Haar measure on compact groups

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Contents

1 Notation and Conventions

The following is a list of symbols frequently used in the text. Most of them are common and those that are not will also be introduced as they occur in the text.

For a topological space X, $x \in X$, we denote by

$C(X)$	...	the space of continuous (complex-valued) functions on X
$C_c(X)$	...	$\{f \in C(X) \,\vert\, f \text{ has compact support}\}$
$C_0(X)$	...	$\{f \in C(X) \,\vert\, \forall \epsilon > 0 \exists K \text{ compact such that} f(x) < \epsilon \forall x \in K^c\}$
$\mathcal{U}(x)$	...	the filter of neighborhoods of x.

If X is even a topological vector space, $M \subset X$, we define

X^*	...	the algebraic dual of X, i.e. all linear functionals on X
X'	...	the topological dual of X, i.e. all *continuous* linear functionals on X
(X, τ_w)	...	X with the weak topology τ_w
(X', τ_{w^*})	...	X' with the weak-star topology τ_{w^*}
$\operatorname{co} M$	...	the convex hull of M
$\overline{\operatorname{co}}\, M$	...	$\overline{\operatorname{co} M}$
$E(M)$	...	the set of extremal points of M.

For arbitrary sets A, B we write $A \subset B$ to say that $A \subsetneq B$ or $A = B$.

We use the following terminology concerning measures on a topological Hausdorff space X:

Borel measure: A measure (i.e. a non-negative σ-additive function) defined on the Borel sets of X.

Radon measure: A Borel measure that is inner regular and finite on compact sets.

For a group $(G, \cdot, ^{-1}, e)$ we usually write gh instead of $\cdot(g, h)$ to denote multiplication of elements $g, h \in G$.

2 Introduction

A topological group is a group $(G, \cdot, ^{-1}, e)$ equipped with a topology such that the mappings

$$G \times G \to G : \ (g, h) \mapsto gh$$
$$G \to G : \ g \mapsto g^{-1}$$

are continuous. For the discussion in this section, we also assume that, viewed as a topological space, G is Hausdorff and locally compact.[1]

An interesting question is whether it is possible to find a measure μ on the Borel sets of G that is compatible with both the topological and algebraic structure of G. By this we mean that

a) μ is a Radon measure (i.e. a measure which is inner regular and finite on compact sets)

b) μ is invariant under translation, i.e. $\mu(Ag) = \mu(A) = \mu(gA)$ for all Borel sets $A \subset G$ and elements $g \in G$.[2]

We will also require μ to be non-trivial: $\mu \neq 0$. A measure with these properties is called **Haar measure**. If in b), only $\mu(gA) = \mu(A)$ (and not necessarily $\mu(Ag) = \mu(A)$) is satisfied, we call μ a **left Haar measure**. A right Haar measure is defined in an analogous way.

In 1933, Alfréd Haar proved the existence of a left Haar measure on topological groups that are Hausdorff, compact and separable. André Weil generalized this result to arbitrary locally compact topological Hausdorff groups and showed that (left) Haar measures are unique up to multiplication by a positive constant. In the present paper we present a proof for the existence and uniqueness of a Haar measure on compact topological Hausdorff groups based on the Ryll-Nardzweski fixed point theorem from functional analysis.

Before doing so, let us consider a few familiar examples of locally compact groups and their Haar measures:

Examples. [Els09] [Wik11]

[1] This is a convenient setting for studying Borel measures on G. It allows us to understand a large class of them (*all* Radon measures) as continuous linear functionals on $C_c(G)$. In fact, the definition of Radon measures as it is given below would not even make sense for arbitrary topological spaces G, since compact sets need not be Borel measurable in non-Hausdorff spaces. (In Hausdorff spaces compact sets are closed and therefore measurable.)

[2] Note that $\mu(gA)$ and $\mu(Ag)$ are well-defined since the translation by elements of G is continuous and, therefore, gA and Ag are Borel-measurable if A is.

(i) The counting measure on any finite group.

(ii) The standard Lebesgue measure on $(\mathbb{R}^n, +)$.

(iii) For the multiplicative group $(\mathbb{R}\backslash\{0\}, \cdot)$ the measure

$$A \mapsto \int_A \frac{1}{|t|} d\lambda(t)$$

is translation invariant by the change of variable formula.

(iv) For the complex unit circle $\partial U_1(0) = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ with standard multiplication as group operation, a Haar measure is given by

$$A \mapsto \lambda\left(\{t \in (0, 2\pi) \,\big|\, e^{it} \in A\}\right),$$

where λ is the usual Lebesgue measure on $\mathbb{R}$.

In the examples above all measures are both left- and right-invariant. For compact Hausdorff groups and locally compact abelian Hausdorff groups left-invariance always implies right-invariance and vice versa. (This is trivial for abelian groups and, for compact groups, will be established in this paper.) However, for arbitrary locally compact Hausdorff groups, a measure that is left-invariant need not be right-invariant (and vice versa).

As in this section, all topological groups we consider in the rest of this paper are Hausdorff (and locally compact). Since "locally compact topological Hausdorff group" is tedious to write we agree upon the following abbreviation:

> In the rest of this paper **"group"** is short for **"topological Hausdorff group"** .

3 Fixed point formulation

Let G be a compact group[3]. Every Haar measure μ for G must be finite, so it is not a restriction to demand $\mu(G) = 1$. We will impose this normalization condition for Haar measures on compact groups in all subsequent chapters without explicitly referring to μ as "*normalized* Haar measure".

With this convention a Haar measure for G is a point in the set

$$Q := \{\mu \,|\, \mu \text{ is Radon measure and } \mu(G) = 1\}$$

[3] See the box above for the terminology used.

that is fixed under the family of mappings

$$\mathcal{F} := \{R_g \,|\, g \in G\} \cup \{L_g \,|\, g \in G\},$$

where[4]

$$\begin{aligned} R_g &: Q \to Q : \mu \mapsto \mu(\cdot\, g) \\ L_g &: Q \to Q : \mu \mapsto \mu(g\, \cdot). \end{aligned}$$

At the moment Q is simply a set without any topological or algebraic properties (such as compactness or convexity). Having in mind to apply a fixed point theorem from functional analysis, we would like to view Q as a subset of an appropriate topological vector space. In the following, we define the appropriate setting and derive a few properties of Q and $\mathcal{F}$.

The set Q of "points"

The Riesz representation theorem for Radon measures on G allows us to interpret Q as a subset $\hat{Q}$ of $C(G)'$ and to translate all results obtained for $\hat{Q}$ back to Q. We recall the precise connection:

Theorem 3.1 (Riesz representation theorem)**.** *Let G be a locally compact Hausdorff space. The mapping*

$$\Phi : \mu \mapsto I_\mu := \int_G \cdot \, d\mu$$

is a bijection from the set of Radon measures on G to the set of positive linear functionals on $C_c(G)$.

A proof can be found in [Els09].

Let $\hat{Q} := \Phi(Q) \subset C_c(G)^*$. Because G is compact, $C_c(G) = C(G)$. Moreover, every Radon measure μ on G is finite and, therefore, the corresponding functional I_μ is continuous on $C(G)$ with respect to the supremum norm: $\hat{Q} \subset C(G)'$. More specifically,[5]

$$\begin{aligned} \hat{Q} = \Phi(Q) &= \{I \in C(G)' \,|\, I \text{ is positive and } I(1) = 1\} = \\ &= \overline{U_1(0)} \cap \iota_1^{-1}(1) \cap \bigcap_{f \geq 0} \iota_f^{-1}(\mathbb{R}_0^+). \end{aligned}$$

Here, $U_1(0)$ is the unit ball in $C(G)'$ (with respect to the operator norm) and, for $f \in C(G)$, ι_f is the linear functional

$$\iota_f : C(G)' \to \mathbb{C} : I \mapsto I(f).$$

[4]We write $\mu(\cdot\, g)$ for the measure $A \mapsto \mu(A\,g)$, defined on the Borel sets of G. This measure indeed lies in Q, since translation by g is a homeomorphism and the topological properties of μ are therefore preserved. Moreover, $\mu(Gg) = \mu(G) = 1$.

[5]By $I(1)$ we mean I evaluated at the constant function $(G \ni x \mapsto 1) \in C(G)$.

Since ι_f is w^*-continuous for all $f \in C(G)$ and $\overline{U_1(0)}$ is w^*-compact by Alaoglu's theorem, the set $\hat{Q}$ is w^*-compact. Clearly, $\hat{Q}$ is convex. In summary, $\hat{Q}$ is a compact, convex subset of the locally convex topological vector space $(C(G)', \tau_{w^*})$.

The set $\mathcal{F}$ of "functions"

The measure $\mu \in Q$ is a fixed point of the family $\mathcal{F}$, i.e. a Haar measure, iff $I_\mu \equiv \Phi(\mu)$ is a fixed point of

$$\hat{\mathcal{F}} := \{\hat{R}_g \,|\, g \in G\} \cup \{\hat{L}_g \,|\, g \in G\},$$

where $\hat{R}_g := \Phi \circ R_g \circ \Phi^{-1}\big|_{\hat{Q}}$ and $\hat{L}_g := \Phi \circ L_g \circ \Phi^{-1}\big|_{\hat{Q}}$. More explicitly, $\hat{R}_g$ maps the functional $I_\mu \in \hat{Q}$ to the functional $\Phi(R_g\mu) = \int_G \cdot \, d(R_g\mu)$, so using the change of variable formula for image measures we obtain $\hat{R}_g I_\mu(f) = I_\mu(f(g\,\cdot))$, $f \in C(G)$. In summary, for $I \in \hat{Q}, f \in C(G)$:

$$\hat{R}_g I(f) = I(f(g\,\cdot))$$
$$\hat{L}_g I(f) = I(f(\cdot\, g)).$$

We want to apply the Ryll-Nardzewski fixed point theorem to an appropriate set of functions $\mathcal{S} \supset \hat{\mathcal{F}}$ from $\hat{Q}$ into itself. The conditions of the theorem (see Theorem 5.1 below) are, among others, that $\mathcal{S}$ be a semigroup with respect to composition: $S_1 S_2 \in \mathcal{S}$ if $S_1, S_2 \in \mathcal{S}$. The set $\mathcal{F}$ is not a semigroup, but we can consider, instead of $\hat{\mathcal{F}}$, the semigroup $\mathcal{S}$ generated by $\hat{\mathcal{F}}$, which has the same fixed points:

$$\mathcal{S} := \langle \hat{\mathcal{F}} \rangle = \{\hat{R}_g \hat{L}_h \,|\, g, h \in G\}.$$

Note that the set on the right hand side is indeed a semigroup, since $\hat{R}_g$ and $\hat{L}_h$ commute and $\hat{R}_{g_1}\hat{R}_{g_2} = \hat{R}_{g_2 g_1}, \hat{L}_{h_1}\hat{L}_{h_2} = \hat{L}_{h_1 h_2}$. Obviously, it is the smallest semigroup containing all $\hat{R}_g, \hat{L}_h$, so the equality holds.

Clearly, all functions in $\mathcal{S}$ map $\hat{Q}$ into itself. We now verify the remaining properties of $\mathcal{S}$, viewed as a family of functions on $\hat{Q} \subset (C(G)', \tau_{w^*})$, that are needed for the Ryll-Nardzewski theorem:

- *Every $S \in \mathcal{S}$ is affine*[6]*: $S(\sum_{i=1}^n \alpha_i I_i) = \sum_{i=1}^n \alpha_i S(I_i)$ for $I_i \in \hat{Q}, \alpha_i \in [0,1], \sum_{i=1}^n \alpha_i = 1$.*

  Clearly, this relation is satisfied for all $\hat{R}_g$ and $\hat{L}_h$. Therefore, every $S \in \mathcal{S}$ is affine as a composition of affine maps.

[6] Actually the notion of affinity is a little overtechnical in this context, since the functions $S \in \mathcal{S}$, extended on $C(G)'$ in the obvious way, are linear and in particular affine on $\hat{Q}$. However, we have introduced $\hat{R}_g, \hat{L}_g$ as functions on $\hat{Q}$, which is not a vector space, so linearity is not defined and we must call them "affine".

- *Every $S \in \mathcal{S}$ is continuous:*
  As mentioned above, we consider $\hat{Q}$ as a subset of $(C(G)', \tau_{w^*})$, so the topology on $\hat{Q}$ is the relative w^*-topology. It is easy to see that $\hat{R}_g$ is continuous: Let $(I_k)_{k\in K}$ be a net in $\hat{Q}$. Then $I_k \to 0 \Rightarrow \big[\iota_f(I_k) = I_k(f) \to 0\ \forall f \in C(G)\big] \Rightarrow \big[I_k(f(g\,\cdot)) = R_g I_k(f) \to 0\ \forall f \in C(G)\big] \Rightarrow R_g I_k \to 0$. In the same way we see that $\hat{L}_g$ is continuous for all $g \in G$. Hence, every element in $\mathcal{S}$ is continuous as a composition of continuous maps.

- *The family $\mathcal{S}$ is noncontracting: $0 \notin \overline{\{S(I) - S(J) \,|\, S \in \mathcal{S}\}}$ for all $I, J \in \hat{Q}, I \neq J$.*
  This is the only property which is non-trivial to verify and where compactness of G finally comes into play. Let $I \neq J$ be arbitrary elements in $\hat{Q}$. Since every $S \in \mathcal{S}$ is injective, we certainly have $0 \notin \{S(I) - S(J) \,|\, S \in \mathcal{S}\} =: M$. We show that M is already closed[7]: By definition of $\mathcal{S}$ we have $M = \{\hat{R}_g\hat{L}_h(I) - \hat{R}_g\hat{L}_h(J) \,|\, g, h \in G\}$, so M is the image of $G \times G$ under the mapping $(g, h) \mapsto \hat{R}_g\hat{L}_h(I) - \hat{R}_g\hat{L}_h(J) \in C(G)'$. Lemma 3.2 below implies that this map is continuous. Hence, M is closed as the continuous image of a compact set in the Hausdorff space $C(G)'$.

Lemma 3.2. *Let G be a compact group, $I \in \hat{Q} \subset C(G)'$. Then*

$$\rho : G \times G \to (C(G)', \tau_{w^*}) : (g, h) \mapsto \hat{R}_g\hat{L}_h(I)$$

is continuous.

Proof. We need to show that for all $f \in C(G)$ the map $(g, h) \mapsto I(\,f(h \cdot g)\,)$ is continuous[8]. Since I is continuous on $C(G)$, it suffices to show that $(g, h) \mapsto f(h \cdot g) \in C(G)$ is continuous for all $f \in C(G)$. Fix $f \in C(G)$. We have to show that for all $g, h \in G, \epsilon > 0$ there exist $U_g \in \mathcal{U}(g), U_h \in \mathcal{U}(h)$ such that $|f(\tilde{h}x\tilde{g}) - f(hxg)| < \epsilon$ for all $\tilde{g} \in U_g, \tilde{h} \in U_h, x \in G$. This is the case if, for all $\epsilon > 0$, there exists $V \in \mathcal{U}(e)$ such that, for all $y \in G$, $|f(\tilde{y}) - f(y)| < \epsilon$ for all $\tilde{y} \in VyV$, where e is the unit element of G.[9]

This is precisely the assertion that f is uniformly continuous and the proof is similar to that for functions on compact sets in $\mathbb{R}$: Let $\epsilon > 0$. Because f is continuous, for all $y \in G$ there exists $U_y \in \mathcal{U}(y)$ such that $|f(\tilde{y}) - f(y)| < \frac{\epsilon}{2}$ for all $\tilde{y} \in U_y$. Continuity of $(x, z) \mapsto xyz$ at (e, e) implies that we can find $W_y \in \mathcal{U}(e)$ such that $W_y y W_y \subset U_y$. Again using

[7] Once again we emphasize that all topological notions refer to the w^*-topology on $C(G)'$.

[8] To avoid misunderstandings, we remark that $f(h \cdot g)$ is the map $G \to \mathbb{C} : x \mapsto f(hxg)$, not f evaluated at hg.

[9] Indeed, given $g, h \in G, \epsilon > 0$ and choosing V as in the second condition, this latter condition tells us that, for all $x \in G$, $|f(\tilde{y}) - f(hxg)| < \epsilon$ for all $\tilde{y} \in VhxgV$. Setting $U_h = Vh, U_g = gV$, this implies the first condition.

the continuity of $(x, z) \mapsto xz$, there exists $V_y \in \mathcal{U}(0)$ such that $V_y^2 \subset W_y$ and $V_y \subset W_y$. The set G is compact and $(V_y y V_y)_{y \in G}$ is an open cover. Therefore, there exist $y_1, \ldots, y_n \in G$ such that $G = \bigcup_{i=1}^n V_{y_i} y_i V_{y_i}$. Set $V := \bigcap_{i=1}^n V_{y_i} \in \mathcal{U}(0)$. Let $y \in G$, $\tilde{y} \in VyV$. Choose $\ell \in \{1, \ldots, n\}$ with $y \in V_{y_\ell} y_\ell V_{y_\ell}$. Then $y \in U_{y_\ell}$, $\tilde{y} \in V V_{y_\ell} y_\ell V_{y_\ell} V \subset W_{y_\ell} y_\ell W_{y_\ell} \subset U_{y_\ell}$ and hence

$$|f(\tilde{y}) - f(y)| \leq |f(\tilde{y}) - f(y_\ell)| + |f(y_\ell) - f(y)| < \epsilon.$$

□

Conclusion

To summarize, we have shown that $\mathcal{S}$ is a noncontracting semigroup of continuous affine maps on a nonempty compact convex subset, $\hat{Q}$, of the locally convex topological vector space $(C(G)', \tau_{w^*})$ into itself. The Ryll-Nardzweski fixed point theorem implies that there exists a fixed point $I_\mu \in \hat{Q}$ of $\mathcal{S}$ and, as discussed above, the corresponding measure μ is a Haar measure for G.

Remark. If G is a non-compact locally compact group, the above argument does not work. We can define Q and $\mathcal{F}$ as above and, using Riesz' representation theorem for positive linear functionals on $C_0(G)$, we can regard Q as a subset $\hat{Q}$ of $C_0(G)'$. However, the condition $\mu(G) = 1$, which translates to $\|I_\mu\| = 1$, now cannot be stated in terms of evaluation at $1 \in C(G) \backslash C_0(G)$ because I_μ is only defined on $C_0(G)$. Thus, the set $\hat{Q} = \overline{U_1(0)} \cap \partial U_1(0) \cap \bigcap_{f \geq 0} \iota_f^{-1}(\mathbb{R}_0^+)$ need not be (w^*-)compact because $\partial U_1(0)$ may not be (w^*-)closed. For instance, if $G = \mathbb{R}$, the functionals $I_n := \int_{[n,n+1]} \cdot \, d\lambda$, $n \in \mathbb{N}$ belong to $\hat{Q}$, but for all $f \in C_0(G)$ we have $I_n f \to 0$ and therefore $I_n \overset{w^*}{\to} 0 \notin \hat{Q}$.
Indeed, for $G = \mathbb{R}$, the set Q cannot contain a fixed point of $\mathcal{F}$ at all, because every Haar measure on $\mathbb{R}$ is unbounded.[10]

One might hope to meet the conditions by choosing an alternative definition of Q, e.g. by fixing a compact neighborhood K of e and taking all Radon measures μ that satisfy $\mu(gK) = 1$ or $\mu(Kg) = 1$ for some $g \in G$. However, such an attempt fails — it *must*, because there are locally compact groups with a left Haar measure that is not right-invariant. For the definition of Q just given the problem is that $\mathcal{S}$ need not be noncontracting on Q.

In Section 5.1 we will briefly explain how a different fixed point theorem, the Markov-Kakutani fixed point theorem, which avoids the requirement that the family of functions be noncontracting, can be used to establish the existence of Haar measure on locally compact *abelian* groups.

[10] Let μ be a Haar measure on $\mathbb{R}$. Since μ is inner regular and $\mu(\mathbb{R}) > 0$, there exists $N \in \mathbb{N}$ such that $\mu([-N, N]) > 0$. Because $\mathbb{R}$ contains an infinite number of disjoint sets of the form $a + [-N, N]$, $a \in \mathbb{R}$, and since μ is translation invariant, we conclude $\mu(\mathbb{R}) = \infty$.

4 Uniqueness

We supply the short proof that the Haar measure on a compact group G, which exists according to the previous section, is unique. Let μ, ν be Haar measures on G. Then for all $f \in C(G)$

$$\begin{aligned} I_\mu f = \int f d\mu = \int \int f(g) d\mu(g) d\nu(h) = \\ = \int \int f(hg) d\mu(g) d\nu(h) = \int \int f(hg) d\nu(h) d\mu(g) = \\ = \int \int f(h) d\nu(h) d\mu(g) = \int f d\nu = I_\nu f, \end{aligned}$$

so $\mu = \nu$. In going from the first to the second line we used the right-invariance of μ and the equality in the second line holds because of Fubini's theorem (which can be applied because μ and ν are finite and f is bounded on G). To establish the third line we used the left-invariance of ν.

For general locally compact groups uniqueness holds in the sense that two left Haar measures may only differ by a positive multiplicative constant: $\mu = c\nu, c \in \mathbb{R}^+$. However, the proof of this is more involved. [Fei09]

In Section 2 we already mentioned that a left Haar measure on a compact group is automatically a Haar measure. This is another consequence of the calculation above, where we only used the invariance of μ under left translation. Therefore, if μ is a left Haar measure and ν is *the* Haar measure for G, the calculation above implies $\mu = \nu$, so μ is Haar measure.

5 The Ryll-Nardzewski fixed point theorem

The following theorem, due to C. Ryll-Nardzewski, is the main result used in our proof of existence of Haar measure on a compact group (Section 3):

Theorem 5.1 (Ryll-Nardzewski)**.** *Let K be a nonempty compact convex subset of a locally convex topological vector space X and let $\mathcal{S}$ be a noncontracting semigroup of continuous affine functions of K into itself. Then $\mathcal{S}$ has a fixed point in K.*

Recall that the property of $\mathcal{S}$ being noncontracting means that

$$0 \notin \overline{\{S(x) - S(y) \mid S \in \mathcal{S}\}},$$

if x and y are different points in K.

Remark. The original version of the theorem, as stated by Ryll-Nardzewski in [RN67], is more general and requires compactness of K and continuity of the functions in $\mathcal{S}$ only with respect to the weak topology on X. The noncontracting property remains formulated with respect to the original topology on X, so altogether the statement is stronger and not, as one might believe at first glance, the same theorem for the space (X, τ_w) instead of X with its original (locally convex) topology. Since we do not need the theorem in its full generality, we shall only prove the simplified version, Theorem 5.1. The interested reader is referred to [Con85] for a (not too complicated) proof of Ryll-Nardzewski's original theorem.

Our proof of Theorem 5.1, which is based on the one in [Con85], requires two major results from functional analysis that are very interesting in their own right. The first is the **Markov-Kakutani fixed point theorem**, presented in Section 5.1, which is a fixed point theorem for *commuting* families of continuous affine functions on compact convex subsets of (not necessarily locally convex) topological vector spaces. The proof, although by no means trivial, only requires elementary properties of topological (vector) spaces.
The second fundamental result used to prove Theorem 5.1 is the **Krein-Milman theorem** whose proof is, in essence, a clever combination of the Hahn-Banach theorem (hence the requirement of local convexity) and Zorn's lemma. It states that a compact convex set K in a locally convex topological vector space is in a sense "generated" by its extremal points: $K = \overline{\text{co}}\, E(K)$. Since the content of the theorem is very well-known and intuitively quite "believable", we present it in a separate section after the proof of Theorem 5.1.

5.1 The Markov-Kakutani fixed point theorem

Theorem 5.2 (Markov-Kakutani)**.** *Let K be a nonempty compact convex subset of a topological vector space X and let $\mathcal{S}$ be a commuting set of continuous affine functions of K into itself. Then $\mathcal{S}$ has a fixed point in K.*

Proof. [Con85] For $S \in \mathcal{S}, n \in \mathbb{N}$, we define

$$S^{(n)} := \frac{I + S + \cdots + S^{n-1}}{n}$$

and $\mathcal{T} := \{S^{(n)} \,|\, n \in \mathbb{N}, S \in \mathcal{S}\}$. Because K is convex it is invariant under all $T \in \mathcal{T}$: $T(K) \subset K$. Therefore, $\mathcal{K} := \{T(K) \,|\, T \in \mathcal{T}\}$ is a family of closed subsets of K. We show that it has the finite intersection property: Let $T_1, \ldots, T_N \in \mathcal{T}$, $N \in \mathbb{N}$. Note that the maps T_i commute, so

$$T_1 \cdots T_N(K) = T_n T_1 \cdots T_{n-1} T_{n+1} \cdots T_N(K) \subset T_n(K).$$

Therefore, $\emptyset \neq T_1 \cdots T_N(K) \subset \bigcap_{n=1}^{N} T_n(K)$ and $\mathcal{K}$ has the finite intersection property. Because K is compact, this implies the existence of an element

$x_0 \in \bigcap_{T\in\mathcal{T}} T(K)$.

We show that x_0 is a fixed point of $\mathcal{S}$: Let $S \in \mathcal{S}$, $n \in \mathbb{N}$. The previous result implies that there exists $x \in K$ such that $x_0 = S^{(n)}(x)$. Therefore,

$$\begin{aligned} S(x_0) - x_0 &= S^{(n+1)}(x) - S^{(n)}(x) = \\ &= \frac{S + S^2 + \cdots + S^n}{n}(x) - \frac{I + S + \cdots + S^{n-1}}{n}(x) = \\ &= \frac{S^n(x) - x}{n} \in \frac{1}{n}(K - K). \end{aligned}$$

The set $K-K$ is compact and in particular bounded, so for any neighborhood $U \in \mathcal{U}(0)$ there exists $n \in \mathbb{N}$ such that $U \supset \frac{1}{n}(K - K) \ni S(x_0) - x_0$. Since X is Hausdorff this can only be the case if $S(x_0) = x_0$. □

In particular, every continuous affine function on a compact convex subset of a topological vector space has a fixed point.

Remark. The Markov-Kakutani fixed point theorem can be used to prove the existence of Haar measure on locally compact abelian groups, see [Izz92] for details. The idea is to consider a family $\mathcal{F}$ of translations R_g ($= L_g$ for an abelian group), $g \in G$, defined in the same way as in Section 3 but on a different set Q of "Haar measure candidates":

$$Q = \{\mu \text{ Radon measure} \,\big|\, \forall g \in G : \, \mu(gN) \le 1 \le \mu(gN^2)\},$$

where N is an arbitrary open symmetric neighborhood of e with compact closure. Using Riesz' theorem for positive linear functionals on $C_c(G)$, we can identify Q with a subset $\hat{Q}$ of $C_c^*(G)$, which is a topological vector space when equipped with the w^*-topology. In the definition of Q above, the first "$\le$" ensures that $\hat{Q}$ is compact in $(C_c^*(G), \tau_{w^*})$, while the second guarantees $0 \notin \hat{Q}$ without destroying compactness. The details, as well as a proof that $Q \ne \emptyset$, can be found in [Izz92]. In summary, the set $\hat{Q}$ is nonempty, convex and compact in $(C_c^*(G), \tau_{w^*})$.

The family $\mathcal{F}$ of translations, viewed as a family $\hat{\mathcal{F}}$ of functions on $C_c^*(G)$, is commuting and all of its elements are affine and continuous on $\hat{Q}$. Therefore, the Markov-Kakutani theorem implies that there is a point $I_\mu \in \hat{Q}$ that is fixed under all translations in $\hat{\mathcal{F}}$, so G has a Haar measure.

5.2 Proof of the Ryll-Nardzewski fixed point theorem

We make use of the following simple lemma:

Lemma 5.3. *Let X be a vector space, $K \subset X$, $x \in K$. Let $S_1, \ldots, S_n$ be functions from K into X. If*

$$S_0(x) := \frac{S_1 + \cdots + S_n}{n}(x) = x$$

and $m \in \{1, \ldots n-1\}$ *such that*

$$S_1(x) = x, \ldots, S_m(x) = x$$
$$S_{m+1}(x) \neq x, \ldots, S_n(x) \neq x$$

then

$$S_0'(x) := \frac{S_{m+1} + \cdots + S_n}{n-m}(x) = x.$$

Proof. The proof is a simple calculation:

$$\begin{aligned} x = S_0(x) &= \frac{mx + S_{m+1}(x) + \ldots + S_n(x)}{n}(x) = \\ &= \frac{m}{n}x + \frac{n-m}{n}S_0'(x). \end{aligned}$$

Solving for $S_0'(x)$ yields $S_0'(x) = x$. □

For the reader unfamiliar with the Krein-Milman theorem we summarize the results of Section 6 that we need for the proof of the Ryll-Nardzewski fixed point theorem in the following lemma:

Lemma 5.4. *Let* X *be a locally convex topological vector space. Let* K *be a compact convex subset of* X *and let* M *be a nonempty subset of* K*. Then there exists* $y \in \overline{M}$ *with the following property: If* $y_1, \ldots, y_n$ *are arbitrary elements in* $\overline{M}$ *such that*

$$y = \frac{y_1 + \cdots + y_n}{n}$$

then

$$y = y_1 = \cdots = y_n.$$

Proof. The lemma is a trivial consequence of the Krein-Milman theorem and its inversion: Let $L = \overline{\text{co}}\, M$. Clearly, L is nonempty, compact and convex. By the Krein-Milman theorem $E(L) \neq \emptyset$ and by Theorem 6.4 $E(L) \subset \overline{M}$. So $\overline{M} \cap E(L) \neq \emptyset$ and any element y of this set has the above stated property. □

We are finally ready to prove the Ryll-Nardzewski fixed point theorem:

Proof of Theorem 5.1. It suffices to show that for every finite number of functions $S_1, \ldots, S_n \in \mathcal{S}$ there exists a common fixed point. Indeed, if this is the case, the family $(\{x \in K \mid S(x) = x\})_{S \in \mathcal{S}}$ of closed subsets of K has the finite intersection property and hence $\bigcap_{S \in \mathcal{S}} \{x \in K \mid S(x) = x\} \neq \emptyset$ because K is compact.

Let $S_1, \dots, S_n \in \mathcal{S}$. According to the Markov-Kakutani theorem the function

$$S_0 := \frac{S_1 + \dots + S_n}{n}$$

has a fixed point $x_0 \in K$: $S_0(x_0) = x_0$. We prove by contradiction that x_0 is also a fixed point of $S_1, \dots, S_n$. If $S_i(x_0) \neq x_0$ for some $i \in \{1, \dots, n\}$, the previous lemma tells us that we can assume $S_j(x_0) \neq x_0$ for *all* $j \in \{1, \dots, n\}$. (Simply take the function denoted S_0' in the lemma instead of S_0.)

With this assumption let $\tilde{\mathcal{S}}$ be the semigroup generated by $\{S_1, \dots, S_n\}$ and let $M \subset K$ be the set of images of x_0 under functions in $\tilde{\mathcal{S}}$:

$$M := \{S(x_0) \,\big|\, S \in \tilde{\mathcal{S}}\}.$$

We use Lemma 5.4 and find $y \in \overline{M}$ with the property stated in the lemma. Let $(T_i)_{i \in I}$ be a net in $\tilde{\mathcal{S}}$ with $T_i(x_0) \to y$. Using $x_0 = S_0(x_0)$

$$y = \lim_{i \in I} T_i(x_0) = \lim_{i \in I} \frac{T_i S_1(x_0) + \dots + T_i S_n(x_0)}{n}. \tag{1}$$

The nets $(T_i S_\ell(x_0))_{i \in I}$, $\ell = 1, \dots, n$, all lie in M. Using the compactness of $\overline{M}$ we find a subnet $(U_j)_{j \in J}$ of $(T_i)_{i \in I}$ such that the nets $(U_j S_\ell(x_0))_{j \in J}$, $\ell = 1, \dots, n$, converge. Let

$$y_\ell := \lim_{j \in J} U_j S_\ell(x_0) \in \overline{M}, \ \ell = 1, \dots, n.$$

By equation (1), $y = \frac{1}{n}(y_1 + \dots + y_n)$, so our choice of y implies $y = y_1 = y_2 = \dots = y_n$. In particular,

$$0 = y_1 - y = \lim_{j \in J} (U_j S_1(x_0) - U_j(x_0))$$

and therefore

$$0 \in \overline{\{S(S_1(x_0)) - S(x_0) \,\big|\, S \in \mathcal{S}\}}.$$

Since $S_1(x_0) \neq x_0$ this contradicts the fact that $\mathcal{S}$ is noncontracting. $\square$

6 The Krein-Milman theorem

In the proof of the Ryll-Nardzewski theorem, we derived a contradiction by constructing a non-empty compact subset $\overline{M}$ of a locally convex topological vector space X with the property that every element of $\overline{M}$ could be written as a convex combination of other elements of $\overline{M}$. At least for $X = \mathbb{C}$ it is intuitively (and also mathematically) clear that such a set cannot exist: If $K \subset \mathbb{C}$ is closed and bounded, we can take the smallest disk D containing

K and no element of $K \cap \partial D$ (which is nonempty) can lie on an *open* line segment whose endpoints are in K.

In Lemma 5.4 we presented a proof of the general case based on two fundamental theorems of functional analysis: the Krein-Milman theorem and its inversion. It is the aim of this section to prove these two theorems.

To avoid the clumsy expression of "points that cannot be written as a convex combination of other points" used above, we make the following definition:

Definition (Extremal point)**.** Let X be a vector space, $K \subset X$. We say that $x \in K$ is an **extremal point** of K if it cannot be written as a proper convex combination of elements in $K \backslash \{x\}$. In other words,

$$x = \lambda x_1 + (1-\lambda)x_2, \quad \lambda \in (0,1),\, x_1, x_2 \in K$$

implies $x_1 = x_2 = x$. The set of extremal points of K is denoted $E(K)$.

For instance, if $K \subset \mathbb{C}$ is a convex polygon $E(K)$ is the set of its corners. If K is a closed disk $E(K) = \partial K$. For these simple examples, it also holds that the entire set K can be reconstructed from its extremal points by taking their convex hull: $K = \operatorname{co} E(K)$. The Krein-Milman theorem, which is the first of the two theorems we will prove in this section, generalizes this result to closed convex subsets K of arbitrary locally convex topological vector spaces: $K = \overline{\operatorname{co}}\, E(K)$. Note that the relation would not hold if we did not take the closure of $\operatorname{co} E(K)$, as there exist closed convex sets K for which $\operatorname{co} E(K)$ is not closed (see [Con85] for an example).

The second theorem that we shall prove, Milman's inversion, states that every closed set A satisfying $K = \overline{\operatorname{co}}\, A$ must contain $\overline{E(K)}$. In other words, $\overline{E(K)}$ is the smallest closed set from which we can "build" K by taking the closed convex hull.

We now turn to the proof of the theorems and a few useful lemmas.

Lemma 6.1. *Let X be a topological vector space and let $U \subset X$ be convex. Then, for fixed $u \in U$, $\lambda \in (0,1)$, the set*

$$W := \{x \in X \,|\, \lambda u + (1-\lambda)x \in U\}$$

is convex and contains $\overline{U}$.

Proof. Clearly, W is convex and contains U. To show that $\overline{U} \subset W$, let $x \in \overline{U} \backslash U$ and let S be the line segment connecting x and u, $S := \operatorname{co}\{x, u\}$. The function

$$f : [0,1] \to S : \mu \mapsto \mu u + (1-\mu)x$$

is a homeomorphism and $f(0) = x, f(1) = u$. Because U and S are connected with nonempty intersection, $S \cap U$ is connected, so $I := f^{-1}(S \cap U)$ must be an interval. Since $1 \in I, 0 \in \overline{I}$, the only possibilities for I are

$$I = (0, 1] \quad \text{or} \quad I = [0, 1].$$

In either case, $(0, 1) \subset f^{-1}(S \cap U)$, so $f(\lambda) \in S \cap U \subset U$, which is what we wanted to show. □

Theorem 6.2 (Krein-Milman)**.** *Let X be a locally convex topological vector space and let K be a nonempty compact convex subset of X. Then K is the closed convex hull of its extremal points:*

$$K = \overline{\text{co}}\, E(K).$$

Proof. [Con85] Clearly $K \supset \overline{\text{co}}\, E(K)$. By contradiction we show that $K \subset \overline{\text{co}}\, E(K)$. If there exists $x_0 \in K \backslash \overline{\text{co}}\, E(K)$, we can use the Hahn-Banach theorem to find a functional $f \in X'$ and $\gamma \in \mathbb{R}$ such that

$$\text{Re} f(x_0) < \gamma < \text{Re} f(\overline{\text{co}}\, E(K)).$$

So $V := \{x \in K \,|\, \text{Re} f(x) > \gamma\}$ is a proper open[11] convex subset of K that contains $E(K)$. We show that such a set cannot exist by looking at the maximal elements of the family $\mathcal{M}$ of proper open convex subsets of K which contain V:

$$\mathcal{M} := \{W \subset K \,|\, W \text{ is a proper open convex subset of } K \text{ and } V \subset W\}.$$

The family $\mathcal{M}$, partially ordered by set inclusion, satisfies the conditions of Zorn's lemma: It is nonempty and given a totally ordered subset $\mathcal{N} \subset \mathcal{M}$, the union $\bigcup_{N \in \mathcal{N}} N \in \mathcal{M}$ is an upper bound for $\mathcal{N}$. Note that $\bigcup_{N \in \mathcal{N}} N$ is indeed a proper subset of K because K is compact and therefore, if $\bigcup_{N \in \mathcal{N}} N = K$, we could find $N_1, \ldots, N_n \in \mathcal{N}$ such that $K = \bigcup_{i=1}^{n} N_i$. Since $\mathcal{N}$ is totally ordered this would imply $N_j = K$ for some $j \in \{1, \ldots, n\}$, contradicting the fact that N_j is proper.

So there indeed exist maximal elements of $\mathcal{M}$. Let U be one of them. We show that $K \backslash U$ is a singleton. By definition of $\mathcal{M}$, $K \backslash U \neq \emptyset$. If there exist two different points $a, b \in K \backslash U$, we can find relatively open convex neighborhoods $U_a, U_b \subset K$ of a respectively b such that $U_a \cap U_b = \emptyset$. Clearly, $U_a \cup U$ is open in K and proper. Moreover, $U_a \cup U$ is convex, which can be seen

[11]If we say that a set V is an open subset of K we mean that V is a subset of K that is open with respect to the relative topology on K.

as follows: Let $\lambda \in (0,1)$, $u \in U$. We have to show that $\lambda u+(1-\lambda)x \in U_a \cup U$ for all $x \in U_a$. In other words, $U_a \cup U \subset W$, where W is the set

$$W := \{x \in K \,\big|\, \lambda u + (1-\lambda)x \in U\}.$$

Clearly, W is open in K. From Lemma 6.1 it follows that W is convex and $\overline{U} \subset W$. Since K is connected, we have $U \neq \overline{U}$, so W is a proper superset of U. Because U is maximal in $\mathcal{M}$, this can only be the case if $W = K$. In particular, $U_a \cup U \subset W$.

Hence, $U_a \cup U$ is a proper open convex subset of K, which is also a proper superset of U since $a \notin U$, a contradiction. We conclude that $K \backslash U$ is a singleton. The point $p \in K\backslash U$ must be an extremal point of K, since otherwise it could be written as $p = \mu x + (1-\mu)y$ with $\mu \in (0,1)$, $x_1, x_2 \in K\backslash\{p\}$, contradicting the convexity of U. Therefore, U does not contain all extremal points of K and since $V \subset U$ the set V cannot either, contradicting the definition of V. □

We now turn to the Milman inversion of the Krein-Milman theorem.

Lemma 6.3. *Let $K_1, \ldots, K_n$ be compact convex subsets of a topological vector space X. Then*

$$\overline{\mathrm{co}}\,(K_1 \cup \cdots \cup K_n) = \mathrm{co}\,(K_1 \cup \cdots \cup K_n).$$

Proof. We only need to show this for the case $n = 2$, the general result follows by induction. Clearly, $K := \mathrm{co}\,(K_1 \cup K_2)$ is closed if we can show that

$$K = \{\lambda x + (1-\lambda)y \,\big|\, \lambda \in [0,1], x \in K_1, y \in K_2\}, \tag{2}$$

because then K is the image of the compact set $[0,1] \times K_1 \times K_2$ under the continuous function

$$f:\ [0,1] \times K_1 \times K_2 \to S:\ \lambda \mapsto \lambda x + (1-\lambda)y.$$

To prove (2), note that the set on the right hand side is contained in every convex superset of $K_1 \cup K_2$. Therefore, if we can show that it is convex, it is the smallest convex set containing $K_1 \cup K_2$. Let $z_1 := \lambda_1 x_1 + (1-\lambda_1)y_1$ and $z_2 := \lambda_2 x_2 + (1-\lambda_2)y_2$ where $\lambda_i \in [0,1], x_i \in K_1, y_i \in K_2$ for $i \in \{1,2\}$. Then, for $\mu \in [0,1]$,

$$\mu z_1 + (1-\mu)z_2 = \lambda x + (1-\lambda)y,$$

where

$$\begin{aligned}
\lambda &= \mu\lambda_1 + (1-\mu)\lambda_2 \in [0,1] \\
x &= \frac{\mu\lambda_1}{\lambda}x_1 + (1 - \frac{\mu\lambda_1}{\lambda})x_2 \in K_1 \\
y &= \frac{\mu(1-\lambda_1)}{1-\lambda}y_1 + (1 - \frac{\mu(1-\lambda_1)}{1-\lambda})y_2 \in K_2.
\end{aligned}$$

Hence, $\mu z_1 + (1-\mu) z_2 \in \{\lambda x + (1-\lambda) y \,\big|\, \lambda \in [0,1], x \in K_1, y \in K_2\}$.

□

Theorem 6.4 (Milman inversion)**.** *Let X be a locally convex topological vector space and let K be a nonempty compact convex subset of X. Let $F \subset K$ be such that*

$$K = \overline{\text{co}}\, F.$$

Then $E(K) \subset \overline{F}$.

Proof. [Con85] Clearly it can be assumed that F is closed and $K \neq F$. Let $x_0 \in K \backslash F$. We show that there exist compact convex sets $K_1, \dots, K_n \subset K$ such that

$$F \subset K_1 \cup \dots \cup K_n \quad \text{and} \quad x_0 \notin K_1 \cup \dots \cup K_n. \tag{3}$$

This implies that x_0 is not an extremal point of K, because using Lemma 6.3 above and the first property in (3)

$$K = \overline{\text{co}}\, F = \text{co}\,(K_1 \cup \dots \cup K_n),$$

so $x_0 \in K$ can be expressed as a proper convex combination of elements of K. Since $x_0 \in K \backslash F$ was arbitrary, $K \backslash F \subset K \backslash E(K)$.

To see that compact convex sets $K_1, \dots, K_n$ with property (3) exist, let U be an open convex neighborhood of 0 that separates x_0 from F in the following manner:

$$(x_0 + U) \cap (F + U) = \emptyset.$$

In particular, $x_0 \notin \overline{F+U}$. Because F is compact, we can find $n \in \mathbb{N}, y_1, \dots, y_n \in F$ such that

$$F \subset \bigcup_{i=1}^{n} (y_i + U).$$

Set $K_i := \overline{\text{co}}\,(F \cap (y_i + U))$, $i \in \{1, \dots, n\}$. Clearly, the sets K_i are compact and convex and $F \subset \bigcup_{i=1}^{n} K_i$. Moreover, if there existed $j \in \{1, \dots, n\}$ such that $x_0 \in K_j$, this would imply $x_0 \in \overline{\text{co}}\,(y_j + U) \subset \overline{F+U}$, a contradiction. Therefore, the sets K_i have the properties we required. □

Remark. The results above are very much in line with intuitive expectations, but this is only so because the conditions have been adapted accordingly. Unexpected things can happen if one of the conditions is dropped. For instance, the closed unit ball in a normed vector space, which one would consider a very "simple" convex set, need not have any extremal points at all. This is for instance the case in the space $L_1([0,1], \lambda)$ with the usual $\|\cdot\|_1$-norm. It may also have far too few for $\overline{U_1(0)} = \overline{\text{co}}\, E(\overline{U_1(0)})$ to hold, which is the case in $C[0,1]$, where $E(\overline{U_1(0)}) = \{-1, 1\}$, so clearly $\overline{U_1(0)} \neq \overline{\text{co}}\, E(\overline{U_1(0)})$.

These two examples do not contradict our general results, because the condition of compactness is not satisfied — at least not with respect to the usual (norm) topology. Indeed, the Krein-Milman theorem implies that $\overline{U_1(0)} \subset L^1(0,1)$ (resp. $\subset C[0,1]$) cannot be compact with respect to *any* topology that makes $L^1(0,1)$ (resp. $C[0,1]$) a locally convex topological vector space. In particular, $L^1(0,1)$ (resp. $C[0,1]$) cannot be isomorphic (in the category of topological vector spaces) to the dual of a normed vector space. Hence, an immediate corollary of the Krein-Milman theorem is that the usual identification of $L^p(0,1)'$ and $L^q(0,1)$ for $1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, cannot be extended to the case $p = \infty, q = 1$.

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