goal is to extend Theorem 2.4.8 by one more equivalence. This will be achieved by Corollary 2.4.12.

Definition 2.4.9. Let $(X, \mathcal{T})$ be a topological space. A set $\mathfrak{F} \subseteq \mathrm{Hom}\left((X, \mathcal{T}), ([0,1], \mathcal{E}([0,1]))\right)$ is called a *partition of unity*, if and only if for every $x \in X$ the sum $\sum_{f \in \mathfrak{F}} f(x)$ converges unconditionally to 1. This partition of unity $\mathfrak{F}$ is said to be *subordinate* to a cover $\mathcal{U}$ of X if for every $f \in \mathfrak{F}$ there is a $U \in \mathcal{U}$ such that $f^{-1}((0,1]) \subseteq U$. The partition of unity $\mathfrak{F}$ is called *locally finite* if for every $x \in X$ there exists a neighborhood V of x such that $\left\{f \in \mathfrak{F} \mid V \cap f^{-1}((0,1]) \neq \emptyset\right\}$ is finite.

Lemma 2.4.10. If $\mathfrak{F}$ is a locally finite partition of unity of a topological space $(X, \mathcal{T})$, then $\mathcal{U} := \left\{f^{-1}((0,1]) \mid f \in \mathfrak{F}\right\}$ is a locally finite, open cover of $(X, \mathcal{T})$.

Proof. Since $(0,1]$ is an open set in $([0,1], \mathcal{E}([0,1]))$ and $f \in \mathrm{Hom}\left((X, \mathcal{T}), ([0,1], \mathcal{E}([0,1]))\right)$, $\mathfrak{F}$ clearly consists of open sets.

In order to show that $\mathcal{U}$ is a cover of X, consider an arbitrary $x \in X$. By $\sum_{f \in \mathfrak{F}} f(x) = 1$ there exists $g \in \mathfrak{F}$ such that $g(x) > 0$. Therefore, $x \in g^{-1}((0,1])$ and by definition we have $g^{-1}((0,1]) \in \mathcal{U}$. Since $\mathfrak{F}$ is a locally finite partition of unity, there is a neighborhood V of x, such that $\left\{f \in \mathfrak{F} \mid V \cap f^{-1}((0,1]) \neq \emptyset\right\}$ is finite. This shows that $\mathcal{U}$ is locally finite. □

Lemma 2.4.11. Let $(X, \mathcal{T})$ be a topological space that satisfies (T_4) and let $\mathcal{U}$ be an open, locally finite cover of $(X, \mathcal{T})$. Then there exists a locally finite partition of unity $\mathfrak{F} = \{f_U \mid U \in \mathcal{U}\}$, such that for all $U \in \mathcal{U}$ the inclusion $f_U^{-1}((0,1]) \subseteq U$ is satisfied.

Proof. By Lemma 2.3.8 there exists an open cover $\mathcal{V} = \{V_U \mid U \in \mathcal{U}\}$ of $(X, \mathcal{T})$, such that for all $U \in \mathcal{U}$ the inclusion $\mathrm{cl}_{\mathcal{T}}(V_U) \subseteq U$ holds true. In accordance with Lemma 2.2.8, there exists for every $U \in \mathcal{U}$ a function $g_U \in \mathrm{Hom}\,((X, \mathcal{T}), ([0,1], \mathcal{E}([0,1])))$ such that $g_U[\mathrm{cl}_{\mathcal{T}}(V_U)] \subseteq \{1\}$ and $g_U^{-1}((0,1]) \subseteq U$. For every $W \in \mathcal{U}$ we set

$$f_W := g_W \left(\sum_{U \in \mathcal{U}} g_U \right)^{-1}.$$

Given $x \in X$ there exists $U_x \in \mathcal{U}$, such that $x \in U_x$. Furthermore, there exists a neighborhood N_x of x in $(X, \mathcal{T})$ such that $\mathcal{U}_x := \{U \in \mathcal{U} \mid N_x \cap U \neq \emptyset\}$ is finite. Consequently,

$$0 < g_{U_x}(x) \leq \sum_{U \in \mathcal{U}} g_U(x) = \sum_{U \in \mathcal{U}_x} g_U(x) < +\infty.$$

Hence, for every $W \in \mathcal{U}$, the function f_W is well defined. Since f_W restricted to N_x consists of sums and quotients of finitely many non-zero, continuous functions,

$$f_W|_{N_x} = g_W \left(\sum_{U \in \mathcal{U}_x} g_U \right)^{-1},$$

it is itself continuous. As continuity is a local property we obtain

$$f_W \in \mathrm{Hom}\,((X, \mathcal{T}), ([0,1], \mathcal{E}([0,1]))).$$

For all $x \in X$ we clearly have $\sum_{U \in \mathcal{U}} f_U(x) = 1$ and $f_U^{-1}((0,1]) \subseteq U$ for all $U \in \mathcal{U}$. From the fact that $\mathfrak{F} := \{f_W \mid W \in \mathcal{U}\}$ is subordinate to $\mathcal{U}$, we immediately obtain that $\mathfrak{F}$ is a locally finite partition of unity. □

Corollary 2.4.12. A topological space $(X, \mathcal{T})$ that satisfies (T_2) is paracompact if and only if every open cover $\mathcal{U}$ of $(X, \mathcal{T})$ admits a locally finite partition of unity which is subordinate to $\mathcal{U}$.

Proof. Let $(X, \mathcal{T})$ be a paracompact topological space that satisfies (T_2) and let $\mathcal{U}$ be an arbitrary open cover of $(X, \mathcal{T})$. Choose a locally finite, open cover $\mathcal{V}$ of $(X, \mathcal{T})$, such that $\mathcal{V}$ is a refinement of $\mathcal{U}$. By Corollary 2.4.7, the space $(X, \mathcal{T})$ satisfies (T_4). Hence, we can apply Lemma 2.4.11.

For the converse, consider an open cover $\mathcal{U}$ of a given topological space $(X, \mathcal{T})$. Assume that there exists a locally finite partition of unity $\mathfrak{F}$ which is subordinate to $\mathcal{U}$. By Lemma 2.4.10 the set $\mathcal{V} := \left\{ f^{-1}((0,1]) \mid f \in \mathfrak{F} \right\}$ is a locally finite, open cover of $(X, \mathcal{T})$. Since $\mathfrak{F}$ is subordinate to $\mathcal{U}$, we conclude that $\mathcal{V}$ is a refinement of $\mathcal{U}$. □

2.5 Abstract simplicial complexes

Definition 2.5.1. A set A is said to be an *abstract simplex* if $\#A \in \mathbb{Z}^+$. A set $\mathcal{K}$ is said to be an *abstract simplicial complex* if it contains only abstract simplices and satisfies that with $A \in \mathcal{K}$ all non-vanishing $B \subseteq A$ belong to $\mathcal{K}$. The set $I := \bigcup \mathcal{K}$ is called the *vertex set* of $\mathcal{K}$ and the elements of I are called *vertices* of $\mathcal{K}$. The elements $A \in \mathcal{K}$ are called *faces* of $\mathcal{K}$ and the *abstract dimension* of a face is defined by $\operatorname{adim} A := \#A - 1$. The *abstract star* of a face is defined as $\operatorname{aSt} A := \{B \in \mathcal{K} \mid A \subseteq B\}$. Although the abstract star of a face clearly depends on the abstract simplicial complex under consideration, we do not include this in the notation since it should always be clear which abstract simplicial complex is meant. An abstract simplicial complex $\mathcal{K}' \subseteq \mathcal{K}$ is called an *abstract subcomplex* of $\mathcal{K}$. The *abstract closure* of $\mathcal{L} \subseteq \mathcal{K}$ is defined as $\operatorname{acl} \mathcal{L} := \bigcap \{\mathcal{K}' \mid \mathcal{K}' \text{ is subcomplex of } \mathcal{K} \wedge \mathcal{L} \subseteq \mathcal{K}'\}$.

Definition 2.5.2. Let $\mathcal{K}$ be an abstract simplicial complex and $I := \bigcup \mathcal{K}$ the vertex set of $\mathcal{K}$. Define

$$Y := \left\{ (\lambda_i)_{i \in I} \in [0,1]^I \mid \{i \in I \mid \lambda_i > 0\} \in \mathcal{K} \wedge \sum_{i \in I} \lambda_i = 1 \right\}$$

and for every $A \in \mathcal{K}$ the set $S_A := \left\{ (\lambda_i)_{i \in I} \in Y \mid \{i \in I \mid \lambda_i > 0\} \subseteq A \right\}$. If $\mathcal{O}$ is the finest topology on Y such that for all $A \in \mathcal{K}$ we have $\iota_A \in \operatorname{Hom}((S_A, \mathcal{E}(S_A)), (Y, \mathcal{O}))$, where ι_A is the inclusion map, then $(Y, \mathcal{O})$ is said to be the *geometric realization* of $\mathcal{K}$. For arbitrary $J \in \mathcal{K}$ the set $\operatorname{gSt} J := \left\{ (\lambda_i)_{i \in I} \in Y \mid \forall j \in J : \lambda_j > 0 \right\}$ is called the *geometric star* of J.

The set S_A considered as a subset of $\mathbb{R}^{\#A}$ can be endowed with the euclidean metric. Accordingly, we denote a ball with radius $\varepsilon \in \mathbb{R}^+$ around a point $(\lambda_i)_{i \in I} \in S_A$ by

$$\mathrm{B}_{S_A}\big((\lambda_i)_{i \in I}, \varepsilon\big) := \left\{ (\mu_i)_{i \in I} \in S_A \mid \left(\sum_{a \in A} (\mu_a - \lambda_a)^2 \right)^{1/2} < \varepsilon \right\}.$$

Lemma 2.5.3. If $\mathcal{K}$ is an abstract simplicial complex with vertex set I and the geometric realization $(Y, \mathcal{O})$ of $\mathcal{K}$, then for every $A \in \mathcal{K}$ the function $f_A : Y \to \mathbb{R}$ defined by $f_A\big((\lambda_i)_{i \in I}\big) := \sum_{a \in A} \lambda_a$ belongs to $\operatorname{Hom}((Y, \mathcal{O}), (\mathbb{R}, \mathcal{E}(\mathbb{R})))$.

Proof. Let $A, B \in \mathcal{K}$ be given and define $\iota_B : S_B \to Y$ as the inclusion function. Consider arbitrary $(\lambda_i)_{i \in I} \in S_B$, $\varepsilon \in \mathbb{R}^+$ and $(\mu_i)_{i \in I} \in \mathrm{B}_{S_B}\Big((\lambda_i)_{i \in I}, \frac{\varepsilon}{\#B}\Big)$. We obtain

$$\begin{aligned} \left| f_A\big(\iota_B\big((\mu_i)_{i \in I}\big)\big) - f_A\big(\iota_B\big((\lambda_i)_{i \in I}\big)\big) \right| &= \left| \sum_{a \in A} (\mu_a - \lambda_a) \right| \leq \sum_{a \in A} |\mu_a - \lambda_a| \\ &= \sum_{b \in A \cap B} |\mu_b - \lambda_b| \leq \sum_{b \in B} |\mu_b - \lambda_b| \\ &\leq \#B \max\{|\mu_b - \lambda_b| \mid b \in B\} \\ &\leq \#B \left(\sum_{b \in B} (\mu_b - \lambda_b)^2 \right)^{1/2} < \#B \frac{\varepsilon}{\#B} = \varepsilon. \end{aligned}$$

Therefore, $f_A \circ \iota_B \in \operatorname{Hom}((S_B, \mathcal{E}(S_B)), (\mathbb{R}, \mathcal{E}(\mathbb{R})))$. Since $B \in \mathcal{K}$ was arbitrary and $\mathcal{O}$ is the final topology of all ι_B, we obtain $f_A \in \operatorname{Hom}((Y, \mathcal{O}), (\mathbb{R}, \mathcal{E}(\mathbb{R})))$. $\square$

Remark 2.5.4. As a consequence of Lemma 2.5.3, $\operatorname{gSt}\{i\} = f_{\{i\}}^{-1}(\mathbb{R}^+)$ is open in $(Y, \mathcal{O})$ for every $i \in I$.

Definition 2.5.5. If X is a set and $\mathcal{U} \subseteq \mathfrak{P}(X)$, then we call

$$\mathcal{K} := \left\{\mathcal{U}' \subseteq \mathcal{U} \mid \bigcap \mathcal{U}' \neq \emptyset \wedge \#\mathcal{U}' \in \mathbb{Z}^+\right\}$$

the *abstract nerve* of $\mathcal{U}$.

Remark 2.5.6. The abstract nerve of $\mathcal{U}$ from Definition 2.5.5 is an abstract simplicial complex with vertex set $\mathcal{U}$. In order to show this, consider $\mathcal{U}' \in \mathcal{K}$ and $\emptyset \neq \mathcal{U}'' \subseteq \mathcal{U}'$. We clearly have $\mathcal{U}'' \subseteq \mathcal{U}$, $\#\mathcal{U}'' \in \mathbb{Z}^+$ as well as $\bigcap \mathcal{U}'' \supseteq \bigcap \mathcal{U}' \neq \emptyset$. Therefore, $\mathcal{U}'' \in \mathcal{K}$.

Lemma 2.5.7. Let $(X, \mathcal{T})$ is a topological space that satisfies (T_4) and let $\mathcal{U}$ be an open, locally finite cover of $(X, \mathcal{T})$ with abstract nerve $\mathcal{K}$ of $\mathcal{U}$. If $(Y, \mathcal{O})$ is the geometric realization of $\mathcal{K}$, then there exists $f \in \operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$ such that for all $U \in \mathcal{U}$ the inclusion $f^{-1}(\operatorname{gSt}\{U\}) \subseteq U$ is satisfied.

Proof. By Lemma 2.4.11 there exists a partition of unity

$$\mathcal{F} = \{f_U \mid U \in \mathcal{U}\} \subseteq \operatorname{Hom}((X, \mathcal{T}), ([0,1], \mathcal{E}([0,1]))), \tag{2.1}$$

such that for every $U \in \mathcal{U}$ we have $f_U^{-1}((0,1]) \subseteq U$. Local finiteness of $\mathcal{U}$ in $(X, \mathcal{T})$ allows us to define a function $f : X \to Y$ by $f(x) := (f_U(x))_{U \in \mathcal{U}}$.

By $\iota : Y \to [0,1]^{\mathcal{U}}$ we denote the embedding and for every $V \in \mathcal{U}$ by $\pi_V : [0,1]^{\mathcal{U}} \to [0,1]$ we denote the projection map defined by $\pi_V\big((\lambda_U)_{U \in \mathcal{U}}\big) := \lambda_V$. Furthermore, let $\mathcal{V} \in \mathcal{K}$ and $\varepsilon \in \mathbb{R}^+$ be given. Recall

$$S_{\mathcal{V}} = \big\{(\lambda_U)_{U \in \mathcal{U}} \in Y \mid \{U \in \mathcal{U} \mid \lambda_U > 0\} \subseteq \mathcal{V}\big\}$$

and the embedding $\iota_{\mathcal{V}} : S_{\mathcal{V}} \to Y$. Consider an arbitrary $(\mu_U)_{U \in \mathcal{U}} \in S_{\mathcal{V}}$ and $W \in \mathcal{U}$. For arbitrary $(\nu_U)_{U \in \mathcal{U}} \in \mathrm{B}_{S_{\mathcal{V}}}\big((\mu_U)_{U \in \mathcal{U}}, \varepsilon\big)$ we obtain

$$\big|\pi_W\big(\iota\big(\iota_{\mathcal{V}}\big((\nu_U)_{U \in \mathcal{U}}\big)\big)\big) - \pi_W\big(\iota\big(\iota_{\mathcal{V}}\big((\mu_U)_{U \in \mathcal{U}}\big)\big)\big)\big| = |\nu_W - \mu_W| \leq \left(\sum_{V \in \mathcal{V}} |\nu_V - \mu_V|^2\right)^{1/2} < \varepsilon.$$

Therefore, $\pi_W \circ \iota \circ \iota_{\mathcal{V}} \in \operatorname{Hom}((S_{\mathcal{V}}, \mathcal{E}(S_{\mathcal{V}})), ([0,1], \mathcal{E}([0,1])))$. Since W and $\mathcal{V}$ were arbitrary, we obtain $\iota \in \operatorname{Hom}\Big((Y, \mathcal{O}), \Big([0,1]^{\mathcal{U}}, \prod_{U \in \mathcal{U}} \mathcal{E}([0,1])\Big)\Big)$.

In order to show the continuity of f, consider $x \in X$. Since $\mathcal{U}$ is a locally finite cover of $(X, \mathcal{T})$, there exists a neighborhood N of x in $(X, \mathcal{T})$, such that $\mathcal{U}_x := \{U \in \mathcal{U} \mid U \cap N \neq \emptyset\}$ is finite. For every $\mathcal{V} \in \mathcal{K}$ the topological space $(S_{\mathcal{V}}, \mathcal{E}(S_{\mathcal{V}}))$ is compact. Hence, $S_{\mathcal{V}} = \iota_{\mathcal{V}}(S_{\mathcal{V}})$ is compact in $(Y, \mathcal{O})$. Consequently, $Z := \bigcup\{S_{\mathcal{V}} \mid \mathcal{V} \in \mathcal{K}, \mathcal{V} \subseteq \mathcal{U}_x\}$ is a finite union of compact sets and therefore a compact subset of $(Y, \mathcal{O})$.

Let $\iota' : Z \to Y$ be the embedding and $\mathrm{id}_Z : Z \to Z$ the identity map. From $\iota \circ \iota' \circ \mathrm{id}_Z \in \operatorname{Hom}\Big((Z, \mathcal{O}|_Z), \Big([0,1]^{\mathcal{U}}, \prod_{U \in \mathcal{U}} \mathcal{E}([0,1])\Big)\Big)$ we conclude

$$\mathrm{id}_Z \in \operatorname{Hom}\left((Z, \mathcal{O}|_Z), \left(Z, \left(\prod_{U \in \mathcal{U}} \mathcal{E}([0,1])\right)|_Z\right)\right).$$

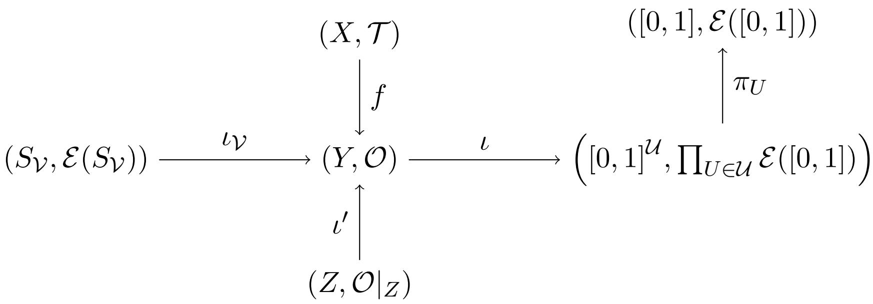

Figure 2.1: Diagram for $\mathcal{V} \in \mathcal{K}$.

Since $(Z, \mathcal{O}|_Z)$ is a compact space, id_Z is a bijection from Z to Z and $\left(Z, \left(\prod_{U\in\mathcal{U}} \mathcal{E}([0,1])\right)|_Z\right)$ is a Hausdorff space, by a Corollary in [Kal14, p.452] id_Z is a homeomorphism and therefore, $\mathcal{O}|_Z = \left(\prod_{U\in\mathcal{U}} \mathcal{E}([0,1])\right)|_Z$.

For $y \in N$ we have $\mathcal{V}_y := \{U \in \mathcal{U} \mid y \in U\} \subseteq \mathcal{U}_x$ which implies $f(y) \in Z$. Since for arbitrary $W \in \mathcal{U}$ we have $\pi_W \circ f = f_W$ and because of (2.1) we obtain

$$f \in \mathrm{Hom}\left((N, \mathcal{T}|_N), \left(Z, \left(\prod_{U\in\mathcal{U}} \mathcal{E}([0,1])\right)|_Z\right)\right) = \mathrm{Hom}\left((N, \mathcal{T}|_N), (Z, \mathcal{O}|_Z)\right).$$

Therefore, we also have $f \in \mathrm{Hom}\left((N, \mathcal{T}|_N), (Y, \mathcal{T})\right)$. Since continuity is a local property, we obtain $f \in \mathrm{Hom}\left((X, \mathcal{T}), (Y, \mathcal{O})\right)$.

It remains to show that for a given $U \in \mathcal{U}$ we have $f^{-1}(\mathrm{gSt}\,\{U\}) \subseteq U$. Consider an arbitrary $x \in f^{-1}(\mathrm{gSt}\,\{U\})$. By definition of f we have $f_U(x) > 0$ and in turn $x \in f_U^{-1}((0,1]) \subseteq U$. Since x was arbitrary, we obtain $f^{-1}(\mathrm{gSt}\,\{U\}) \subseteq U$. ☐

Definition 2.5.8. A subset $\mathcal{K}'$ of an abstract simplicial complex is said to be *finite dimensional* if there exists $n \in \mathbb{N}$ such that for all $A \in \mathcal{K}'$ the inequality $\mathrm{adim}\, A < n$ is satisfied.

The following two lemmas are taken from [Dow47, p. 207-209].

Lemma 2.5.9. Let $\mathcal{K}$ be an abstract simplicial complex with vertex set I and $(Y, \mathcal{O})$ be its geometric realization. Then $\mathcal{U} := \{\mathrm{gSt}\,\{i\} \mid i \in I\}$ is an open cover of $(Y, \mathcal{O})$ that has a refinement $\mathcal{W}$ with abstract nerve $\mathcal{L}$, such that for all $j \in \bigcup \mathcal{L}$ the subset $\mathrm{aSt}\,\{j\}$ of $\mathcal{L}$ is finite dimensional.

Proof. By Lemma 2.5.3, for every $A \in \mathcal{K}$ the function $f_A : Y \to \mathbb{R}$ defined by $f_A\big((\lambda_i)_{i\in I}\big) := \sum_{a\in A} \lambda_a$ belongs to $\mathrm{Hom}\left((Y, \mathcal{O}), (\mathbb{R}, \mathcal{E}(\mathbb{R}))\right)$. We define for every $A \in \mathcal{K}$ and every $\alpha \in [0, +\infty]$ the following subsets of Y

$$G_\alpha(A) := f_A^{-1}\left(\left(1 - \frac{\alpha+1}{\alpha+2} 2^{-(\#A+1)}, +\infty\right)\right), \qquad G_\infty(A) := f_A^{-1}\left(\left(1 - 2^{-(\#A+1)}, +\infty\right)\right)$$

$$\overline{G}_\alpha(A) := f_A^{-1}\left(\left[1 - \frac{\alpha+1}{\alpha+2} 2^{-(\#A+1)}, +\infty\right)\right), \qquad \overline{G}_\infty(A) := f_A^{-1}\left(\left[1 - 2^{-(\#A+1)}, +\infty\right)\right).$$

Clearly, $G_\alpha(A)$ and $G_\infty(A)$ are open and $\overline{G}_\alpha(A)$, $\overline{G}_\infty(A)$ are closed subsets of $(Y,\mathcal{O})$. For every subcomplex $\mathcal{K}'$ of $\mathcal{K}$ we set

$$G_\alpha(\mathcal{K}') := \bigcup\{G_\alpha(A) \mid A \in \mathcal{K}'\}, \qquad G_\infty(\mathcal{K}') := \bigcup\{G_\infty(A) \mid A \in \mathcal{K}'\}$$
$$\overline{G}_\alpha(\mathcal{K}') := \bigcup\{\overline{G}_\alpha(A) \mid A \in \mathcal{K}'\}, \qquad \overline{G}_\infty(\mathcal{K}') := \bigcup\{\overline{G}_\infty(A) \mid A \in \mathcal{K}'\}.$$

$G_\alpha(\mathcal{K}')$ and $G_\infty(\mathcal{K}')$ are open subsets of $(Y,\mathcal{O})$. We claim that $\overline{G}_\alpha(\mathcal{K}')$ is a closed subset of $(Y,\mathcal{O})$. In order to show this, let $(\lambda_i)_{i\in I} \in Y \setminus \overline{G}_\alpha(\mathcal{K}')$ be given and define $A := \{i \in I \mid \lambda_i > 0\} \in \mathcal{K}$. Furthermore we set $\tilde{\mathcal{K}} := \mathcal{K}' \cap (\operatorname{acl}\{A\})$ and $U := G_\alpha(A) \setminus \overline{G}_\alpha\left(\tilde{\mathcal{K}}\right)$. Since A is a finite set and $\operatorname{acl}\{A\}$ consists of all nonempty subsets of A, the set $\tilde{\mathcal{K}}$ is finite. Therefore, $\overline{G}_\alpha\left(\tilde{\mathcal{K}}\right)$ is closed and hence, U is open in $(Y,\mathcal{O})$. $\overline{G}_\alpha\left(\tilde{\mathcal{K}}\right) \subseteq \overline{G}_\alpha(\mathcal{K}')$ yields $(\lambda_i)_{i\in I} \notin \overline{G}_\alpha\left(\tilde{\mathcal{K}}\right)$. By $\sum_{a\in A}\lambda_a = 1$ we have $(\lambda_i)_{i\in I} \in G_\alpha(A)$ implying $(\lambda_i)_{i\in I} \in U$.

Suppose there exists a point $(\mu_i)_{i\in I} \in U \cap \overline{G}_\alpha(\mathcal{K}')$. Then $(\mu_i)_{i\in I} \in \overline{G}_\alpha(B)$ for some $B \in \mathcal{K}' \setminus \tilde{K}$. In the case $C := A \cap B \neq \emptyset$ we have $C \in \operatorname{acl}\{A\}$ and $C \in \mathcal{K}'$. Therefore, $C \in \tilde{\mathcal{K}}$ and hence, $(\mu_i)_{i\in I} \notin \overline{G}_\alpha(C)$. Since $B \notin \tilde{\mathcal{K}}$, we have $\#C < \#B$ and $A \notin \mathcal{K}'$ implies $\#C < \#A$. We obtain

$$\begin{aligned}
\sum_{a\in A}\mu_a &> 1 - \frac{\alpha+1}{\alpha+2}2^{-(\#A+1)} \geq 1 - \frac{\alpha+1}{\alpha+2}2^{-(\#C+2)} \\
\sum_{b\in B}\mu_b &\geq 1 - \frac{\alpha+1}{\alpha+2}2^{-(\#B+1)} \geq 1 - \frac{\alpha+1}{\alpha+2}2^{-(\#C+2)} \\
&\qquad\qquad \sum_{c\in C}\mu_c < 1 - \frac{\alpha+1}{\alpha+2}2^{-(\#C+1)}.
\end{aligned}$$

Adding the first two lines and subtracting the third line gives $\sum_{i\in A\cup B}\mu_i > 1$, which contradicts $\sum_{i\in A\cup B}\mu_i \leq 1$. If $A \cap B = \emptyset$, then we obtain the same contradiction just by adding the first inequalities of the first two lines without introducing C and the third line. Therefore, $U \subseteq Y \setminus \overline{G}_\alpha(\mathcal{K}')$. Since $(\lambda_i)_{i\in I}$ was arbitrarily chosen, $\overline{G}_\alpha(\mathcal{K}')$ is closed in $(Y,\mathcal{O})$. By a similar reasoning $\overline{G}_\infty(\mathcal{K}')$ is closed in $(Y,\mathcal{O})$.

For every $i \in I$ we set

$$\mathcal{K}_i := \operatorname{acl}(\operatorname{aSt}\{i\}) \setminus \operatorname{aSt}\{i\} \quad \text{and} \quad V_i := \operatorname{gSt}\{i\} \setminus \overline{G}_\infty(\mathcal{K}_i).$$

Let $A \in \mathcal{K}_i$ and $\emptyset \neq B \subseteq A$ be given. Since A is an element of the abstract subcomplex $\operatorname{acl}(\operatorname{aSt}\{i\})$ of $\mathcal{K}$, so is B. From $i \notin A$ and $B \subseteq A$ we conclude $i \notin B$ and $B \notin \operatorname{aSt}\{i\}$. Since we just showed that $B \in \mathcal{K}_i$, $\mathcal{K}_i$ is an abstract subcomplex of $\mathcal{K}$. Therefore, the set V_i is well defined and, in accordance with Remark 2.5.4, open.

We claim that $\mathcal{V} := \{V_i \mid i \in I\}$ is a cover of Y. In order to show this, consider $(\lambda_i)_{i\in I} \in Y$. Define $A := \{i \in I \mid \lambda_i > 0\}$ and let $j \in A$ be such that $\lambda_j = \max\{\lambda_i \mid i \in I\}$. Suppose $(\lambda_i)_{i\in I} \notin V_j$. From $(\lambda_i)_{i\in I} \in \operatorname{gSt}\{j\}$, we obtain the existence of a $B \in \mathcal{K}_j$ such that $(\lambda_i)_{i\in I} \in \overline{G}_\infty(B)$. Therefore, $\sum_{b\in B}\lambda_b \geq 1 - 2^{-(\#B+1)}$ and hence,

$$\lambda_j \geq \frac{1}{\#B}\sum_{b\in B}\lambda_b \geq \frac{1}{\#B}\left(1 - 2^{-(\#B+1)}\right) = \frac{2^{\#B+1}-1}{\#B2^{\#B+1}} > 2^{-(\#B+1)}.$$

$B \in \mathcal{K}_j$ implies $j \notin B$. We obtain

$$\sum_{i \in I} \lambda_i = \sum_{b \in B} \lambda_b + \sum_{i \in I \setminus B} \lambda_i > \left(1 - 2^{-(\#B+1)}\right) + 2^{-(\#B+1)} = 1$$

in contradiction to $\sum_{i \in I} \lambda_i = 1$. Hence, $(\lambda_i)_{i \in I} \in V_j$.

We already know that $\mathcal{V}$ is an open cover of $(Y, \mathcal{O})$. In order to show that $\mathcal{V}$ is also locally finite let $(\lambda_i)_{i \in I} \in Y$ be given and set $A := \{i \in I \mid \lambda_i > 0\}$. The set $G_\infty(A)$ is clearly a neighborhood of $(\lambda_i)_{i \in I}$ in $(Y, \mathcal{O})$. Suppose there exists $j \in I \setminus A$ such that $V_j \cap G_\infty(A) \neq \emptyset$. We find $(\mu_i)_{i \in I} \in V_j \cap G_\infty(A)$ and define $B := \{i \in I \mid \mu_i > 0\}$. In the case $A \cap B \neq \emptyset$ we define $C := A \cap B$. From $B \in \operatorname{aSt}\{j\}$ and $j \notin A$ we conclude $C \in \mathcal{K}_j$ implying $(\mu_i)_{i \in I} \notin \overline{G}_\infty(\mathcal{K}_j) \supseteq \overline{G}_\infty(C)$. Therefore, we have

$$\sum_{a \in A} \mu_a = \sum_{c \in C} \mu_c < 1 - 2^{-(\#C+1)} \leq 1 - 2^{-(\#A+1)}, \tag{2.2}$$

which clearly contradicts $(\mu_i)_{i \in I} \in G_\infty(A)$. In the case $A \cap B = \emptyset$, we have the contradiction $0 = \sum_{a \in A} \mu_a > 1 - 2^{-(\#A+1)}$. Hence, such a $j \in I \setminus A$ can not exist. The fact that A is a finite set implies local finiteness of $\mathcal{V}$.

For every $n \in \mathbb{N}$ we define the abstract subcomplex $\mathcal{K}^{(n)} := \{A \in \mathcal{K} \mid \operatorname{adim} A \leq n\}$ of $\mathcal{K}$. Based on these sets we define $H_0 := G_0\big(\mathcal{K}^{(0)}\big)$ as well as $H_1 := G_1\big(\mathcal{K}^{(1)}\big)$ and for all $n \in \mathbb{N}$ with $n > 1$ the set $H_n := G_n\big(\mathcal{K}^{(n)}\big) \setminus \overline{G}_{n-2}\big(\mathcal{K}^{(n-2)}\big)$. $\mathcal{H} := \{H_n \mid n \in \mathbb{N}\}$ clearly consists of open sets. We claim that $\mathcal{H}$ is also a locally finite cover of $(Y, \mathcal{O})$. In order to show this, define $A := \{i \in I \mid \lambda_i > 0\}$ for a given point $(\lambda_i)_{i \in I} \in Y$. Clearly, $(\lambda_i)_{i \in I} \in \overline{G}_{\#A}\big(\mathcal{K}^{(\#A)}\big)$. Defining $m := \min\big\{n \in \mathbb{N} \mid (\lambda_i)_{i \in I} \in \overline{G}_n\big(\mathcal{K}^{(n)}\big)\big\}$ there exists $B \in \mathcal{K}^{(m)}$ such that $(\lambda_i)_{i \in I} \in \overline{G}_m(B)$. From

$$\sum_{b \in B} \lambda_b \geq 1 - \frac{m+1}{m+2} 2^{-(\#B+1)} > 1 - \frac{m+2}{m+3} 2^{-(\#B+1)}$$

we derive $(\lambda_i)_{i \in I} \in G_{m+1}(B) \subseteq G_{m+1}\big(\mathcal{K}^{(m+1)}\big)$. For $m = 0$ we have $(\lambda_i)_{i \in I} \in H_1$. If $m > 0$, $(\lambda_i)_{i \in I} \notin \overline{G}_{m-1}\big(\mathcal{K}^{(m-1)}\big)$ by definition of m and therefore $(\lambda_i)_{i \in I} \in H_{m+1}$. Consider $l \in \mathbb{N}$ with $l < m$. For $(\mu_i)_{i \in I} \in G_l(C)$ with $C \in \mathcal{K}^{(l)}$ we have

$$\sum_{c \in C} \mu_c > 1 - \frac{l+1}{l+2} 2^{-(\#C+1)} \geq 1 - \frac{m}{m+1} 2^{-(\#C+1)}.$$

Hence, $G_l\big(\mathcal{K}^{(l)}\big) \subseteq \overline{G}_{m-1}\big(\mathcal{K}^{(m-1)}\big)$ and we obtain $H_l \cap H_{m+1} = \emptyset$. If $l > m+2$, then we have $G_{m+1}\big(\mathcal{K}^{(m+1)}\big) \subseteq \overline{G}_{l-2}\big(\mathcal{K}^{(l-2)}\big)$ and consequently, $H_{m+1} \cap H_l = \emptyset$. Since we just showed that H_{m+1} is a neighborhood of $(\lambda_i)_{i \in I}$ that meets at most three sets of $\mathcal{H}$, namely H_m, H_{m+1} and H_{m+2}, the cover $\mathcal{H}$ is locally finite.

For $i \in I$ and $n \in \mathbb{N}$ we define $W_{i,n} := V_i \cap H_n$. Since $\mathcal{V}$ and $\mathcal{H}$ are locally finite, open covers of $(Y, \mathcal{O})$, so is $\mathcal{W} := \{W_{i,n} \mid i \in I, n \in \mathbb{N}\}$. By $W_{i,n} \subseteq V_i \subseteq \operatorname{gSt}\{i\}$, $\mathcal{W}$ is a refinement of $\mathcal{U}$.

Denote by $\mathcal{L} := \{M \subseteq I \times \mathbb{N} \mid \#M \in \mathbb{Z}^+ \wedge \bigcap\{W_{i,n} \mid (i,n) \in M\} \neq \emptyset\}$ a set that will represent the abstract nerve of $\mathcal{W}$ and consider $(j,p) \in I \times \mathbb{N}$ and $M \in \operatorname{aSt}\{(j,p)\}$ with respect to $\mathcal{L}$. There exists $(\lambda_i)_{i\in I} \in \bigcap\{W_{l,m} \mid (l,m) \in M\}$. In particular, $(\lambda_i)_{i\in I} \in H_p \cap (\bigcap\{V_l \mid (l,m) \in M\})$. By definition of H_p, there exists $A \in \mathcal{K}^{(p)}$ such that

$$(\lambda_i)_{i\in I} \in G_p(A) \cap \Big(\bigcap\{V_l \mid (l,m) \in M\}\Big) \tag{2.3}$$

Let $(k,q) \in M$ be given. Suppose $k \in I \setminus A$ and consider an arbitrary $(\mu_i)_{i\in I} \in V_k = \mathrm{gSt}\,\{k\} \setminus \overline{G}_\infty(\mathcal{K}_k)$. We define $B := \{i \in I \mid \mu_i > 0\}$ and observe that $\mu_k > 0$ yields $B \in \mathrm{aSt}\,\{k\}$. Therefore, if $C := A \cap B \neq \emptyset$, then it satisfies $C \in \mathrm{acl}\,(\mathrm{aSt}\,\{k\})$. Furthermore, $k \notin C$ and in turn $C \in \mathcal{K}_k$. Hence, $(\mu_i)_{i\in I} \notin \overline{G}_\infty(\mathcal{K}_k) \supseteq \overline{G}_\infty(C)$. Because of (2.2) we derive $(\mu_i)_{i\in I} \notin \overline{G}_\infty(A) \supseteq G_p(A)$. If $A \cap B = \emptyset$, then $\sum_{a\in A} \mu_a = 0 < 1 - 2^{-(\#A+1)}$ and hence, $(\mu_i)_{i\in I} \notin \overline{G}_\infty(A) \supseteq G_p(A)$ as well. Therefore, $V_k \cap G_p(A) = \emptyset$, which clearly contradicts (2.3). Thus, $k \in A$.

$H_p \cap H_q \neq \emptyset$ yields $q \in \{p-1, p, p+1\}$ and further

$$\#M \leq \#(A \times \{p-1, p, p+1\}) = 3\#A \leq 3p.$$

This shows that $\mathrm{aSt}\,\{(j,p)\}$ is finite dimensional. □

Lemma 2.5.10. If $\mathcal{U}$ is an open, locally finite cover of a topological space $(X, \mathcal{T})$ that satisfies (T_4), then there exists a refinement $\mathcal{V}$ of $\mathcal{U}$, such that $\mathcal{V}$ is an open, locally finite cover of $(X, \mathcal{T})$ and the abstract star of each vertex of the abstract nerve of $\mathcal{V}$ is finite dimensional.

Proof. Let $\mathcal{K}$ be the abstract nerve of $\mathcal{U}$ and $(Y, \mathcal{O})$ the geometric realization of $\mathcal{K}$. By Lemma 2.5.7, there exists a function $f \in \mathrm{Hom}\,((X,\mathcal{T}), (Y,\mathcal{O}))$, such that for all $U \in \mathcal{U}$ we have $f^{-1}(\mathrm{gSt}\,\{U\}) \subseteq U$. By Lemma 2.5.9, the set $\mathcal{H} := \{\mathrm{gSt}\,\{U\} \mid U \in \mathcal{U}\}$ is an open cover of $(Y, \mathcal{O})$ and there exists an open cover $\mathcal{W}$ of $(Y, \mathcal{O})$, such that $\mathcal{W}$ is a refinement of $\mathcal{H}$ and the abstract star of each vertex of the abstract nerve of $\mathcal{W}$ is finite dimensional. Define $\mathcal{V} := f^{-1}[\mathcal{W}]$. For any given $W \in \mathcal{W}$ there exists $U \in \mathcal{U}$ such that $W \subseteq \mathrm{gSt}\,\{U\}$. Hence, $f^{-1}(W) \subseteq f^{-1}(\mathrm{gSt}\,\{U\}) \subseteq U$. Therefore, $\mathcal{V}$ is a refinement of $\mathcal{U}$. Since $\mathcal{W}$ is a locally finite open cover of $(Y, \mathcal{O})$, we conclude that $\mathcal{V}$ is a locally finite, open cover of $(X, \mathcal{T})$.

It remains to show that the abstract star of each vertex of the abstract nerve of $\mathcal{V}$ is finite dimensional. In order to do this, let a vertex $V \in \mathcal{V}$ be given and consider an arbitrary $\mathcal{V}' \in \mathrm{aSt}\,\{V\}$. Let $W_V \in \mathcal{W}$, such that $f^{-1}(W_V) = V$ and for every $V' \in \mathcal{V}' \setminus \{V\}$ let $W_{V'} \in \mathcal{W}$ such that $f^{-1}(W_{V'}) = V'$. There exists $x \in \bigcap \mathcal{V}' = f^{-1}(\bigcap\{W_{V'} \mid V' \in \mathcal{V}'\})$, which implies $f(x) \in \bigcap\{W_{V'} \mid V' \in \mathcal{V}'\} \neq \emptyset$. Hence, we have $\{W_{V'} \mid V' \in \mathcal{V}'\} \in \mathrm{aSt}\,\{W_V\}$. Since $\#\mathcal{V}' = \#\{W_{V'} \mid V' \in \mathcal{V}'\}$ and $\mathrm{aSt}\,\{W_V\}$ is finite dimensional, so is $\mathrm{aSt}\,\{V\}$. □

3 Absolute extensors and absolute neighborhood extensors

Definition 3.0.1. A topological space $(Y, \mathcal{O})$ is called *absolute extensor* for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ if for every closed set A of an object $(X, \mathcal{T})$ of $\mathfrak{C}$ and every $f \in \operatorname{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$ there exists an extension $\overline{f} \in \operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$ of f.

Definition 3.0.2. A topological space $(Y, \mathcal{O})$ is called *absolute neighborhood extensor* for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ if for every closed subset A of an object $(X, \mathcal{T})$ of $\mathfrak{C}$ and every $f \in \operatorname{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$ there exists a neighborhood U of A in $(X, \mathcal{T})$ and an extension $\overline{f} \in \operatorname{Hom}((U, \mathcal{T}|_U), (Y, \mathcal{O}))$ of f.

In this chapter we will study absolute extensors and absolute neighborhood extensors, because they are strongly connected to absolute retracts and absolute neighborhood retracts. The connection is stated in Theorem 4.1.1 and Theorem 4.1.4.

3.1 Basic properties

The following two Propositions obviously hold true.

Proposition 3.1.1. Every absolute extensor for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is an absolute neighborhood extensor for $\mathfrak{C}$.

Proposition 3.1.2. Every absolute extensor (respectively absolute neighborhood extensor) for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is also an absolute extensor (respectively absolute neighborhood extensor) for every full subcategory $\mathcal{C}$ of $\mathfrak{C}$.

Requiring that topological spaces are not empty, we obtain the following Proposition.

Proposition 3.1.3. If a subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ contains an object which does not fulfill (T_4), then every absolute neighborhood extensor for $\mathfrak{C}$, which satisfies (T_2), consists of a single point.

Proof. Suppose there exists a Hausdorff space $(Y, \mathcal{O})$ which is an absolute neighborhood extensor for $\mathfrak{C}$ that contains two distinct points $y_1, y_2 \in Y$. Let $(X, \mathcal{T})$ be an object of $\mathfrak{C}$ which is does not satisfy (T_4). Hence, we find two disjoint closed sets B and C of $(X, \mathcal{T})$ which can not be separated by neighborhoods. The set $A := B \cup C$ is closed in $(X, \mathcal{T})$. We define a function $f : A \to Y$ by

$$f(x) := \begin{cases} y_1 & , \text{if } x \in B, \\ y_2 & , \text{if } x \in C, \end{cases}$$

which by Lemma 2.3.2 belongs to $\mathrm{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$. Since $(Y, \mathcal{O})$ is an absolute neighborhood extensor for the class $\mathfrak{C}$, there exists a neighborhood U of A in $(X, \mathcal{T})$ and an extension $\overline{f} \in \mathrm{Hom}((U, \mathcal{T}|_U), (Y, \mathcal{O}))$ of f. Since $(Y, \mathcal{O})$ is Hausdorff, we find a neighborhood V of y_1 and a neighborhood W of y_2 such that $V \cap W = \emptyset$. Therefore, $\overline{f}^{-1}(V)$ is a neighborhood of B in $(U, \mathcal{T}|_U)$, the set $\overline{f}^{-1}(W)$ is a neighborhood of C in $(U, \mathcal{T}|_U)$ and $\overline{f}^{-1}(V) \cap \overline{f}^{-1}(W) = \emptyset$. Thus we separated B and C by neighborhoods, which is a contradiction to our assumptions. □

Proposition 3.1.4. The product space of absolute extensors for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is an absolute extensor for $\mathfrak{C}$.

Proof. Let $(Y, \mathcal{O})$ be the product space of a family $((Y_i, \mathcal{O}_i))_{i \in I}$ of absolute extensors for $\mathfrak{C}$. Consider a closed set A of any arbitrary object $(X, \mathcal{T})$ of $\mathfrak{C}$ and $f \in \mathrm{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$. For any $i \in I$ we define $f_i := \pi_i \circ f$, where $\pi_i : Y \to Y_i$ denotes the projection. Since $f_i \in \mathrm{Hom}((A, \mathcal{T}|_A), (Y_i, \mathcal{O}_i))$ and $(Y, \mathcal{O}|_Y)$ is an absolute extensor for $\mathfrak{C}$, there exists an extension $\overline{f}_i \in \mathrm{Hom}((X, \mathcal{T}), (Y_i, \mathcal{O}_i))$ of f_i. This allows us to define an extension $\overline{f} \in \mathrm{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$ of f by $\overline{f}(x) := \left(\overline{f}_i(x)\right)_{i \in I}$. □

Proposition 3.1.5. Every product space of finitely many absolute neighborhood extensors for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is an absolute neighborhood extensor for $\mathfrak{C}$.

Proof. Given $n \in \mathbb{N}$ set $I := \{1, \ldots, n\}$ and let $((Y_i, \mathcal{O}_i))_{i \in I}$ be absolute neighborhood extensors for $\mathfrak{C}$ and $(Y, \mathcal{O})$ be the corresponding product space. Consider any closed set A of an object $(X, \mathcal{T})$ of $\mathfrak{C}$ and $f \in \mathrm{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$. For every $i \in I$ we define $f_i : A \to Y_i$ by $f_i(x) := \pi_i(f(x))$ satisfying $f_i \in \mathrm{Hom}((A, \mathcal{T}|_A), (Y_i, \mathcal{O}_i))$. Since $(Y_i, \mathcal{O}_i)$ is an absolute neighborhood extensor for $\mathfrak{C}$, there exists a neighborhood U_i of A in $(X, \mathcal{T})$ and an extension $\overline{f}_i \in \mathrm{Hom}((U_i, \mathcal{T}|_{U_i}), (Y_i, \mathcal{O}_i))$ of f_i. We define $U := \bigcap \{U_i \mid i \in I\}$ and $\overline{f} : U \to Y$ by $\overline{f}(x) := \left(\overline{f}_i(x)\right)_{i \in I}$. Clearly, U is a neighborhood of A in $(X, \mathcal{T})$ and $\overline{f} \in \mathrm{Hom}((U, \mathcal{T}|_U), (Y, \mathcal{O}))$ is an extension of f. □

Proposition 3.1.6. Every open subspace of an absolute neighborhood extensor for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is an absolute neighborhood extensor for $\mathfrak{C}$.

Proof. Let $(Y, \mathcal{O})$ be a given absolute neighborhood extensor for $\mathfrak{C}$ and V an arbitrary open subset of $(Y, \mathcal{O})$. Consider any closed subset A of a given object $(X, \mathcal{T})$ of $\mathfrak{C}$ and a function $f \in \mathrm{Hom}((A, \mathcal{T}|_A), (V, \mathcal{O}|_V))$. We clearly also have $f \in \mathrm{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$. Since $(Y, \mathcal{O})$ is an absolute neighborhood extensor for $\mathfrak{C}$, there exists a neighborhood U of $(A, \mathcal{T}|_A)$ in $(X, \mathcal{T})$ and an extension $g \in \mathrm{Hom}((U, \mathcal{T}|_U), (Y, \mathcal{O}))$ of f. The set $g^{-1}(V)$ is an open neighborhood of A in $(X, \mathcal{T})$. The function $\overline{f} \in \mathrm{Hom}\left((g^{-1}(V), \mathcal{T}|_{g^{-1}(V)}), (V, \mathcal{O}|_V)\right)$ defined by $\overline{f}(x) := g(x)$ is an extension of f and hence, $(V, \mathcal{O}|_V)$ is an absolute neighborhood extensor for $\mathfrak{C}$. □

Definition 3.1.7. Let $(X, \mathcal{T})$ be a topological space. A set $A \subseteq X$ is said to be a *retract* of $(X, \mathcal{T})$ if there exists $r \in \mathrm{Hom}((X, \mathcal{T}), (A, \mathcal{T}|_A))$ such that for all $a \in A$ we have $r(a) = a$. The function r is called *retraction*. The set A is called a *neighborhood retract* of $(X, \mathcal{T})$ if there exists a neighborhood U of A in $(X, \mathcal{T})$ such that A is a retract of $(U, \mathcal{T}|_U)$.

Proposition 3.1.8. Every neighborhood retract of an absolute neighborhood extensor for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is an absolute neighborhood extensor for $\mathfrak{C}$.

Proof. Let $(Y,\mathcal{O})$ be an absolute neighborhood extensor for $\mathfrak{C}$ and A a neighborhood retract of $(Y,\mathcal{O})$. Therefore, there exists a neighborhood U of A in $(Y,\mathcal{O})$ and a retraction $r \in \operatorname{Hom}((U,\mathcal{O}|_U),(A,\mathcal{O}|_A))$. Let B be a given closed subset of some object $(X,\mathcal{T})$ of $\mathfrak{C}$ and $f \in \operatorname{Hom}((B,\mathcal{T}|_B),(A,\mathcal{O}|_A))$. Clearly, we have $g \in \operatorname{Hom}((B,\mathcal{T}|_B),(Y,\mathcal{O}))$. Since $(Y,\mathcal{O})$ is an absolute neighborhood extensor, there exists a neighborhood V of B in $(X,\mathcal{T})$ and an extension $\overline{g} \in \operatorname{Hom}((V,\mathcal{T}|_V),(Y,\mathcal{O}))$ of g. We define $W := \overline{g}^{-1}(U)$ and $\overline{f} \in \operatorname{Hom}((W,\mathcal{T}|_W),(A,\mathcal{O}|_A))$ by $\overline{f}(x) := r(\overline{g}(x))$. Since for $x \in B$ we have $\overline{f}(x) = r(\overline{g}(x)) = g(x)$, $\overline{f}$ is an extension of f. ☐

3.2 Examples of absolute extensors and absolute neighborhood extensors

Example 3.2.1. The Tietze extension Theorem, a proof of which can be found in [Kal14, p.446] states that all intervals of the form $[-\lambda,\lambda]$, where $\lambda \in \mathbb{R}^+$, as well as $\mathbb{R}$ are absolute extensors for $\mathfrak{T}4$. As a homeomorphic copy of one of the sets before, every interval $[\mu,\nu]$, where $\mu,\nu \in \mathbb{R}$ and $\mu < \nu$, is an absolute extensor for $\mathfrak{T}4$. By Proposition 3.1.4, all product spaces of these spaces, in particular every $\mathbb{R}^n$ is an absolute extensor for $\mathfrak{T}4$. In accordance with Proposition 3.1.2 and Proposition 3.1.6, every open subspace of $\mathbb{R}^n$ is an absolute neighborhood extensor for $\mathfrak{T}4$.

The following theorem provides us with an even greater number of examples for absolute extensors.

Theorem 3.2.2 (Dugundji extension theorem)**.** Every convex subset of a locally convex topological vector space is an absolute extensor for $\mathfrak{Met}$.

Proof. Let C be a convex subset of an object $\big(L,+,(\omega_\lambda)_{\lambda\in\mathbb{K}},\mathcal{O}\big)$ of $\mathfrak{LCTVS}_{\mathbb{K}}$ and let $(X,\mathcal{T})$ be an object of $\mathfrak{Met}$ with a metric d that induces $\mathcal{T}$. Consider an arbitrary closed subset A of $(X,\mathcal{T})$ and a function $f \in \operatorname{Hom}((A,\mathcal{T}|_A),(C,\mathcal{O}|_C))$. The set $\mathcal{U} := \big\{\mathrm{B_d}\big(x, 4^{-1}\operatorname{dist}(x,A)\big) \mid x \in X\setminus A\big\}$ is an open cover of $\big(X\setminus A,\mathcal{T}|_{X\setminus A}\big)$.

The metrizable space $\big(X\setminus A,\mathcal{T}|_{X\setminus A}\big)$ is paracompact, see Theorem 2.4.4. Hence, there exists a locally finite, open refinement $\mathcal{V}$ of $\mathcal{U}$ which covers $X\setminus A$. By Lemma 2.4.11, for every $V \in \mathcal{V}$ we find a function $g_V \in \operatorname{Hom}\big(\big(X\setminus A,\mathcal{T}|_{X\setminus A}\big),([0,1],\mathcal{E}([0,1]))\big)$ such that $g_V^{-1}((0,1]) \subseteq V$ and $\mathfrak{G} := \{g_V \mid V \in \mathcal{V}\}$ is a locally finite partition of unity of $\big(X\setminus A,\mathcal{T}|_{X\setminus A}\big)$.

Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, for a given $V \in \mathcal{V}$ there exists $x \in X\setminus A$ such that $V \subseteq \mathrm{B_d}\big(x,4^{-1}\operatorname{dist}(x,A)\big)$. Furthermore, there exists $a_V \in A$ such that $\mathrm{d}(x,a_V) < \frac{5}{4}\operatorname{dist}(x,A)$. As for arbitrary $y \in V$ and $a \in A$

$$\operatorname{dist}(x,A) \le \mathrm{d}(x,a) \le \mathrm{d}(x,y) + \mathrm{d}(y,a) \le \frac{1}{4}\operatorname{dist}(x,A) + \mathrm{d}(y,a)$$

we have $\operatorname{dist}(x, A) \leq \frac{4}{3} \operatorname{dist}(y, A)$. Consequently,

$$\begin{aligned}\mathrm{d}(y, a_V) &\leq \mathrm{d}(y, x) + \mathrm{d}(x, a_V) \leq \frac{1}{4} \operatorname{dist}(x, A) + \frac{5}{4} \operatorname{dist}(x, A) = \frac{6}{4} \operatorname{dist}(x, A) \\ &\leq \frac{6}{4}\frac{4}{3} \operatorname{dist}(y, A) = 2 \operatorname{dist}(y, A).\end{aligned}$$

Hence, for every $V \in \mathcal{V}$ we find a point $a_V \in A$ such that for all $y \in V$ we have $\mathrm{d}(y, a_V) \leq 2\,\mathrm{d}(y, A)$. We define a function $\overline{f} : X \to C$ by

$$\overline{f}(x) := \begin{cases} f(x) & \text{, if } x \in A, \\ \sum_{V \in \mathcal{V}} g_V(x) f(a_V) & \text{, if } x \in X \setminus A, \end{cases}$$

and show that $\overline{f} \in \operatorname{Hom}((X, \mathcal{T}), (C, \mathcal{O}|_C))$. In order to prove this, consider some $x \in X \setminus A$. Since $(L, \mathcal{O})$ is locally convex, there exists a neighborhood W of x in $(X \setminus A, \mathcal{T}|_{X \setminus A})$ that only meets the elements of a finite subset $\mathcal{V}'$ of $\mathcal{V}$. Hence, for an arbitrary $y \in W$

$$\overline{f}(y) = \sum_{V \in \mathcal{V}} g_V(y) f(a_V) = \sum_{V \in \mathcal{V}'} g_V(y) f(a_V).$$

Since the last sum is a convex combination, we verified $\overline{f}(y) \in C$. Furthermore, we see that $\overline{f}$ restricted to W is continuous, because it can be written as a finite sum of continuous functions. Therefore, $\overline{f}$ is continuous at any point in $X \setminus A$.

It remains to show continuity at any point $x \in A$. In order to prove this, let W be a given neighborhood of $f(x)$ in $(C, \mathcal{O}|_C)$. Since $(L, \mathcal{O})$ is locally convex, we can assume without loss of generality that W is convex. Since f is continuous, there exists $\delta \in \mathbb{R}^+$ such that $f[\mathrm{B}_{\mathrm{d}}(x, \delta) \cap A] \subseteq W$.

Consider any $y \in \mathrm{B}_{\mathrm{d}}(x, 3^{-1}\delta)$. For $y \in A$ we obviously have $\overline{f}(y) = f(y) \in W$. Let us assume from now on that $y \notin A$. Since $\mathfrak{G}$ is locally finite, the set $\mathcal{V}' := \{V \in \mathcal{V} \mid y \in V\}$ is finite. From $\mathrm{d}(y, A) \leq \mathrm{d}(y, x) < 3^{-1}\delta$ we conclude for any $V \in \mathcal{V}'$

$$\mathrm{d}(x, a_V) \leq \mathrm{d}(x, y) + \mathrm{d}(y, a_V) \leq \mathrm{d}(x, y) + 2\,\mathrm{d}(y, A) < \delta.$$

Therefore, $a_V \in \mathrm{B}_{\mathrm{d}}(x, \delta) \cap A$, which yields $f(a_V) \in W$. We finally obtain

$$\overline{f}(y) = \sum_{V \in \mathcal{V}} g_V(y) f(a_V) = \sum_{V \in \mathcal{V}'} g_V(y) f(a_V) \in W,$$

because the last sum is a convex combination. □

3.3 Local absolute neighborhood extensors

Definition 3.3.1. A topological space $(Y, \mathcal{O})$ is said to be a *local absolute neighborhood extensor* for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ if for all $y \in Y$ there exists a neighborhood U of y in $(Y, \mathcal{O})$, such that $(U, \mathcal{O}|_U)$ is an absolute neighborhood extensor for $\mathfrak{C}$.

Theorem 3.3.2. If all objects of a weakly hereditary, full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ are fully T_4 and satisfy (T_1), then every local absolute neighborhood extensor for $\mathfrak{C}$ is an absolute neighborhood extensor for $\mathfrak{C}$.

Proof. Let $(Y, \mathcal{O})$ be a given local absolute neighborhood extensor for $\mathfrak{C}$, let A be a closed subset of an object $(X, \mathcal{T})$ of $\mathfrak{C}$ and $f \in \operatorname{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$. Since $(Y, \mathcal{O})$ is a local absolute neighborhood extensor, there exists an open cover $\mathcal{U}$ of $(Y, \mathcal{O})$ that only consists of absolute neighborhood extensors for $\mathfrak{C}$. In accordance with Lemma 2.4.3, $(A, \mathcal{T}|_A)$ is fully T_4 and satisfies (T_4). Since $(A, \mathcal{T}|_A)$ also satisfies (T_1), we can apply Theorem 2.4.8 and obtain that $(A, \mathcal{T}|_A)$ is paracompact. Since $f^{-1}[\mathcal{U}]$ is a cover of $(A, \mathcal{T}|_A)$, there exists an open, locally finite cover $\tilde{\mathcal{V}}$ of $(A, \mathcal{T}|_A)$ that is a refinement of $f^{-1}[\mathcal{U}]$. By Lemma 2.5.10 we may assume that the abstract star of each vertex of the abstract nerve of $\tilde{\mathcal{V}}$ is finite dimensional. For every $\tilde{V} \in \tilde{\mathcal{V}}$ we choose $U_{\tilde{V}} \in \mathcal{U}$ such that

$$\tilde{V} \subseteq f^{-1}(U_{\tilde{V}}). \tag{3.1}$$

In accordance with Lemma 2.4.5 there exists a locally finite open cover $\mathcal{W} = \left\{ W_{\tilde{V}} \mid \tilde{V} \in \tilde{\mathcal{V}} \right\}$ of $(X, \mathcal{T})$ such that for all $\tilde{V} \in \tilde{\mathcal{V}}$ the equality $W_{\tilde{V}} \cap A = \tilde{V}$ is satisfied. Applying Lemma 2.3.9 we obtain a closed neighborhood F of A in $(X, \mathcal{T})$ and a locally finite closed cover $\mathcal{F} = \left\{ F_{\tilde{V}} \mid \tilde{V} \in \tilde{\mathcal{V}} \right\}$ of $(F, \mathcal{T}|_F)$ such that for all $\tilde{V} \in \mathcal{V}$ we have

$$F_{\tilde{V}} \cap A \subseteq \tilde{V} \tag{3.2}$$

and for all $\tilde{\mathcal{V}}' \subseteq \tilde{\mathcal{V}}$ with $\#\tilde{\mathcal{V}}' \in \mathbb{Z}^+$ the implication

$$\bigcap \tilde{\mathcal{V}}' = \emptyset \Rightarrow \bigcap \left\{ F_{\tilde{V}} \mid \tilde{V} \in \tilde{\mathcal{V}}' \right\} = \emptyset \tag{3.3}$$

holds true.

We choose a subset $\mathcal{V} \subset \tilde{\mathcal{V}}$ such that for all $\tilde{V} \in \tilde{\mathcal{V}}$ there exists exactly one $V \in \mathcal{V}$ such that $F_V = F_{\tilde{V}}$. Hence, $\mathcal{V} \ni V \mapsto F_V \in \mathcal{F}$ constitutes a bijection. Let $\mathcal{K}$ be the abstract nerve of $\mathcal{F}$ and consider an arbitrary $V \in \mathcal{V}$ and $\mathcal{F}' \in \mathcal{K}$, such that $\mathcal{F}' \in \operatorname{aSt}\{F_V\}$. Due to our choice of $\mathcal{V}$, there is a unique $\mathcal{V}' \subseteq \mathcal{V}$, such that $\mathcal{F}' = \{F_{V'} \mid V' \in \mathcal{V}'\}$. The fact that by (3.3) we have $\bigcap \mathcal{V}' \neq \emptyset$ together with $\#\mathcal{V}' = \#\mathcal{F}' \in \mathbb{Z}^+$ implies $\mathcal{V}' \in \operatorname{aSt}\{V\}$. Since $\operatorname{aSt}\{V\}$ is finite dimensional, so is $\operatorname{aSt}\{F_V\}$. Therefore, recalling that $\hat{\mathcal{F}} \in \operatorname{aSt}\mathcal{F}' \Leftrightarrow \mathcal{F}' \subseteq \hat{\mathcal{F}}$, we can define $p : \mathcal{K} \to \mathbb{N}$ by

$$p(\mathcal{F}') := \sup \left\{ \#\hat{\mathcal{F}} - \#\mathcal{F}' \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subseteq \hat{\mathcal{F}} \right\}.$$

For every $\mathcal{F}' \in \mathcal{K}$, we define

$$D_{\mathcal{F}'} := \left(\bigcap \mathcal{F}' \right) \setminus \left(\bigcup (\mathcal{F} \setminus \mathcal{F}') \right), \quad D_{\operatorname{aSt}\mathcal{F}'} := \bigcup \left\{ D_{\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \operatorname{aSt}\mathcal{F}' \right\}$$

and $A_{\mathcal{F}'} := D_{\mathcal{F}'} \cap A$ as well as $A_{\operatorname{aSt}\mathcal{F}'} := D_{\operatorname{aSt}\mathcal{F}'} \cap A$. Define $\mathcal{D} := \{D_{\mathcal{F}'} \mid \mathcal{F}' \in \mathcal{K}\}$ and $\mathcal{D}_{\operatorname{aSt}} := \{D_{\operatorname{aSt}\mathcal{F}'} \mid \mathcal{F}' \in \mathcal{K}\}$. For a given $x \in F$, there exists a neighborhood N_x of x in $(F, \mathcal{T}|_F)$, such that

$$\mathcal{F}_x := \{F_V \in \mathcal{F} \mid V \in \mathcal{V} \wedge F_V \cap N_x \neq \emptyset\} \quad \text{and} \quad \mathcal{F}'_x := \{F_V \in \mathcal{F} \mid V \in \mathcal{V} \wedge x \in F_V\}$$

are finite. We claim that $\mathcal{D}$ covers F by mutually disjoint sets. We clearly have $\mathcal{F}'_x \in \mathcal{K}$ and $x \in D_{\mathcal{F}'_x}$ and hence, we conclude that $\mathcal{D}$ is a cover of F. Let $\mathcal{F}' \in \mathcal{K} \setminus \{\mathcal{F}'_x\}$ be given.

In case there exists $F' \in \mathcal{F}'$ such that $x \notin F'$, we clearly have $x \notin D_{\mathcal{F}'}$. Otherwise, there exists $F \in \mathcal{F} \setminus \mathcal{F}'$ such that $x \in F$, which implies $x \notin D_{\mathcal{F}'}$. Hence, we showed that the arbitrary $x \in F$ is contained in exactly one element of $\mathcal{D}$, namely $D_{\mathcal{F}'_x}$.

If $\mathcal{F}' \in \mathcal{K}$ and $x \in \bigcap \mathcal{F}'$, then $\mathcal{F}'_x \in \operatorname{aSt} \mathcal{F}'$, which implies $x \in D_{\mathcal{F}'_x} \subseteq D_{\operatorname{aSt} \mathcal{F}'}$. If, on the other hand, $x \in D_{\operatorname{aSt} \mathcal{F}'}$, then we obtain $x \in \bigcap \mathcal{F}'$. Therefore, $D_{\operatorname{aSt} \mathcal{F}'} = \bigcap \mathcal{F}'$, showing that $\mathcal{D}_{\operatorname{aSt}}$ is a closed cover of $(F, \mathcal{T}|_F)$. We claim that $\mathcal{D}_{\operatorname{aSt}}$ is also locally finite. Given $x \in F$ we conclude from $N_x \cap D_{\operatorname{aSt} \mathcal{F}'} \neq \emptyset$ that there exists $\hat{\mathcal{F}} \supseteq \mathcal{F}'$ such that $N_x \cap D_{\hat{\mathcal{F}}} \neq \emptyset$. This implies that for all $\hat{F} \in \hat{\mathcal{F}}$ we have $N_x \cap \hat{F} \neq \emptyset$ and therefore, $\mathcal{F}' \subseteq \hat{\mathcal{F}} \subseteq \mathcal{F}_x$. Since $\mathcal{F}_x$ contains only finitely many subsets, we conclude that $\mathcal{D}_{\operatorname{aSt}}$ is locally finite.

For $\mathcal{F}' \in \mathcal{K}$ we define $U_{\mathcal{F}'} := \bigcap \{U_V \mid V \in \mathcal{V} \wedge F_V \in \mathcal{F}'\}$. As a finite intersection of open sets $U_{\mathcal{F}'}$ is open in $(U_V, \mathcal{O}|_{U_V})$ for every $V \in \mathcal{V}$ with $F_V \in \mathcal{F}'$. Since $(U_V, \mathcal{O}|_{U_V})$ is an absolute neighborhood extensor for $\mathfrak{C}$, in accordance with Proposition 3.1.6 also $U_{\mathcal{F}'}$ is an absolute neighborhood extensor for $\mathfrak{C}$. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{K}$ with $\mathcal{F}_1 \subseteq \mathcal{F}_2$ we clearly have $U_{\mathcal{F}_2} \subseteq U_{\mathcal{F}_1}$. Due to (3.1) and (3.2) for arbitrary $\mathcal{F}' \in \mathcal{K}$ we obtain

$$\begin{aligned} f[A_{\mathcal{F}'}] &\subseteq f\Big[\bigcap \big\{F_V \cap A \mid V \in \mathcal{V}, F_V \in \mathcal{F}'\big\}\Big] \subseteq f\Big[\bigcap \big\{V \in \mathcal{V} \mid F_V \in \mathcal{F}'\big\}\Big] \\ &\subseteq \bigcap \big\{f[V] \mid V \in \mathcal{V} \wedge F_V \in \mathcal{F}'\big\} \subseteq U_{\mathcal{F}'}. \end{aligned} \tag{3.4}$$

Now we start with the main part of the proof. It consists of a long induction, showing the following claim for all $n \in \mathbb{N}$:

If $\mathcal{F}' \in \mathcal{K}$ and $p(\mathcal{F}') \leq n$, then there exists $B_{\mathcal{F}'} \subseteq F$, a function

$$g_{\mathcal{F}'} \in \operatorname{Hom}\big((B_{\mathcal{F}'}, \mathcal{T}|_{B_{\mathcal{F}'}}), (Y, \mathcal{O})\big),$$

a set $B_{\operatorname{aSt} \mathcal{F}'}$ and another function $g_{\operatorname{aSt} \mathcal{F}'} \in \operatorname{Hom}\big((B_{\operatorname{aSt} \mathcal{F}'}, \mathcal{T}|_{B_{\operatorname{aSt} \mathcal{F}'}}), (Y, \mathcal{O})\big)$ such that $B_{\operatorname{aSt} \mathcal{F}'} = \bigcup \Big\{B_{\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \operatorname{aSt} \mathcal{F}'\Big\}$, $g_{\operatorname{aSt} \mathcal{F}'}$ is an extension of $g_{\hat{\mathcal{F}}}$ for all $\hat{\mathcal{F}} \in \operatorname{aSt} \mathcal{F}'$ as well as

$$A_{\mathcal{F}'} \subseteq B_{\mathcal{F}'} \subseteq D_{\mathcal{F}'}, \tag{3.5}$$

$$g_{\mathcal{F}'}|_{A_{\mathcal{F}'}} = f|_{A_{\mathcal{F}'}}, \tag{3.6}$$

$$g_{\mathcal{F}'}[B_{\mathcal{F}'}] \subseteq U_{\mathcal{F}'}, \tag{3.7}$$

$$B_{\operatorname{aSt} \mathcal{F}'} \text{ is a neighborhood of } A_{\operatorname{aSt} \mathcal{F}'} \text{ in } \big(D_{\operatorname{aSt} \mathcal{F}'}, \mathcal{T}|_{D_{\operatorname{aSt} \mathcal{F}'}}\big), \tag{3.8}$$

$$B_{\operatorname{aSt} \mathcal{F}'} \text{ is closed in } (F, \mathcal{T}|_F). \tag{3.9}$$

We start with the base case $n = 0$ and $\mathcal{F}' \in \mathcal{K}$ with $p(\mathcal{F}') = 0$. This implies $\operatorname{aSt} \mathcal{F}' = \{\mathcal{F}'\}$ and consequently, $D_{\operatorname{aSt} \mathcal{F}'} = D_{\mathcal{F}'}$ as well as $A_{\operatorname{aSt} \mathcal{F}'} = A_{\mathcal{F}'}$. Since $D_{\operatorname{aSt} \mathcal{F}'}$ is a closed subset of $(F, \mathcal{T}|_F)$ and F is a closed subset of $(X, \mathcal{T})$, $D_{\operatorname{aSt} \mathcal{F}'}$ is a closed subset of $(X, \mathcal{T})$. Since $\mathfrak{C}$ is weakly hereditary, $\big(D_{\operatorname{aSt} \mathcal{F}'}, \mathcal{T}|_{D_{\operatorname{aSt} \mathcal{F}'}}\big)$ is an object of $\mathfrak{C}$. Hence, it satisfies (T_4), see Lemma 2.4.3. Since $(Y, \mathcal{O})$ is an absolute neighborhood extensor for $\mathfrak{C}$, and $A_{\operatorname{aSt} \mathcal{F}'} = A_{\mathcal{F}'}$ is a closed subset of $\big(D_{\operatorname{aSt} \mathcal{F}'}, \mathcal{T}|_{D_{\operatorname{aSt} \mathcal{F}'}}\big)$, there exists a closed neighborhood $B_{\mathcal{F}'}$ of $A_{\mathcal{F}'}$ in $\big(D_{\operatorname{aSt} \mathcal{F}'}, \mathcal{T}|_{D_{\operatorname{aSt} \mathcal{F}'}}\big)$ and an extension $g_{\mathcal{F}'} \in \operatorname{Hom}\big((B_{\mathcal{F}'}, \mathcal{T}|_{B_{\mathcal{F}'}}), (U_{\mathcal{F}'}, \mathcal{O}|_{U_{\mathcal{F}'}})\big)$ of $\tilde{f} \in \operatorname{Hom}\big((A_{\mathcal{F}'}, \mathcal{T}|_{A_{\mathcal{F}'}}), (U_{\mathcal{F}'}, \mathcal{O}|_{U_{\mathcal{F}'}})\big)$ defined by $\tilde{f}(x) = f(x)$, which is well defined because of (3.4). The properties (3.5), (3.6) and (3.7) are obviously satisfied. With the definition $B_{\operatorname{aSt} \mathcal{F}'} := B_{\mathcal{F}'}$ and $g_{\operatorname{aSt} \mathcal{F}'} := g_{\mathcal{F}'}$ it is clear that (3.8) and (3.9) are satisfied as well. This finishes the base case.

It remains to prove the induction step. Assume that our claim is true for some $n \in \mathbb{N}$ and consider $\mathcal{F}' \in \mathcal{K}$ with $p(\mathcal{F}') = n+1$. As $p\left(\hat{\mathcal{F}}\right) < p(\mathcal{F}')$ for any $\hat{\mathcal{F}} \supsetneq \mathcal{F}'$, we can define

$$\tilde{A}_{\mathcal{F}'} := \bigcup \left\{ A_{\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subsetneq \hat{\mathcal{F}} \right\} = \bigcup \left\{ A_{\mathrm{aSt}\,\hat{F}} \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subsetneq \hat{\mathcal{F}} \right\}, \tag{3.10}$$

$$\tilde{B}_{\mathcal{F}'} := \bigcup \left\{ B_{\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subsetneq \hat{\mathcal{F}} \right\} = \bigcup \left\{ B_{\mathrm{aSt}\,\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subsetneq \hat{\mathcal{F}} \right\}, \tag{3.11}$$

$$\tilde{D}_{\mathcal{F}'} := \bigcup \left\{ D_{\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subsetneq \hat{\mathcal{F}} \right\} = \bigcup \left\{ D_{\mathrm{aSt}\,\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subsetneq \hat{\mathcal{F}} \right\}. \tag{3.12}$$

The first equality from (3.10) together with (3.5) and the fact that $\mathcal{D}$ consists of mutually disjoint sets assures that $\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'}$ is a union of mutually disjoint sets as well. The inclusions (3.4) and (3.7) allow us to define $\tilde{g}_{\mathcal{F}'} : \tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'} \to U_{\mathcal{F}'}$ by

$$\tilde{g}_{\mathcal{F}'}(x) := \begin{cases} f(x) & \text{, if } x \in A_{\mathcal{F}'}, \\ g_{\hat{\mathcal{F}}}(x) & \text{, if } x \in B_{\hat{\mathcal{F}}} \text{ for } \mathcal{F}' \subsetneq \hat{\mathcal{F}} \in \mathcal{K}. \end{cases}$$

Property (3.5) assures $A_{\mathrm{aSt}\,\mathcal{F}'} \subseteq \tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'}$. Using (3.6), we observe $\tilde{g}_{\mathcal{F}'}|_{A_{\mathrm{aSt}\,\mathcal{F}'}} = f|_{A_{\mathrm{aSt}\,\mathcal{F}'}}$ and $\tilde{g}_{\mathcal{F}'}|_{B_{\mathrm{aSt}\,\hat{\mathcal{F}}}} = g_{\mathrm{aSt}\,\hat{\mathcal{F}}}$ for all $\hat{\mathcal{F}} \supsetneq \mathcal{F}'$. Recall that $g_{\mathrm{aSt}\,\hat{\mathcal{F}}}$ is continuous for $\hat{\mathcal{F}} \supsetneq \mathcal{F}'$. Since $\mathcal{D}_{\mathrm{aSt}}$ is locally finite, the second equality of (3.11) is a representation of $\tilde{B}_{\mathcal{F}'}$ as a union of a locally finite set consisting of closed subsets of $\left(\tilde{D}_{\mathcal{F}'}, \mathcal{T}|_{\tilde{D}_{\mathcal{F}'}}\right)$ Therefore, we can apply Lemma 2.3.4 showing that $\tilde{B}_{\mathcal{F}'}$ is closed in $\left(\tilde{D}_{\mathcal{F}'}, \mathcal{T}|_{\tilde{D}_{\mathcal{F}'}}\right)$. Furthermore, all $B_{\mathrm{aSt}\,\hat{\mathcal{F}}}$ for $\hat{\mathcal{F}} \supsetneq \mathcal{F}'$ together with $A_{\mathrm{aSt}\,\mathcal{F}'}$ form a locally finite, closed cover of $\left(\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'}, \mathcal{T}|_{\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'}}\right)$. Thus, we can apply Lemma 2.3.2 and obtain

$$\tilde{g}_{\mathcal{F}'} \in \mathrm{Hom}\left(\left(\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'}, \mathcal{T}|_{\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'}}\right), \left(U_{\mathcal{F}'}, \mathcal{O}|_{U_{\mathcal{F}'}}\right)\right).$$

Recall that $\mathcal{D}_{\mathrm{aSt}}$ is locally finite and $\left\{ D_{\mathrm{aSt}\,\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \mathcal{K}, \mathcal{F}' \subsetneq \hat{\mathcal{F}} \right\} \subseteq \mathcal{D}_{\mathrm{aSt}}$ only consists of closed subsets of $\left(\tilde{D}_{\mathcal{F}'}, \mathcal{T}|_{\tilde{D}_{\mathcal{F}'}}\right)$. The second equality from (3.12) and an application of Lemma 2.3.4 assert that $\tilde{D}_{\mathcal{F}'}$ is a closed subset of $(F, \mathcal{T}|_F)$. By (3.8) and Lemma 2.3.6, we obtain that $\tilde{B}_{\mathcal{F}'}$ is a neighborhood of $\tilde{A}_{\mathcal{F}'}$ in $\left(\tilde{D}_{\mathcal{F}'}, \mathcal{T}|_{\tilde{D}_{\mathcal{F}'}}\right)$. As a closed subspace of $(F, \mathcal{T}|_F)$, the space $\left(D_{\mathrm{aSt}\,\mathcal{F}'}, \mathcal{T}|_{D_{\mathrm{aSt}\,\mathcal{F}'}}\right)$ satisfies (T_4). Since we clearly have $\tilde{D}_{\mathcal{F}'} \subseteq D_{\mathrm{aSt}\,\mathcal{F}'}$, we obtain that $\tilde{D}_{\mathcal{F}'}$ is a closed subset of $\left(D_{\mathrm{aSt}\,\mathcal{F}'}, \mathcal{T}|_{D_{\mathrm{aSt}\,\mathcal{F}'}}\right)$. By definition $A_{\mathrm{aSt}\,\mathcal{F}'}$ is a closed subset of $\left(D_{\mathrm{aSt}\,\mathcal{F}'}, \mathcal{T}|_{D_{\mathrm{aSt}\,\mathcal{F}'}}\right)$ and it satisfies $\tilde{A}_{\mathcal{F}'} = \tilde{D}_{\mathcal{F}'} \cap A_{\mathrm{aSt}\,\mathcal{F}'}$. Therefore, in accordance with Lemma 2.1.7 for $X = D_{\mathrm{aSt}\,\mathcal{F}'}$, $X' = \tilde{D}_{\mathcal{F}'}$, $A = A_{\mathrm{aSt}\,\mathcal{F}'}$, $A' = \tilde{A}_{\mathcal{F}'}$ and $B' = \tilde{B}_{\mathcal{F}'}$, we obtain a closed neighborhood $E_{\mathcal{F}'}$ of $A_{\mathrm{aSt}\,\mathcal{F}'}$ in $\left(D_{\mathrm{aSt}\,\mathcal{F}'}, \mathcal{T}|_{D_{\mathrm{aSt}\,\mathcal{F}'}}\right)$ such that $\tilde{B}_{\mathcal{F}'} = \tilde{D}_{\mathcal{F}'} \cap E_{\mathcal{F}'}$. Defining $B^*_{\mathcal{F}'} := E_{\mathcal{F}'} \setminus \tilde{D}_{\mathcal{F}'}$ we have $\tilde{B}_{\mathcal{F}'} \cup B^*_{\mathcal{F}'} = E_{\mathcal{F}'}$ and $B^*_{\mathcal{F}'} \subseteq D_{\mathrm{aSt}\,\mathcal{F}'} \setminus \tilde{D}_{\mathcal{F}'}$.

Since $E_{\mathcal{F}'}$ is a closed subset of $(X, \mathcal{T})$, we conclude that $\left(E_{\mathcal{F}'}, \mathcal{T}|_{E_{\mathcal{F}'}}\right)$ is an object of $\mathfrak{C}$. Since $\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'} = \tilde{B}_{\mathcal{F}'} \cup A_{\mathrm{aSt}\,\mathcal{F}'}$ is a closed subset of $\left(E_{\mathcal{F}'}, \mathcal{T}|_{E_{\mathcal{F}'}}\right)$ and $\left(U_{\mathcal{F}'}, \mathcal{O}|_{U_{\mathcal{F}'}}\right)$ is an absolute neighborhood extensor for $\mathfrak{C}$, there exists a closed neighborhood $E'_{\mathcal{F}'}$ of $\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'}$ in $\left(E_{\mathcal{F}'}, \mathcal{T}|_{E_{\mathcal{F}'}}\right)$ and an extension $g'_{\mathcal{F}'} \in \mathrm{Hom}\left(\left(E'_{\mathcal{F}'}, \mathcal{T}|_{E'_{\mathcal{F}'}}\right), \left(U_{\mathcal{F}'}, \mathcal{O}|_{U_{\mathcal{F}'}}\right)\right)$ of $\tilde{g}_{\mathcal{F}'}$. We define $B_{\mathcal{F}'} := E'_{\mathcal{F}'} \setminus \tilde{B}_{\mathcal{F}'}$ and $B_{\mathrm{aSt}\,\mathcal{F}'} := \bigcup \left\{ B_{\hat{\mathcal{F}}} \mid \hat{\mathcal{F}} \in \mathrm{aSt}\,\mathcal{F}' \right\}$. Finally, we define

$g_{\mathcal{F}'} \in \operatorname{Hom}\left(\left(B_{\mathcal{F}'}, \mathcal{T}|_{B_{\mathcal{F}'}}\right), (Y, \mathcal{O})\right)$ by $g_{\mathcal{F}'}(x) := g'_{\mathcal{F}'}(x)$ and, in accordance with Lemma 2.3.2, the function $g_{\mathrm{aSt}\,\mathcal{F}'} \in \operatorname{Hom}\left(\left(B_{\mathrm{aSt}\,\mathcal{F}'}, \mathcal{T}|_{B_{\mathrm{aSt}\,\mathcal{F}'}}\right), (Y, \mathcal{O})\right)$ by $g_{\mathrm{aSt}\,\mathcal{F}'}(x) := g_{\hat{\mathcal{F}}}(x)$ for $\mathcal{F}' \subseteq \hat{\mathcal{F}} \in \mathcal{K}$ with $x \in B_{\hat{\mathcal{F}}}$. Since $\tilde{B}_{\mathcal{F}'} \cup A_{\mathcal{F}'} \subseteq \tilde{B}_{\mathcal{F}'} \cup B_{\mathcal{F}'}$ and $\tilde{B}_{\mathcal{F}'} \cap A_{\mathcal{F}'} = \emptyset$, we obtain $A_{\mathcal{F}'} \subseteq B_{\mathcal{F}'} \subseteq B^*_{\mathcal{F}'} \subseteq D_{\mathcal{F}'}$. Hence, (3.5) holds true. Property (3.6) is clearly satisfied as well. We have $g_{\mathcal{F}'}[B_{\mathcal{F}'}] \subseteq g'_{\mathcal{F}'}\left[E'_{\mathcal{F}'}\right] \subseteq U_{\mathcal{F}'}$ and hence, (3.7) is satisfied. Since $B_{\mathrm{aSt}\,\mathcal{F}'}$ is a neighborhood of $A_{\mathrm{aSt}\,\mathcal{F}'}$ in $\left(\tilde{B}_{\mathcal{F}'} \cup B^*_{\mathcal{F}'}, \mathcal{T}|_{\tilde{B}_{\mathcal{F}'} \cup B^*_{\mathcal{F}'}}\right)$ and $\tilde{B}_{\mathcal{F}'} \cup B^*_{\mathcal{F}'} = E_{\mathcal{F}'}$ is a neighborhood of $A_{\mathrm{aSt}\,\mathcal{F}'}$ in $\left(D_{\mathrm{aSt}\,\mathcal{F}'}, \mathcal{T}|_{D_{\mathrm{aSt}\,\mathcal{F}'}}\right)$, we obtain (3.8). Lastly, since we can write $B_{\mathrm{aSt}\,\mathcal{F}'} = \tilde{B}_{\mathcal{F}'} \cup E'_{\mathcal{F}'}$, the property (3.9) is satisfied. This finishes the induction.

The set $\mathcal{B} := \{B_{\mathcal{F}'} \mid \mathcal{F}' \in \mathcal{K}\}$ consists of mutually disjoint sets. Hence, for $B := \bigcup \mathcal{B}$ the function $g : B \to Y$ is well defined by $g(x) := g_{\mathcal{F}'}(x)$ if $x \in B_{\mathcal{F}'}$. By Lemma 2.3.6, B is a neighborhood of A in $(F, \mathcal{T})$. Since $\mathcal{B}_{\mathrm{aSt}} := \{B_{\mathrm{aSt}\,\mathcal{F}'} \mid \mathcal{F}' \in \mathcal{K}\}$ is a locally finite, closed cover of $(B, \mathcal{T}|_B)$ and $g|_{B_{\mathrm{aSt}\,\mathcal{F}'}} = g_{\mathrm{aSt}\,\mathcal{F}'}$, we obtain from Lemma 2.3.2 that $g \in \operatorname{Hom}((B, \mathcal{T}|_B), (Y, \mathcal{O}))$. Since $\mathcal{A} := \{A_{\mathcal{F}'} \mid \mathcal{F}' \in \mathcal{K}\}$ is a cover of A and $g|_{A_{\mathcal{F}'}} = f|_{A_{\mathcal{F}'}}$, we have $g|_A = f|_A$. □

Example 3.3.3. It follows from Theorem 3.3.2 together with Example 3.2.1 that every manifold is an absolute neighborhood extensor for $\mathfrak{T}4$.

4 Absolute retracts and absolute neighborhood retracts

Definition 4.0.1. A topological space $(X,\mathcal{T})$ is called an *absolute retract* for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ if $(X,\mathcal{T})$ is itself an object of $\mathfrak{C}$ and every closed subset A of an object $(Y,\mathcal{O})$ of $\mathfrak{C}$, where $(A,\mathcal{O}|_A)$ is homeomorphic to $(X,\mathcal{T})$, is a retract of $(Y,\mathcal{O})$.

Definition 4.0.2. A topological space $(X,\mathcal{T})$ is called an *absolute neighborhood retract* for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ if $(X,\mathcal{T})$ is an object of $\mathfrak{C}$ itself and every closed set A of an object $(Y,\mathcal{O})$ of $\mathfrak{C}$, where $(A,\mathcal{O}|_A)$ is homeomorphic to $(X,\mathcal{T})$, is a neighborhood retract of $(Y,\mathcal{O})$.

4.1 Relation to absolute extensors and absolute neighborhood extensors

Theorem 4.1.1. Every object of a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ that is an absolute neighborhood extensor (respectively absolute extensor) for $\mathfrak{C}$ is an absolute neighborhood retract (respectively absolute retract) for $\mathfrak{C}$.

Proof. We will proof the Theorem only for absolute neighborhood extensors and absolute neighborhood retracts. The proof for absolute extensors and absolute retracts is similar. Let $(X,\mathcal{T})$ be an absolute neighborhood extensor for $\mathfrak{C}$ and let A be some closed subset of an object $(Y,\mathcal{O})$ of $\mathfrak{C}$, such that $(A,\mathcal{O}|_A)$ is homeomorphic to $(X,\mathcal{T})$. Let $f \in \operatorname{Hom}((X,\mathcal{T}),(A,\mathcal{O}|_A))$ be a homeomorphism and $g \in \operatorname{Hom}((A,\mathcal{O}|_A),(X,\mathcal{T}))$ its inverse. Since $(X,\mathcal{T})$ is an absolute neighborhood extensor there exists a neighborhood U of A in $(Y,\mathcal{O})$ and an extension $\overline{g} \in \operatorname{Hom}((U,\mathcal{O}|_U),(X,\mathcal{T}))$ of g. The function $f \circ \overline{g} \in \operatorname{Hom}((U,\mathcal{O}|_U),(A,\mathcal{O}|_A))$ is a retraction, since for $a \in A$ we have $\overline{g}(a) = g(a)$ and in turn $f(\overline{g}(a)) = f(g(a)) = a$. □

Remark 4.1.2. Theorem 4.1.1 asserts that all examples of absolute extensors and absolute neighborhood extensors for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ that are objects of $\mathfrak{C}$ themselves are also examples of absolute retracts and absolute neighborhood retracts for $\mathfrak{C}$.

Theorem 4.1.3. For every metrizable space $(X,\mathcal{T})$ there exists convex subset C of a topological vector space $\left(L,+,(\omega_\lambda)_{\lambda\in\mathbb{R}},\mathcal{O}\right)$, a norm $\|\cdot\|$ on $\left(L,+,(\omega_\lambda)_{\lambda\in\mathbb{R}}\right)$ that induces $\mathcal{O}$ and makes $\left(L,+,(\omega_\lambda)_{\lambda\in\mathbb{R}},\|\cdot\|\right)$ a Banach space and a closed subset Y of $(C,\mathcal{O}|_C)$ such that $(X,\mathcal{T})$ and $(Y,\mathcal{O}|_Y)$ are homeomorphic. If $(X,\mathcal{T})$ is separable, so is $(C,\mathcal{O}|_C)$.

Proof. Given any metrizable topological space $(X,\mathcal{T})$, we consider the set of all continuous and bounded real valued functions

$$L := \{f \in \operatorname{Hom}((X,\mathcal{T}),(\mathbb{R},\mathcal{E}(\mathbb{R}))) : \|f\|_\infty < +\infty\}.$$

It is a well known fact, which one can find for example in [Kal14] that $\left(L, +, (\omega_\lambda)_{\lambda\in\mathbb{R}}, \|\cdot\|_\infty\right)$ is a Banach space. Let $\mathcal{O}$ be the topology induced by $\|\cdot\|_\infty$. In accordance with Lemma 2.1.3 we find a bounded metric $\mathrm{d} : X \times X \to \mathbb{R}$ that induces $\mathcal{T}$. We define a function $\iota \in \operatorname{Hom}((X, \mathcal{T}), (L, \mathcal{O}))$ by $\iota(x_1)(x_2) := \mathrm{d}(x_1, x_2)$.

According to the triangle inequality, for arbitrary $x_1, x_2, x \in X$ we obtain

$$|\iota(x_1)(x) - \iota(x_2)(x)| = |\mathrm{d}(x_1, x) - \mathrm{d}(x_2, x)| \leq \mathrm{d}(x_1, x_2)$$

and therefore, $\|\iota(x_1) - \iota(x_2)\|_\infty \leq \mathrm{d}(x_1, x_2)$. Since we also have

$$|\iota(x_1)(x_2) - \iota(x_2)(x_2)| = |\mathrm{d}(x_1, x_2) - \mathrm{d}(x_2, x_2)| = \mathrm{d}(x_1, x_2),$$

we even obtain $\|\iota(x_1) - \iota(x_2)\|_\infty = \mathrm{d}(x_1, x_2)$, which means that ι is an isometry.

The set $C := \operatorname{co}(\iota[X])$ is clearly a convex subset of L. We define $Y := \iota[X]$ and claim that it is a closed subset of $(C, \mathcal{O}|_C)$. In order to show this, let $f \in C \setminus Y$ be a given function. By definition of C there exists a finite set I and there are $(\lambda_i)_{i\in I} \in [0,1]^I$ and $(z_i)_{i\in I} \in X^I$ such that

$$\sum_{i\in I} \lambda_i = 1 \quad \text{and} \quad f = \sum_{i\in I} \lambda_i \iota(z_i).$$

As $f \notin Y$ we have $\|f - \iota(z_i)\|_\infty > 0$ for all $i \in I$. Hence, we can choose $\delta \in \mathbb{R}^+$, such that for all $i \in I$ the inequality $2\delta < \|f - \iota(z_i)\|_\infty$ is satisfied.

We claim that $C \cap B_{\|\cdot\|_\infty}(f, \delta) \subseteq C \setminus Y$. In order to show this, assume there exists $g \in C \cap B_{\|\cdot\|_\infty}(f, \delta)$ such that $g \in Y$. By the triangle inequality for any $i \in I$ we have

$$\|f - \iota(z_i)\|_\infty \leq \|f - g\|_\infty + \|g - \iota(z_i)\|_\infty.$$

Therefore

$$\|g - \iota(z_i)\| \geq \|f - \iota(z_i)\|_\infty - \|f - g\|_\infty > 2\delta - \delta = \delta.$$

Since $g \in Y$, there exists $x \in X$ such that $\iota(x) = g$. We saw above that ι is an isometry. Hence

$$\iota(z_i)(x) = \mathrm{d}(z_i, x) = \|\iota(z_i) - \iota(x)\|_\infty = \|\iota(z_i) - g\|_\infty > \delta.$$

From this we obtain

$$\|f - g\|_\infty \geq |f(x) - \iota(x)(x)| = |f(x) - \mathrm{d}(x, x)| = |f(x)| = \sum_{i\in I} \lambda_i \iota(z_i)(x) > \sum_{i\in I} \lambda_i \delta = \delta.$$

This clearly contradicts $g \in B_{\|\cdot\|_\infty}(f, \delta)$.

Assume that $(X, \mathcal{T})$ is separable. Since $(Y, \mathcal{O}|_Y)$ is homeomorphic to $(X, \mathcal{T})$, there exists a countable, dense subset D of $(Y, \mathcal{O}|_Y)$. The set $\mathcal{A} := \{\operatorname{co} M \mid M \subseteq D \wedge \#M \in \mathbb{Z}^+\}$ is countable and consists of compact subsets of $(L, \mathcal{O})$. By a Corollary in [Kal14, p. 457], all elements of $\mathcal{A}$, endowed with the subspace topology of $\mathcal{O}$, are separable. Therefore, $\operatorname{co} D = \bigcup \mathcal{A}$ is, as a countable union of separable sets, itself separable. Hence, there exists a countable, dense subset E of $(\operatorname{co} D, \mathcal{O}|_{\operatorname{co} D})$.

We claim that E is dense in $(C, \mathcal{O}|_C)$. In order to show this, let $y \in C$ and $\delta \in \mathbb{R}^+$ be given. By definition of C, there exist a finite set I, for every $i \in I$ a point $y_i \in Y$ and a number $\lambda_i \in [0,1]$, such that $y = \sum_{i\in I} \lambda_i y_i$ and $\sum_{i\in I} \lambda_i = 1$. Since D is dense in $(Y, \mathcal{O})$, for every $i \in I$ we find $z_i \in D$, such that $\|z_i - y_i\|_\infty < \delta/2$. The point $z := \sum_{i\in I} \lambda_i z_i$ is obviously an element of $\operatorname{co} D$ and satisfies

$$\|z - y\|_\infty = \left\| \sum_{i\in I} \lambda_i (z_i - y_i) \right\|_\infty \le \sum_{i\in I} \lambda_i \|z_i - y_i\|_\infty < \frac{\delta}{2} \sum_{i\in I} \lambda_i = \frac{\delta}{2}.$$

As $z \in \operatorname{co} D$ there exists $u \in E$, such that $\|u - z\|_\infty < \delta/2$. Therefore, we obtain

$$\|y - u\|_\infty \le \|y - z\|_\infty + \|z - u\|_\infty < \delta.$$

Thus, E is dense in $(C, \mathcal{O}|_C)$ and $(C, \mathcal{O}|_C)$ is separable. □

The following Theorem is from [Hu65, p. 84]. It also holds true for other categories of topological spaces than the ones for which it is stated here.

Theorem 4.1.4. If $\mathfrak{C}$ is one of the categories $\mathfrak{Met}$ or $\mathfrak{sepMet}$, then every absolute neighborhood retract (respectively absolute retract) for $\mathfrak{C}$ is an absolute neighborhood extensor (respectively absolute extensor) for $\mathfrak{C}$.

Proof. We will only proof the theorem for absolute neighborhood retracts and absolute neighborhood extensors. The proof for absolute retracts and absolute extensors is similar. According to Theorem 4.1.3, it suffices to show this Theorem for an absolute neighborhood retract $(Y, \mathcal{O}|_Y)$ for $\mathfrak{C}$, where $\left(L, +, (\omega_\lambda)_{\lambda\in\mathbb{R}}, \|\cdot\|\right)$ is a Banach space, $\mathcal{O}$ is the induced topology of $\|\cdot\|$, the set C is a convex subset of L and Y is a closed subset of $(C, \mathcal{O}|_C)$. Since Theorem 4.1.3 asserts that $(C, \mathcal{O}|_C)$ is an object of $\mathfrak{C}$, there exists a neighborhood V of Y in $(C, \mathcal{O}|_C)$ and a retraction $r \in \operatorname{Hom}((V, \mathcal{O}|_V), (Y, \mathcal{O}|_Y))$.

Let A be a closed subset of a given object $(X, \mathcal{T})$ of $\mathfrak{C}$ and $f \in \operatorname{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}|_Y))$, which implies $f \in \operatorname{Hom}((A, \mathcal{T}|_A), (C, \mathcal{O}|_C))$. Hence, by Theorem 3.2.2 there exists an extension $\tilde{f} \in \operatorname{Hom}((X, \mathcal{T}), (C, \mathcal{O}|_C))$ of f. Obviously, $U := \tilde{f}^{-1}(V)$ is a neighborhood of A in $(X, \mathcal{T})$. The function $\overline{f} \in \operatorname{Hom}((U, \mathcal{T}|_U), (Y, \mathcal{O}|_Y))$ defined by $\overline{f} := r \circ \tilde{f}|_U$ satisfies $\overline{f}(x) = r\left(\tilde{f}(x)\right) = r(f(x)) = f(x)$ for all $x \in A$. Thus, $\overline{f}$ is an extension of f and $(Y, \mathcal{O}|_Y)$ is an absolute neighborhood extensor. □

4.2 Properties of absolute neighborhood retracts

The following two propositions are obviously true.

Proposition 4.2.1. Every absolute retract for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ is an absolute neighborhood retract for $\mathfrak{C}$.

Proposition 4.2.2. Let $\mathfrak{C}$ be a full subcategory of $\mathfrak{Top}$ and $\mathfrak{B}$ a full subcategory of $\mathfrak{C}$. If a topological space $(X, \mathcal{T})$ is an object of $\mathfrak{B}$ and an absolute neighborhood retract (respectively absolute retract) for $\mathfrak{C}$, then it is also an absolute neighborhood retract (respectively absolute retract) for $\mathfrak{B}$.

An application of Theorem 4.1.4, Proposition 3.1.4 and Theorem 4.1.1 yields

Proposition 4.2.3. If the product space of absolute retracts for $\mathfrak{Met}$ is metrizable, then it is an absolute retract for $\mathfrak{Met}$.

Since the product space of finitely many metrizable spaces is itself metrizable, we can apply Theorem 4.1.4, Proposition 3.1.5 and Theorem 4.1.1 and obtain

Proposition 4.2.4. The product space of finitely many absolute neighborhood retracts for $\mathfrak{Met}$ is an absolute neighborhood retract for $\mathfrak{Met}$.

An application of Theorem 4.1.4, Proposition 3.1.6 and Theorem 4.1.1 yields

Proposition 4.2.5. Every open subspace of an absolute neighborhood retract for $\mathfrak{Met}$ is an absolute neighborhood retract for $\mathfrak{Met}$.

An application of Theorem 4.1.4, Proposition 3.1.8 and Theorem 4.1.1 yields

Proposition 4.2.6. Every neighborhood retract of an absolute neighborhood retract for $\mathfrak{Met}$ is an absolute neighborhood retract for $\mathfrak{Met}$.

Definition 4.2.7. A topological space $(X, \mathcal{T})$ is said to be a *local absolute neighborhood retract* for a full subcategory $\mathfrak{C}$ of $\mathfrak{Top}$ if for every point $x \in X$ there exists a neighborhood U of x in $(X, \mathcal{T})$, such that $(U, \mathcal{T}|_U)$ is an absolute neighborhood retract for $\mathfrak{C}$.

An application of Theorem 4.1.4, Theorem 3.3.2 and Theorem 4.1.1 yields

Theorem 4.2.8. If a metrizable space is a local absolute neighborhood retract for $\mathfrak{Met}$, then it also is an absolute neighborhood retract for $\mathfrak{Met}$.

Example 4.2.9. Let $d \in \mathbb{Z}^+$ and $K_d := \left\{x \in \mathbb{R}^d : \|x\|_2 \leq 1\right\}$ be the closed unit ball in $\mathbb{R}^d$ and $S_d := \left\{x \in \mathbb{R}^d : \|x\|_2 = 1\right\}$ the unit sphere in $\mathbb{R}^d$. Since S_d is a metrizable manifold, it is by Remark 4.1.2, Proposition 4.2.5 and Theorem 4.2.8 an absolute neighborhood retract for $\mathfrak{Met}$. On the other hand, by a Corollary in [WKB20, p.103], the set S_d is not a retract of K_d. Thus, S_d is definitely not an absolute retract for $\mathfrak{Met}$.

4.3 Homotopies

Definition 4.3.1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{O})$ be topological spaces and $\mathcal{U}$ an open cover of $(Y, \mathcal{O})$. Two functions $f, g \in \operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$ are called *$\mathcal{U}$-close* if for every $x \in X$ there exists a $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subseteq U$. A homotopy

$$H \in \operatorname{Hom}((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (Y, \mathcal{O})) \tag{4.1}$$

is said to be *limited* by $\mathcal{U}$ if for every $x \in X$ there exists $U \in \mathcal{U}$ such that $H[\{x\} \times [0,1]] \subseteq U$. The functions f and g are called *$\mathcal{U}$-homotopic* if there exists a homotopy (4.1) that is limited by $\mathcal{U}$, such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.

Lemma 4.3.2. Let C be a convex subset of a locally convex, metrizable topological vector space $\left(L, +, (\omega_\lambda)_{\lambda\in\mathbb{R}}, \mathcal{O}\right)$ and Y a closed subspace of $(C, \mathcal{O}|_C)$. If $\mathcal{U}$ is an open cover of $(Y, \mathcal{O}|_Y)$ and $(Y, \mathcal{O}|_Y)$ an absolute neighborhood retract for $\mathfrak{Met}$, then there exists an open neighborhood N of Y in $(C, \mathcal{O}|_C)$ and a retraction $r \in \operatorname{Hom}((N, \mathcal{O}|_N), (Y, \mathcal{O}|_Y))$, such that for every $y \in N$ there exists an open and convex neighborhood C_y of y in $(C, \mathcal{O}|_C)$ and that $\mathcal{V} = \{Y \cap C_y \mid y \in N\}$ is an open cover of $(Y, \mathcal{O}|_Y)$ and a refinement of $\mathcal{U}$. Furthermore, for every $y \in N$ there exists $U_y \in \mathcal{U}$, such that $C_y \subseteq r^{-1}(U_y)$.

Proof. Since $(Y, \mathcal{O}|_Y)$ is an absolute neighborhood retract for $\mathfrak{Met}$, there exists an open neighborhood N of Y in $(C, \mathcal{O}|_C)$ and a retraction $r \in \operatorname{Hom}((N, \mathcal{O}|_N), (Y, \mathcal{O}|_Y))$. The set $\{r^{-1}(U) \mid U \in \mathcal{U}\}$ is obviously an open cover of $(N, \mathcal{O}|_N)$. Hence, for a given $y \in N$ we find a set $U_y \in \mathcal{U}$ such that $y \in r^{-1}(U_y)$. Since N is an open subset of $(C, \mathcal{O}|_C)$, so is $r^{-1}(U_y)$. Therefore, there exists an open subset O_y of $(L, \mathcal{O})$, such that $r^{-1}(U_y) = O_y \cap C$. Since L is locally convex, there exists a convex, open neighborhood $\tilde{C}_y$ of y in $(L, \mathcal{O})$, such that $\tilde{C}_y \subseteq O_y$. As an intersection of two convex sets, $C_y := \tilde{C}_y \cap C$ is convex, where

$$C_y = \tilde{C}_y \cap C \subseteq O_y \cap C = r^{-1}(U_y) \subseteq N.$$

Furthermore, C_y is an open subset of $(C, \mathcal{O}|_C)$. Since r is a retraction, we have $Y \cap C_y \subseteq Y \cap r^{-1}(U_y) \subseteq U_y$. Thus, the set $\mathcal{V} := \{Y \cap C_y \mid y \in N\}$ is clearly an open cover of $(Y, \mathcal{O}|_Y)$ and a refinement of $\mathcal{U}$. □

Theorem 4.3.3. For every open cover $\mathcal{U}$ of a metrizable space $(Y, \mathcal{S})$ which is an absolute neighborhood retract for $\mathfrak{Met}$ there exists an open cover $\mathcal{V}$ of $(Y, \mathcal{S})$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ and for every topological space $(X, \mathcal{T})$ the following holds true. Any two $\mathcal{V}$-close functions $f, g \in \operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{S}))$ are $\mathcal{U}$-homotopic.

Proof. By Theorem 4.1.3 we can assume without loss of generality that there exists a convex subset C of a Banach space $\left(L, +, (\omega_\lambda)_{\lambda\in\mathbb{R}}, \|\cdot\|\right)$, where $\|\cdot\|$ induces the topology $\mathcal{O}$ and such that Y is a closed subset of $(C, \mathcal{O}|_C)$ with $\mathcal{S} = \mathcal{O}|_Y$. Let $\mathcal{U}$ be an open cover of $(Y, \mathcal{O}|_Y)$. In accordance with Lemma 4.3.2 we obtain an open neighborhood N of Y in $(C, \mathcal{O}|_C)$ and a retraction $r \in \operatorname{Hom}((N, \mathcal{O}|_N), (Y, \mathcal{O}|_Y))$, such that for every $y \in N$ there exists an open and convex neighborhood C_y of y in $(C, \mathcal{O}|_C)$, where $\mathcal{V} = \{Y \cap C_y \mid y \in N\}$ is an open cover of $(Y, \mathcal{O}|_Y)$ and a refinement of $\mathcal{U}$. Furthermore, for every $y \in W$, there exists $U_y \in \mathcal{U}$, such that $C_y \subseteq r^{-1}(U_y)$.

Given a topological space $(X, \mathcal{T})$ and two $\mathcal{V}$-close functions $f, g \in \operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$ we define the homotopy $\tilde{H} \in \operatorname{Hom}((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (N, \mathcal{O}|_N))$ by $\tilde{H}(x, \lambda) := \lambda g(x) + (1-\lambda) f(x)$. The function $H \in \operatorname{Hom}((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (Y, \mathcal{O}))$ defined by $H := r \circ \tilde{H}$ is clearly also a homotopy.

It remains to show that H is limited by $\mathcal{U}$ and that $\operatorname{ran}\left(\tilde{H}\right) \subseteq N$. In order to do this, let $x \in X$ be a given point. Since f and g are $\mathcal{V}$-close, there exists $y \in N$, such that $\{f(x), g(x)\} \subseteq Y \cap C_y$. Since C_y is convex, we clearly have $\tilde{H}[\{x\} \times [0,1]] \subseteq C_y$. There exists $U_y \in \mathcal{U}$, such that $C_y \subseteq r^{-1}(U_y) \subseteq N$ and therefore, $\operatorname{ran}\left(\tilde{H}\right) \subseteq N$. Furthermore,

$$H[\{x\} \times [0,1]] = r\left[\tilde{H}[\{x\} \times [0,1]]\right] \subseteq r[C_y] \subseteq r\left[r^{-1}(U_y)\right] \subseteq U_y.$$

□

Theorem 4.3.4. Let $(Y, \mathcal{S})$ be an absolute neighborhood retract for $\mathfrak{Met}$ and A a closed subset of a metrizable topological space $(X, \mathcal{T})$. If $\mathcal{U}$ is an open cover of $(Y, \mathcal{S})$, then there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ that is still an open cover of $(Y, \mathcal{S})$ and such that for any two $\mathcal{V}$-close functions $f, g \in \mathrm{Hom}\,((X, \mathcal{T}), (Y, \mathcal{S}))$ and any homotopy

$$\tilde{H} \in \mathrm{Hom}\,((A \times [0,1], \mathcal{T}|_A \times \mathcal{E}([0,1])), (Y, \mathcal{S}))$$

that is limited by $\mathcal{V}$ and satisfies $\tilde{H}(\cdot, 0) = f|_A$ and $\tilde{H}(\cdot, 1) = g|_A$, there exists a homotopy $H \in \mathrm{Hom}\,((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (Y, \mathcal{S}))$ that is limited by $\mathcal{U}$ and satisfies $H(\cdot, 0) = f$, $H(\cdot, 1) = g$ and $H|_{A \times [0,1]} = \tilde{H}$.

Proof. By Theorem 4.1.3 we can assume without loss of generality that there exists a convex subset C of a Banach space $\left(L, +, (\omega_\lambda)_{\lambda \in \mathbb{R}}, \|\cdot\|\right)$, where $\|\cdot\|$ induces the topology $\mathcal{O}$ and such that Y is a closed subset of $(C, \mathcal{O}|_C)$ with $\mathcal{S} = \mathcal{O}|_Y$. Let $\mathcal{U}$ be an open cover of $(Y, \mathcal{O}|_Y)$. In accordance with Lemma 4.3.2 we obtain an open neighborhood N of Y in $(C, \mathcal{O}|_C)$ and a retraction $r \in \mathrm{Hom}\,((N, \mathcal{O}|_N), (Y, \mathcal{O}|_Y))$, such that for every $y \in N$ there exists an open and convex neighborhood C_y of y in $(C, \mathcal{O}|_C)$, and $\mathcal{V} = \{Y \cap C_y \mid y \in W\}$ is an open cover of $(Y, \mathcal{O}|_Y)$ and a refinement of $\mathcal{U}$. Furthermore, for every $y \in N$, there exists $U_y \in \mathcal{U}$, such that $C_y \subseteq r^{-1}(U_y)$.

Let $(X, \mathcal{T})$ be a metrizable topological space and $f, g \in \mathrm{Hom}\,((X, \mathcal{T}), (Y, \mathcal{O}|_Y))$ be two $\mathcal{V}$-close functions. Furthermore, let $\tilde{H} \in \mathrm{Hom}\,((A \times [0,1], \mathcal{T}|_A \times \mathcal{E}([0,1])), (Y, \mathcal{S}))$ be a homotopy that is limited by $\mathcal{V}$ and satisfies $\tilde{H}(\cdot, 0) = f|_A$ and $\tilde{H}(\cdot, 1) = g|_A$. Define the function $H' \in \mathrm{Hom}\,((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (N, \mathcal{O}|_N))$ by $H'(x, \lambda) := \lambda g(x) + (1-\lambda) f(x)$. For an arbitrary $x \in X$ there exists $y \in N$, such that $f(x), g(x) \in Y \cap C_y$ and $U_y \in \mathcal{U}$, such that $C_y \subseteq r^{-1}(U_y) \subseteq N$. Since C_y is convex, we have

$$H'[\{x\} \times [0,1]] \subseteq C_y \subseteq r^{-1}(U_y) \subseteq N. \tag{4.2}$$

The set

$$Q := (X \times \{0\}) \cup (A \times [0,1]) \cup (X \times \{1\})$$

is clearly a closed subset of $(X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1]))$. By Lemma 2.3.2, we have $F \in \mathrm{Hom}\,((Q, (\mathcal{T} \times \mathcal{E}([0,1]))|_Q), (Y, \mathcal{O}|_Y))$, where

$$F(x, \lambda) := \begin{cases} f(x) & , \text{ if } \lambda = 0 \\ \tilde{H}(x, \lambda) & , \text{ if } x \in A \\ g(x) & , \text{ if } \lambda = 1. \end{cases}$$

Since $(Y, \mathcal{O}|_Y)$ is an absolute neighborhood retract for $\mathfrak{Met}$, we can apply Theorem 4.1.4. Hence, we obtain that there exists an open neighborhood W of Q in $(X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1]))$ and an extension

$$\overline{F} \in \mathrm{Hom}\,((W, (\mathcal{T} \times \mathcal{E}([0,1]))|_W), (Y, \mathcal{O}|_Y))$$

of F.

Let $x \in A$ be a given point and $V_x \in \mathcal{V}$ such that for all $\lambda \in [0,1]$ we have $\overline{F}(x,\lambda) = \tilde{H}(x,\lambda) \in V_x$. By continuity for a given $\lambda \in [0,1]$ there exists an open neighborhood W^x_λ of x in $(X,\mathcal{T})$ and an $\varepsilon_\lambda \in \mathbb{R}^+$, such that $W^x_\lambda \times \mathrm{B}_{[0,1]}(\lambda,\varepsilon_\lambda) \subseteq W$ and for all $(z,\mu) \in W^x_\lambda \times \mathrm{B}_{[0,1]}(\lambda,\varepsilon_\lambda)$ we have $\overline{F}(z,\mu) \in V_x$. Since $([0,1],\mathcal{E}([0,1]))$ is a compact topological space and $\{\mathrm{B}_{[0,1]}(\lambda,\varepsilon_\lambda) \mid \lambda \in [0,1]\}$ constitutes an open cover of $([0,1],\mathcal{E}([0,1]))$, there exists a finite subset E_x of $[0,1]$, such that $\{\mathrm{B}_{[0,1]}(\lambda,\varepsilon_\lambda) \mid \lambda \in E_x\}$ still covers $[0,1]$. Define $W_x := \bigcap\{W^x_\lambda \mid \lambda \in E_x\}$ and consider $(z,\mu) \in W_x \times [0,1]$. There exists $\nu \in E_x$, such that $\mu \in \mathrm{B}_{[0,1]}(\nu,\varepsilon_\nu)$. Therefore $(z,\mu) \in W^x_\nu \times \mathrm{B}_{[0,1]}(\nu,\varepsilon_\nu) \subseteq W$ showing $W_x \times [0,1] \subseteq W$. The set $M := \bigcup\{W_x \mid x \in A\}$ is clearly an open neighborhood of A in $(X,\mathcal{T})$ satisfying $A \times [0,1] \subseteq M \times [0,1] \subseteq W$.

We claim that the homotopy $F' := \overline{F}|_{M\times[0,1]}$ is limited by $\mathcal{V}$. For a given $x \in M$ there exists $z \in A$, such that $x \in W_z = \bigcap\{W^z_\lambda \mid \lambda \in E_z\}$. Let $V_z \in \mathcal{V}$ as before, satisfying in particular $\overline{F}(z,\lambda) = \tilde{H}(z,\lambda) \in V_z$ for all $\lambda \in [0,1]$ we have . For an arbitrary $\mu \in [0,1]$ there exists $\nu \in E_z$, such that $\mu \in \mathrm{B}_{[0,1]}(\nu,\varepsilon_\nu)$. Therefore, $(x,\mu) \in W^z_\nu \times \mathrm{B}_{[0,1]}(\nu,\varepsilon_\nu)$ and consequently, $F'(x,\mu) = \overline{F}(x,\mu) \in V_z$ showing that F' is limited by $\mathcal{V}$.

As a metrizable space, $(X,\mathcal{T})$ satisfies (T_4). There exists an open subset R of $(X,\mathcal{T})$, such that $A \subseteq R \subseteq \mathrm{cl}_{\mathcal{T}}(R) \subseteq M$. In accordance with Lemma 2.2.8, there exists a function $s \in \mathrm{Hom}\,((X,\mathcal{T}),([0,1],\mathcal{E}([0,1])))$, such that $s[X \setminus R] \subseteq \{0\}$ and $s[A] \subseteq \{1\}$. We define a homotopy $G \in \mathrm{Hom}\,((X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1])),(N,\mathcal{O}|_N))$ by

$$G(x,\lambda) := \begin{cases} s(x)F'(x,\lambda) + (1-s(x))H'(x,\lambda) & \text{, if } x \in M \\ H'(x,\lambda) & \text{, if } x \in X \setminus R. \end{cases}$$

We claim that the homotopy $H \in \mathrm{Hom}\,((X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1])),(Y,\mathcal{O}))$ defined by

$$H(x,\lambda) := r(G(x,\lambda))$$

is limited by $\mathcal{U}$. In order to show this, consider an arbitrary $x \in X$. According to (4.2), in case $s(x) = 0$ there exists $U_x \in \mathcal{U}$, such that $H'[\{x\}\times[0,1]] \subseteq r^{-1}(U_x)$ and further $H[\{x\}\times[0,1]] = r[H'[\{x\}\times[0,1]]] \subseteq r\big[r^{-1}(U_x)\big] \subseteq U_x$.

Consider the case that $s(x) \in \mathbb{R}^+$. Since we showed above that F' is limited by $\mathcal{V}$, there exists $y \in N$, such that $F'[\{x\}\times[0,1]] \subseteq Y \cap C_y$. In particular, we have $F'(x,0) = f(x) \in Y \cap C_y$ and $F'(x,1) = g(x) \in Y \cap C_y$. C_y being convex together with $H'[\{x\}\times[0,1]] \subseteq C_y$ implies $G[\{x\}\times[0,1]] \subseteq C_y$ as well. Since there exists $U_y \in \mathcal{U}$, such that $C_y \subseteq r^{-1}(U_y)$, we obtain that $H[\{x\}\times[0,1]] = r[G[\{x\}\times[0,1]]] \subseteq r\big[r^{-1}(U_y)\big] \subseteq U_y$.

Finally, we have $H(\cdot,0) = r\circ G(\cdot,0) = r\circ f = f$, as well as $H(\cdot,1) = r\circ G(\cdot,1) = r\circ g = g$ and $H|_{A\times[0,1]} = r\circ G|_{A\times[0,1]} = r\circ F'|_{A\times[0,1]} = F|_{A\times[0,1]} = \tilde{H}|_{A\times[0,1]}$. □

Theorem 4.3.5. A metrizable space $(Y,\mathcal{O})$ is an absolute neighborhood retract for $\mathfrak{Met}$, if and only if there exists an open cover $\mathcal{V}$ of $(Y,\mathcal{O})$, such that for every metrizable space $(X,\mathcal{T})$, every closed subspace A of $(X,\mathcal{T})$, every two $\mathcal{V}$-close functions

$$f,g \in \mathrm{Hom}\,((X,\mathcal{T}),(Y,\mathcal{O}))$$

and every homotopy $H \in ((A\times[0,1],\mathcal{T}|_A\times\mathcal{E}([0,1])),(Y,\mathcal{O}))$ that is limited by $\mathcal{V}$ with $H(\cdot,0) = f|_A$ and $H(\cdot,1) = g|_A$ there exists a homotopy

$$\overline{H} \in \mathrm{Hom}\,((X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1])),(Y,\mathcal{O})),$$

such that $\overline{H}|_{A\times[0,1]} = H$ as well as $\overline{H}(\cdot,0) = f$ and $\overline{H}(\cdot,1) = g$.

Proof. In order to proof sufficiency, let $(Y,\mathcal{O})$ be an absolute neighborhood retract for $\mathfrak{Met}$. The set $\mathcal{U} := \{Y\}$ is obviously an open cover of $(Y,\mathcal{O})$. Therefore, we can apply Theorem 4.3.4 and are finished.

It remains to show necessity. Let $\mathcal{V}$ be an open cover of the metrizable space $(Y,\mathcal{O})$ that is as required in the statement of the Theorem. Consider an arbitrary $y \in Y$. Since $\mathcal{V}$ is a cover of Y, there exists $V \in \mathcal{V}$ such that $y \in V$. Define the functions $f_1, g_1 \in \operatorname{Hom}((V,\mathcal{O}|_V),(Y,\mathcal{O}))$ and the homotopy

$$H_1 \in \operatorname{Hom}\big((\{y\}\times[0,1],\mathcal{O}|_{\{y\}}\times\mathcal{E}([0,1])),(Y,\mathcal{O})\big)$$

by $f_1(v) := y$, $g_1(v) := v$ and $H_1(y,\lambda) := y$. Since $(Y,\mathcal{O})$ is metrizable, so is $(V,\mathcal{O}|_V)$ and $\{y\}$ is closed in $(Y,\mathcal{O})$. Furthermore, f_1 and g_1 are obviously $\mathcal{V}$-close and H_1 is limited by $\mathcal{V}$. Therefore, there exists a homotopy $\overline{H}_1 \in \operatorname{Hom}((V\times[0,1],\mathcal{O}|_V\times\mathcal{E}([0,1])),(Y,\mathcal{O}))$ such that $\overline{H}_1|_{\{y\}\times[0,1]} = H_1$, as well as $\overline{H}_1(\cdot,0) = f_1$ and $\overline{H}_1(\cdot,1) = g_1$. Since $([0,1],\mathcal{E}([0,1]))$ is a compact space and $\overline{H}_1(y,\lambda) = y$ for all $\lambda \in [0,1]$ there exists an open neighborhood U of y in $(Y,\mathcal{O})$, such that $\overline{H}_1[U\times[0,1]] \subseteq V$.

Let A be an arbitrary closed subspace of a given metrizable space $(X,\mathcal{T})$ and $h \in \operatorname{Hom}((A,\mathcal{T}|_A),(U,\mathcal{O}|_U))$. Define $f_2, g_2 \in \operatorname{Hom}((X,\mathcal{T}),(Y,\mathcal{O}))$ and

$$H_2 \in \operatorname{Hom}((A\times[0,1],\mathcal{T}|_A\times\mathcal{E}([0,1])),(Y,\mathcal{O}))$$

by $f_2(x) := g_2(x) := y$ and

$$H_2(x,\lambda) := \begin{cases} \overline{H}_1(h(x),2\lambda) & ,\text{if } \lambda \in [0,1/2], \\ \overline{H}_1(h(x),2-2\lambda) & ,\text{if } \lambda \in [1/2,1]. \end{cases}$$

By assumption, there exists a homotopy $\overline{H}_2 \in \operatorname{Hom}((X,\mathcal{T}),(Y,\mathcal{O}))$, such that $\overline{H}_2|_{A\times[0,1]} = H_2$, as well as $\overline{H}_2(\cdot,0) = f_2$ and $\overline{H}_2(\cdot,1) = g_2$. Define $\tilde{h} := \overline{H}_2(\cdot,1/2)$. We have

$$\tilde{h}|_A = \overline{H}_2(\cdot,1/2)|_A = \overline{H}_1(h|_A(\cdot),1) = g_1\circ h|_A = h|_A.$$

Hence, $W := \tilde{h}^{-1}(U)$ is an open neighborhood of A in $(X,\mathcal{T})$ and $\overline{h} := \tilde{h}|_W$ is an extension of h over the neighborhood W of A in $(U,\mathcal{O}|_U)$. We conclude that U is an absolute neighborhood extensor for $\mathfrak{Met}$. By Theorem 4.1.4, the topological space $(U,\mathcal{O}|_U)$ is an absolute neighborhood retract for $\mathfrak{Met}$ and $(Y,\mathcal{O})$ is a local absolute neighborhood retract for $\mathfrak{Met}$. An application of Theorem 4.2.8 finishes the proof. □

Lemma 4.3.6. Let A be a topological space $(X,\mathcal{T})$ that satisfies (T_4). Then for every neighborhood U of $B := (X\times\{0\})\cup(A\times[0,1])$ in $(X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1]))$ there exists a function $F \in \operatorname{Hom}((X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1])),(U,(\mathcal{T}\times\mathcal{E}([0,1]))|_U))$ such that $F|_B = \mathrm{id}_B$.

Proof. Since U is a neighborhood of B in $(X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1]))$, there exists an open subset U' of $(X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1]))$ such that $B \subseteq U' \subseteq U$. Since $D := (X\times[0,1])\setminus U'$ is closed in $(X\times[0,1],\mathcal{T}\times\mathcal{E}([0,1]))$, we can apply Lemma 2.2.7 and obtain that $\pi_1[D]$ is closed in $(X,\mathcal{T})$.

By Lemma 2.2.8, which requires $(X, \mathcal{T})$ to satisfy (T_4), there exists

$$f \in \operatorname{Hom}((X, \mathcal{T}), ([0,1], \mathcal{E}([0,1])))$$

such that $f[A] \subseteq \{1\}$ and $f[\pi_1[D]] \subseteq \{0\}$. We claim that the function

$$F \in \operatorname{Hom}((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (U, (\mathcal{T} \times \mathcal{E}([0,1]))|_U))$$

defined by $F(x, \lambda) := (x, f(x)\lambda)$ satisfies $F|_B = \mathrm{id}_B$. In order to show this let $(x, \lambda) \in B$ be given. If $(x, \lambda) \in X \times \{0\}$, then $F(x, \lambda) = F(x, 0) = (x, 0) = (x, \lambda)$. If, on the other hand, $(x, \lambda) \in A \times [0,1]$, then $F(x, \lambda) = (x, f(x)\lambda) = (x, \lambda)$.

It remains to show that $\operatorname{ran} F \subseteq U$. Given any $(x, \lambda) \in X \times [0,1]$ we can distinguish two cases. If $x \in \pi_1[D]$, then $f(x) = 1$ and $F(x, \lambda) = (x, f(x)\lambda) = (x, 0) \in U$. If, on the other hand, $x \in X \setminus \pi_1[D]$, then $F(x, \lambda) = (x, f(x)\lambda) \notin D$. Therefore, $F(x, \lambda) \in U' \subseteq U$. □

Theorem 4.3.7 (Borsuk Homotopy Extension Theorem)**.** If A is a closed subspace of a metrizable space $(X, \mathcal{T})$, if the topological space $(Y, \mathcal{O})$ is an absolute neighborhood retract for $\mathfrak{Met}$ and $F \in \operatorname{Hom}((A \times [0,1], \mathcal{T}|_A \times \mathcal{E}([0,1])), (Y, \mathcal{O}))$ is a homotopy such that $f \in \operatorname{Hom}((A, \mathcal{T}|_A), (Y, \mathcal{O}))$ defined by $f(x) := F(x, 0)$ can be extended to a function $g \in \operatorname{Hom}((X, \mathcal{T}), (Y, \mathcal{O}))$, then there exists a homotopy

$$G \in \operatorname{Hom}((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (Y, \mathcal{O}))$$

such that for all $x \in X$ the equlaity $G(x, 0) = g(x)$ is satisfied and for all $(x, \lambda) \in A \times [0,1]$ we have $G(x, \lambda) = F(x, \lambda)$.

Proof. We define $B := (X \times \{0\}) \cup (A \times [0,1])$ and, in accordance with Lemma 2.3.2,

$$H \in \operatorname{Hom}((B, \mathcal{T} \times \mathcal{E}([0,1])), (Y, \mathcal{O}))$$

by

$$H(x, \lambda) := \begin{cases} F(x, \lambda) & \text{, if } (x, \lambda) \in A \times [0,1], \\ g(x) & \text{, if } (x, \lambda) \in X \times \{0\}. \end{cases}$$

Since $(Y, \mathcal{O})$ is an absolute neighborhood retract for $\mathfrak{Met}$ and B is a closed subspace of $X \times [0,1]$, we can apply Theorem 4.1.4 and obtain a neighborhood V of B in

$$(X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1]))$$

and an extension $H' \in \operatorname{Hom}((V, (\mathcal{T} \times \mathcal{E}([0,1]))|_V), (Y, \mathcal{O}))$ of H. By Lemma 4.3.6 there exists $\tilde{F} \in \operatorname{Hom}((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (V, (\mathcal{T} \times \mathcal{E}([0,1]))|_V))$ such that $\tilde{F}|_B = id_B$. We define the function $G \in \operatorname{Hom}((X \times [0,1], \mathcal{T} \times \mathcal{E}([0,1])), (Y, \mathcal{O}))$ by

$$G(x, \lambda) := H'\Big(\tilde{F}(x, \lambda)\Big).$$

Given $x \in X$ we have

$$G(x, 0) = H'\Big(\tilde{F}(x, 0)\Big) = H'(x, 0) = g(x)$$

and for $(x, \lambda) \in A \times [0,1]$

$$G(x, \lambda) = H'\Big(\tilde{F}(x, \lambda)\Big) = H'(x, \lambda) = F(x, \lambda).$$

□

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