Pointless Topology

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Starting with the motivating example of Stone's representation theorem that allows one to represent Boolean algebras as subalgebras of the poweralgebra of a sufficiently large set, we ask the question of whether it is possible to generalize this to a relationship between lattice theory and topology. This can be done by considering special lattices called locales, which are, in a sense, a suitable algebraic model for a topological space. Every topological space is a locale and we can assign to each locale a topological space, which one can consider as the set of "points" of that lattice. We will make this precise in the form of an adjunction between the category of topological spaces and the category of locales, which is the main point of this paper.

1 Introduction

The concept of a "point" in a continuum has always been a mysterious one. In a naive example, one could say that a point in euclidean space corresponds in the real world to a directed measurement with a ruler, from a certain fixed starting point to another. One would have to object that no matter how precise the measurement, we would still not reach the point itself, we would just get close. By refining the measurement, i.e. using a more precise ruler, we could still get closer to the point we are trying to measure, yet the point itself will never be reached. As such, a point in space might actually be nothing more than a series of increasingly precise measurements. In today's mathematics, specifically analysis and topology, the starting point is always the set of points. Only after the existence of points has been assumed, we assign a notion of "nearness" (as a topology) or "distance" (as a metric space) to the set of points. Now there is a way to turn this logic the other way around, but in order to establish the mathematics behind it, we have to go back in history for a few years.

Stone's representation theorem goes back to the 1930s and was motivated by his study of the spectral theory of operators on a Hilbert Space. The idea is to associate to each boolean algebra B a set of "points" X, where a point is defined to be an ultrafilter in B. One can then find the boolean algebra as a subalgebra of the power algebra 2^X. Let us state the theorem:

Theorem 1.1. ***Stone's Representiation Theorem (classical version)*** *For every boolean algebra B, there is a set X and an injective boolean algebra homomorphism from B into 2^X.*

Viewing B as a sub-boolean algebra of 2^X, we can generate the topology $\mathcal{T}$ on X with B as sub-basis. It turns out that B is already a basis of $\mathcal{T}$ and every element $b \in B$ is a clopen subset of $\mathcal{T}$ (since B is closed under complements). Therefore, $(X, \mathcal{T})$ is totally disconnected. One can further show that it is compact and Hausdorff. Such a space is called a *Stone Space*. Let K be a clopen subset of X. Since K is open, it can be written as a union of elements b_i of B. It is a closed subset of the compact space X, therefore itself compact, hence a finite number of such b_i's suffices. Therefore K is already in B. Since for any topological space the set of clopen sets forms a boolean algebra, it must be equal to B, that means we can recover B from $\mathcal{T}$. This process is an example of a duality.

Theorem 1.2. ***Stone's Representiation Theorem (categorical version)*** *The category of boolean algebras and boolean algebra homomorphisms is dual to the category of Stone spaces and continuous functions between them.*

Now back to our discussion of "points". In the example of Stone's Theorem, a point was identified with an ultrafilter. We can view this filter as an algebraic model of our intuitive notion of a series of increasingly precise measurements of a point. The question arises: Is it possible to extend this analogy between certain topological spaces and algebraic structures to all of topology by weakening the requirements from boolean algebras to a certain type of lattice? It turns out that we can.

The most general type of lattice that can still be considered topological is a complete lattice satisfying a certain distributivity law called a *locale*. A locale is the same thing as a complete *Heyting-Algebra*, a model for intuitionistic logic generalizing classical boolean logic. The only difference lies in their treatment of morphisms, i.e. generalized continuous functions. We will later on prove that topological spaces are adjoint to locales, which reduces to an equivalence on the rather large subcategories of sober spaces and spatial locales. The category of sober spaces includes all Hausdorff spaces as well as spaces arising from the Zariski topology on the spectrum of any commutative ring. Many types of Stone dualities, including the aforementioned Stone Representation Theorem reduce to special cases of this equivalence.

Almost any topological concept can be translated into localic terms. Products, quotients, subobjects, and in general limits and colimits all exist.[1] Compactness, regularity and a variety of separation axioms can be easily formulated, with almost the same properties as for topological spaces.

Now why should the reader care? The best answer can probably be found in Johnstone's Paper "The Point of Pointless Topology" dating 1983, [9]. There are differences between topological spaces and locales. One example is that products are generally bigger (for sober spaces, in general, there exists only a surjection from their localic product to their topological product). John Isbell argues that those deficiencies are actually a feature, stating that the localic product is, in general, better behaved than the topological one [6]. A Tychonoff-theorem exists for compact regular locales, whose proof neither requires the axiom of choice nor the law of the excluded middle. The same applies for a version of the Stone-Czech compactification and a variety of other analogues to point-set theorems. The subcategories of Compact Hausdorff Spaces and Compact Regular Locales can be proven equivalent, but any such proof requires one to invoke a choice principle (most commonly a version of the prime ideal theorem), cf. [15]. Further, there is a close connection between the theory of locales and topos theory. Locales can be formulated in any topos, which allows one to do topology in any mathematical setting where the axiom of choice and/or law of the excluded middle is missing. An example of a frequently encountered topos is the category of sheaves over a topological space X. Much information about X can be gained by looking at the behaviour of its topos of sheaves. Besides the usefulness of being able to do topology in this setting, there is another argument for locales. The assignment from X to the sheaves over X, $\mathrm{Sh}(X)$, gives a functor into the quasi-category of all topoi, yet has the shortcoming of not being an embedding. Restricting to sober spaces, this functor suddenly becomes an embedding that further extends naturally to the full category of locales, cf. [8], [13]. Another interesting and quite recent result is that with the use of locale theory, it is possible to have an isometry-invariant measure on $\mathbb{R}^n$ with its localic structure, for which *all* subsets are measurable, a result that is impossible in classical topology, cf. [16].

A thorough discussion of this would go far beyond the scope of this paper, but we had the feeling it at least had to be mentioned. The most complete account on how topology is done with locales, without the use of deep category theory, is currently found in Picado and Pultr's wonderful book, [15]. Another good introduction is Johnstone's Stone Spaces,

[1]Limits in point-set topology are better known as inverse limits, colimits as direct limits.

[7].

We will need some categorical terminology to describe the main points of the paper. The reader familiar with the concept of natural transformations and adjoint functors can skip the next section. Note that although the author tried to keep this paper self-contained, the treatment is rather short and terse. We refer the reader to [2], [1] and [12] (in roughly ascending order of difficulty and generality).

2 Category Theory

One thing to keep in mind is that, historically, the main reason that category theory was invented is to study functors, and the main reason functors were invented is to study natural transformations and adjoints. As usual, in mathematics, by defining things we are doing it backwards.

Definition A *category* $\mathcal{A}$ consists of a class $\mathcal{O}$, sometimes also denoted $Ob(A)$, whose members are called $\mathcal{A}$-*objects*, together with a set hom(A,B) for each pair of $\mathcal{A}$-objects A and B, whose elements are referred to as $\mathcal{A}$-*morphisms* from A to B [the statement $f \in \hom(A,B)$ is usually written as $A \xrightarrow{f} B$], such that:

1. for each $\mathcal{A}$-Object, there exists a morphism $A \xrightarrow{id_A} A$, called $\mathcal{A}$-*identity* on A,
2. for all morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ there is a morphism $A \xrightarrow{g \circ f} C$, called the *composite* of f and g,

with the following conditions:

1. composition is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$, whenever both sides of this equation are well-defined,[2]
2. $\mathcal{A}$-Identities act as identities with respect to composition, i.e. for $A \xrightarrow{f} B$ we have $id_B \circ f = f = f \circ id_A$,
3. the sets hom(A,B) are pairwise disjoint.

We call $\mathcal{A}$ *small* if $\mathcal{O}$ is a set. If $\mathcal{O}$ is a finite set and all hom-sets are finite, we call $\mathcal{A}$ *finite*.

Example The following are categories with which most mathematicians are familiar:

- The category **Set** with objects sets and as morphisms functions between them.
- The category **Vec** of vectorspaces over a field $\mathbb{K}$, with linear maps between them.
- For those with knowledge in functional analysis: The subcategory $\mathbf{CommC^* - Alg}$ of **Vec** with objects commutative C^*-Algebras and as maps *-algebra homomorphisms. It is a proper subcategory in both objects as well as morphisms. We will use this category later for an example of (non-obvious) duality.

[2]This is already the case if either side is well-defined.

- The category **Top** of topological spaces with continuous maps as morphisms, as well as the (full) subcategory **HComp** of compact Hausdorff Spaces.

Example (Important!) A category P having only at most one morphism between any two objects is called a *partially ordered class.* Writing $A \leq B$ whenever $A \xrightarrow{f} B$ we can infer that $\leq$ is a relation on $Ob(P)$ with the properties:

1. $A \leq A$,
2. if $A \leq B$ and $B \leq A$ then $A = B$,
3. if $A \leq B$ and $B \leq C$ then $A \leq C$.

The first statement is the same as the existence of identities, the second follows from only having at most one morphism in between two objects, and the last is simply the composition of morphism. Conversely, any class with a relation fulfilling those properties can be viewed as a category that is a partially ordered class. We will call a small, partially ordered class a *partially ordered set*, or short *poset*.[3]

Definition A morphism $A \xrightarrow{f} B$ is called an *isomorphism*, iff there exists a morphism $B \xrightarrow{g} A$ s.t. $id_A = g \circ f$ and $id_B = f \circ g$. We call g the Inverse of f and write f^{-1}.

This notation is justified since, if inverses exist, they are unique. [Let g, h be both inverses to f. Then $g = (h \circ f) \circ g = h \circ (f \circ g) = h$ by associativity.]

Definition A *(covariant) functor* $F : \mathcal{A} \to \mathcal{B}$ is a mapping that assigns to every $\mathcal{A}$-object A a $\mathcal{B}$-object $F(A)$ and every $\mathcal{A}$-morphism $A \xrightarrow{f} A'$ a $\mathcal{B}$-morphism $F(A) \xrightarrow{F(f)} F(A')$ s.t.

1. $F(id_A) = id_{F(A)}$
2. $F(f \circ g) = F(f) \circ F(g)$.

Slightly trivial:

Proposition 2.1. *Every category $\mathcal{A}$ has an identity functor $id_{\mathcal{A}}$ defined in the obvious way.*

Proposition 2.2. *The composition of two functors F and G (again) defined in the obvious way is a functor.*

Proof. 1. $GF(id_A) = G(id_{F(A)}) = id_{GF(A)}$ and
2. $GF(f \circ g) = G(F(f) \circ F(g)) = GF(f) \circ GF(g)$. □

[3]Readers already familiar with posets might object that this definition is rather bizarre. The reason for this is that a lot of concepts from lattice theory have generalizations to arbitrary categories. Since we will be using these generalizations anyway there is no need in proving things twice. We will simply view posets as special cases. A more traditional account can be found in [5].

Proposition 2.3. *Functors preserve isomorphisms.*

Proof. Let $A \xrightarrow{f} B$ be an isomorphism, then

$$id_F(A) = F(id_A) = F(f^{-1} \circ f) = F(f^{-1}) \circ F(f) \tag{1}$$

$id_F(B) = F(f) \circ F(f^{-1})$ is proved the same way, which means $F(f^{-1}) = F(f)^{-1}$. □

Remark Taking the conglomerate of all categories as elements and viewing functors in between them as morphisms gives something that has the formal structure of a category, but is not one due to set theoretical difficulties (The "class" of all classes doesn't exist). Typically one speaks of **CAT** as the quasi-category of categories and functors in between them. If we want to view functors as morphisms in a category, it is usually convenient to consider the category **Cat** of small categories.

Example A functor $F : P \to Q$ between two partially ordered classes is also called a *monotone map*. This is justified by

$$A \leq B \implies F(A) \leq F(B). \tag{2}$$

Similarly, any map between two partially ordered classes with this property is a functor.

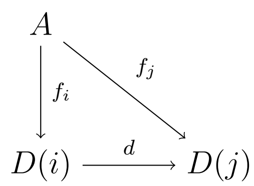
The class of posets together with monotone mappings between them is a (full) subcategory of **Cat**. We will denote it by **Pos**.

Important categorical concepts are those of products, terminal objects, coproducts and initial objects. All of these are special cases of the following definition:

Definition A *diagram* in a category $\mathcal{A}$ is a functor $D : \mathcal{I} \to \mathcal{A}$ with codomain $\mathcal{A}$. The domain $\mathcal{I}$ is also called the *scheme* of the diagram. We call a diagram small (finite) iff its scheme is small (finite).

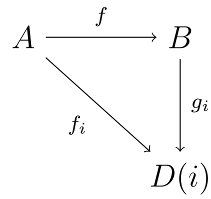
Given a diagram D we can define the *comma category* $\mathcal{A} \downarrow D$ as follows:

An object of $\mathcal{A} \downarrow D$ is an $\mathcal{A}$-Object A together with $\mathcal{A}$-Morphisms $A \xrightarrow{f_i} D(i)$ for all $\mathcal{I}$-Objects i, such that for each $\mathcal{I}$-Morphism $i \xrightarrow{d} j$



commutes. It is also said to be a *natural source* for D.

A morphism in $\mathcal{A} \downarrow D$ between two natural sources $(A \xrightarrow{f_i} D(i))$, $(B \xrightarrow{g_i} D(i))$ is an $\mathcal{A}$-morphism $A \xrightarrow{f} B$ such that



commutes for all i.

A *limit* of D is a natural source $(L \xrightarrow{l_i} D(i))$, s.t. for any other natural source $(A \xrightarrow{f_i} D_i)$ there exists a unique $(\mathcal{A} \downarrow D)-$morphism $A \xrightarrow{f} L$.

Definition A *terminal object* is a limit over the empty diagram. Explicitely, an object 1 is terminal, if for each object A there is a unique morphism $A \to 1$.

Proposition 2.4. *Terminal objects are unique up to unique isomorphism. Explicitly, if* 1 *and* $1'$ *are terminal, the single morphism* $1 \to 1'$ *is also an isomorphism.*

Proof. There exist unique morphisms $1 \xrightarrow{l} 1'$ and $1' \xrightarrow{r} 1$. The composition $r \circ l$ is a morphism $1 \to 1$. Since 1 is terminal, and id_1 is already a morphism $1 \to 1$, $r \circ l = id_1$. Doing the same with $1'$ gives $l \circ r = id_{1'}$. □

Proposition 2.5. *If limits over a diagram D exist, they are unique up to unique isomorphism in $\mathcal{A} \downarrow D$.*

Proof. A limit over D is the same as a terminal object in $\mathcal{A} \downarrow D$. □

Example If a terminal object exists, it will usually be denoted by 1 (it is unique up to unique isomorphism anyway). For $\mathcal{A}$-objects X_i, the *product* $\prod_{i \in I} X_i$ [also denoted $X_1 \times X_2$ if we have only two objects] is the limit over the diagram $i \mapsto X_i$, with $\mathcal{I}$ viewed as a category whose set of objects is I and the only morphisms are the identity morphisms.

Definition For any category $\mathcal{A}$, we define the *dual category* $\mathcal{A}^{\text{op}}$ to be the category consisting of the same objects as $\mathcal{A}$, $\hom_{\mathcal{A}^{\text{op}}}(A, B) := \hom_{\mathcal{A}}(B, A)$ for all A, B and $f \circ^{\text{op}} g := g \circ f$.

Remark The process of dualizing is simply reversing the directions of arrows in all diagrams. Any statement $S_{\mathcal{A}^{\text{op}}}$ concerning objects X_i in the category $\mathcal{A}^{\text{op}}$ can be translated into a logically equivalent statement $S_{\mathcal{A}}^{\text{op}}$ concerning X_i in the category $\mathcal{A}$.

As a simple example: In the category **Vec** of vectorspaces over a field $\mathbb{K}$, a vector v in a vectorspace V is nothing else than a morphism $\mathbb{K} \xrightarrow{v} V$. A dual vector v^* is a morphism $V \xrightarrow{v^*} \mathbb{K}$. The composition $\mathbb{K} \xrightarrow{v^* \circ v} \mathbb{K}$ gives a scalar.

Moreover, for any property P concerning categories we get a corresponding dual property, by stating P in the dual category. If a property P holds for any category, then so does its dual property. This is nothing deep per se, but allows a more economical treatment of categories.

Definition $F : \mathcal{A} \to \mathcal{B}$ is a *contravariant functor* iff $F : \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$ is a functor. *Colimits, coproducts* and *initial objects* are defined as dual notions to limits, products and terminal objects. We will denote an initial object with 0. The coproduct of objects X_i is denoted $\coprod_{i \in I} X_i$, or $X_1 + X_2$ in the case of two objects.

Definition A category is called *(co-) complete* if all small diagrams have (co-) limits.
A category is called *finitely (co-) complete* if all finite diagrams have (co-) limits.

Example Examples of complete categories are abundant. In fact **Set**, **Top**, **Vec**, $\mathbf{CommC^*-Alg}$, **HComp** and **Pos** are complete and cocomplete. As a counter example, consider the category **Field** of fields with field homomorphisms (that is, unital ring-homorphisms). Since field homomorphisms exist only between fields with the same characteristic, neither an initial nor a terminal object can exist.

Example Returning to our example of partially ordered classes: If a terminal object 1 in a partially ordered class exists, it is unique and we will call it *greatest element*. Dually, an initial object 0 is called *smallest element*. The product of objects X_i is also called *meet*, or *infimum*, and the coproduct is also called *join*, or *supremum*. A poset is called a *lattice* if products and coproducts for every pair of objects exist, *bounded* if both terminal and initial objects exist and a *complete lattice*, if it is complete and cocomplete. A *lattice homomorphism* is a monotone map that preserves joins and meets for any two objects. It is further called *bounded* if it preserves terminal and initial objects (and therefore in general finite joins and meets).

Next we want to introduce natural transformations. A natural transformation can be thought of as a morphism between functors with same domain and codomain. The simplest motivating example is the following:

Consider the category **Vec**. The operation $V \mapsto V^*$ and $(V \xrightarrow{f} W) \mapsto (W^* \xrightarrow{f^*} V^*)$ is a contravariant functor $^* : \mathbf{Vec} \to \mathbf{Vec}$, also called the dual functor. There is no "natural" map from a vector space V into it's dual V^* (apart from the zero map), since such a map would require the choice of a basis. But there is a natural map ι_V into the double dual V^{**} given by $v \mapsto (v^* \mapsto v^*(v))$. ι can be considered as a function from the objects of **Vec** to morphisms in **Vec**. Restricting ourselves to the class of vector spaces where ι_V is an isomorphism gives the full subcategory **FinVec** of finite-dimensional vector spaces.

Definition Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation* τ from F to G is a mapping that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-morphism $FA \xrightarrow{\tau_A} GA$ such that the following *naturality square*

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow{\tau_{A'}} & GA' \end{array}$$

commutes. We will write $F \xrightarrow{\tau} G$.

A natural transformation whose components τ_A are all isomorphisms is called *natural isomorphism.*

Definition An *adjoint situation* $F \dashv G$ consist of a pair of functors $\mathcal{B} \xrightarrow{F} \mathcal{A}$, $\mathcal{A} \xrightarrow{G} \mathcal{B}$ and a pair of natural transformations, $id_{\mathcal{B}} \xrightarrow{\eta} GF$ (called unit) and $FG \xrightarrow{\epsilon} id_{\mathcal{A}}$ (called counit) s.t. the counit and unit triangles[4]

$$\begin{array}{ccc} GY & \xrightarrow{\eta_{GY}} & GFGY \\ & \searrow^{id_{GY}} & \downarrow G\epsilon_Y \\ & & GY \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FGFX \\ & \searrow^{id_{FX}} & \downarrow \epsilon_{FX} \\ & & FX \end{array}$$

commute for all $\mathcal{A}$-objects X and $\mathcal{B}$-objects Y. We call F *left adjoint* to G and G *right adjoint* to F. An adjoint situation is called an *equivalence* of $\mathcal{A}$ and $\mathcal{B}$ if both η and ϵ are natural isomorphims. A *duality* is an equivalence between $\mathcal{A}$ and $\mathcal{B}^{\mathrm{op}}$.

Example The Stone-Czech compactification βX for a topological space X is a compact, Hausdorff space and a continuous map $X \xrightarrow{\iota_X} \beta X$ with the following universal property: Given any continuous map $X \xrightarrow{f} K$ with K compact, there exists a unique continuous map f' s.t.

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \beta X \\ & \searrow^{f} & \downarrow f' \\ & & K \end{array}$$

commutes. Assuming the axiom of choice, one can prove that such a compactification always exists. [14]

Let U be the inclusion functor of **CHaus** into **Top**. Using the universal property, we can lift any morphism $X \xrightarrow{f} Y$ in **Top** to a morphism $\beta X \xrightarrow{\beta f} \beta Y$, which makes β a functor, and $id_{\mathbf{Top}} \xrightarrow{\iota} U\beta$ a natural transformation. Since for every compact, Hausdorff space K, βK is uniquely isomorphic to it (simply lift id_K), we also get a natural equivalence $\beta U \to id_{\mathbf{CHaus}}$. We leave it to the reader to check that the triangle identities hold. Altogether, this gives $\beta \dashv U$ with unit ι. Stated in different terms, **CHaus** is a *reflexive* subcategory of **Top**.

Example For every unital commutative C^*-algebra A its Gelfand space M_A, consisting of all unital C^*-Alg-morphisms $A \xrightarrow{m} \mathbb{C}$, $m \neq 0$, is a compact, Hausdorff space in the w^*-topology. Any C^*-Alg-morphism $A \xrightarrow{f} B$ gives a continuous map $M_B \xrightarrow{M_f} M_A$

[4] also called triangle or zigzag identities, the latter refers to their picture when written as string diagrams. We refer the reader to [3] for an extremely lucid introductory account.

by precomposition, $\mathbf{CommC^*} - \mathbf{Alg} \xrightarrow{M} \mathbf{CHaus}$ is thus a contravariant functor. The canonical injection $a \mapsto (m \mapsto m(a))$ turns out to give an isomorphism $A \xrightarrow{\eta_A} C(M_A, \mathbb{C})$. A bit more difficult to prove is that for any space X, $M_{C(X,\mathbb{C})}$ is already isomorphic to the Stone-Czech compactification of X, with ι_X given by the evaluation map $x \mapsto (f \mapsto f(x))$ (note the formal similarity in both cases). Since $C(-, \mathbb{C})$ is also a contravariant functor, we get a duality between $\mathbf{CommC^*} - \mathbf{Alg}$ and $\mathbf{CHaus}$, given by $M \dashv C(-, \mathbb{C})$. The correspondence can be further extended to include non-unital commutative C^*-algebras and locally compact Hausdorff spaces, but care must be given to restrict the morphisms to proper continuous maps and non-degenerate *-homomorphisms accordingly. [17]

Remark An adjoint situation between two posets is also called a *Galois connection.*

3 Locales and Frames

Let us consider a topology $(X, \mathcal{T})$ and its lattice $\Omega(X)$ of open sets, with set inclusion $\subseteq$ as its relation $\leq$. Since both $\emptyset$ as well as X are elements of $\mathcal{T}$, $\Omega(X)$ is bounded. Further, it is complete, since for any collection of open sets U_i the union $\bigcup U_i$ is their supremum $\coprod U_i$ and $(\bigcap U_i)^\circ$ is their infimum $\prod U_i$. We also have an extended law of distributivity

$$(\coprod U_i) \times O = \coprod (U_i \times O) \tag{3}$$

since $\bigcup U_i \cap O$ is already open. Note that for the dual version in general we only have the canonical inclusion $(\prod U_i) + O \subseteq \prod (U_i + O)$, not equality.

Furthermore, if we have a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{O})$ we have an induced map $\Omega(f) : \mathcal{O} \to \mathcal{T}$ by the rule $\Omega(f)(U) := f^{-1}[U]$. Since forming pre-images is stable under arbitrary unions and intersections, and the arbitrary union and finite intersection of open sets are open, we get a bounded lattice homomorphism that preserves arbitrary joins.

We will use those properties as the basic ingredients for our theory:

Definition A *frame* is complete lattice L such that equation 3 is true for all L-objects U_i and O. A *frame homomorphism* is a bounded lattice homomorphism that preserves arbitrary joins. We denote the resulting category of frames and frame homomorphisms by **Frm**. The category of locales **Loc** is defined as the dual to **Frm**, i.e. $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$.

The relationship between locales and frames is rather trivial, every locale is a frame and vice versa, just the direction of the mappings is reversed. The reason for this distinction is to mimic the behaviour of **Top**, as we just saw that the corresponding frame homomorphism to a continues map goes the opposite direction. The general motto "Algebra is dual to topology" applies again.

3.1 The functor Ω

Now, continuing on the previous remarks, every topological space can be considered a frame, and every continuous map can be considered a (in the opposite direction) frame homomorphism, as we just saw. Forming inverse images reverses composition. We, therefore, have a (contravariant) functor Ω : **Top** $\to$ **Frm** [or, equivalently, a covariant functor Lc : **Top** $\to$ **Loc**]. Note that this is not an embedding, since, for example, if we choose the indiscrete topology on two sets of different cardinality, the resulting spaces cannot be homeomorphic, yet will give the same locale. Let us summarize:

$$\Omega : \mathbf{Top} \to \mathbf{Frm} \tag{4}$$

$$(X, \mathcal{T}) \mapsto \Omega(X) := (T, \subseteq) \tag{5}$$

$$f : X \to Y \mapsto \Omega(Y) \xrightarrow{\Omega(f)} \Omega(X),\ \Omega(f)[U] := f^{-1}[U] \tag{6}$$

3.2 The functor pt and the natural transformation Σ

There are two (equivalent) ways to define points in a frame. Since we cannot talk about "elements" of a frame, it is useful to consider the following: A point in a topological space X is the same as a map $(\{\bullet\}, \{\emptyset, \{\bullet\}\}) \xrightarrow{x} X$. Now the topology of $\{\bullet\}$ corresponds to the lattice $\mathbf{2} := (0 \leq 1)$. Dualizing, we can define a *point* in a frame L as a map $L \xrightarrow{x} \mathbf{2}$ [alternatively seen as a localic map $P \to L$, with $P = \mathbf{2}$].

The other way to characterize a point is motivated like this: We can associate to every point $x \in X$ in a topological space its neighbourhood filter $\mathcal{U}(x)$ consisting of all open sets U with $x \in U$. $\mathcal{U}(x)$ has the following properties:

1. $\emptyset \notin \mathcal{U}(x)$
2. if $U_1, U_2 \in \mathcal{U}(x)$, then $U_1 \cap U_2 \in \mathcal{U}(x)$,
3. if $U_1 \in \mathcal{U}(x)$ and $U_1 \leq U_2$ then $U_2 \in \mathcal{U}(x)$,
4. if $\bigcup U_i \in \mathcal{U}(x)$, then $\exists i : U_i \in \mathcal{U}(x)$.

Definition A subset $F \subset L$ of a frame L is called *completely prime filter*, short *c.p.filter*, if

1. $0 \notin F$, and F is not empty,
2. if $a, b \in F$, then $a \times b \in F$,
3. if $a \in F$ and $a \leq b$ then $b \in F$,
4. if $\coprod a_i \in F$, then $\exists i : a_i \in F$.

We will refer to these properties in the following as (1), (2), (3) and (4).

Proposition 3.1. *(Frame homomorphisms reflect c.p.filter) If* $L \xrightarrow{f} M$ *is a frame homomorphism and* $F \subset M$ *a c.p.filter , then* $f^{-1}[F]$ *is also a c.p.filter.*

Proof.

1. f preserves initial objects and F obeys (1),
2. f preserves finite meets and F obeys (2),

3. f is monotone and F obeys (3),

4. f preserves arbitrary joins and F obeys (4).

The details are left to the reader. □

Our two notions of points are equivalent:

Proposition 3.2. *For every frame homomorphism* $L \xrightarrow{x} 2$ *the subset* $F = x^{-1}\{1\}$ *is a c.p.filter. Conversely, given a c.p.filter* F*, there is exactly one frame homomorphism* $L \xrightarrow{x} 2$ *with* $F = x^{-1}\{1\}$ *.*

Proof. The first part is a special case of 3.1. Given any c.p.filter F, define a map $L \xrightarrow{x} 2$ by

$$x(a) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases} \tag{7}$$

(3) guarantees that this is a monotone map, (1) and (2) provide that the map preserves finite infima, and (4) provides that it preservers arbitrary suprema. □

We are now ready to define the set of points of a locale L:

$$pt(L) := \{\text{c.p.filter in } L\} \tag{8}$$

We want to introduce a compatible topology on $pt(L)$. Consider the function

$$\Sigma_L \quad : \quad L \to P(pt(L)) \tag{9}$$

$$\Sigma_L(a) \quad := \quad \{ \text{ F is a c.p.filter in } L \mid a \in F\} \tag{10}$$

Proposition 3.3. *The following properties hold for* Σ

1. $\Sigma_L(0) = \emptyset; \quad \Sigma_L(1) = pt(L)$,

2. $\Sigma_L(a \times b) = \Sigma_L(a) \cap \Sigma_L(b)$,

3. $\Sigma_L(\coprod a_i) = \bigcup \Sigma_L(a_i)$.

Proof.

1. No c.p.filter contains 0, and since no c.p.filter is empty, 1 is element in each c.p.filter.

2. If a c.p.filter contains both a and b then it contains $a \times b$ by (2). On the other hand, if $a \times b$ is element of a c.p.filter, then so is a and b by (3).

3. If at least one a_i is in a c.p.filter, then so is $\coprod a_i$ by (3). On the other hand, if $\coprod a_i$ is in a c.p.filter, then by (4) there's at least one a_i also in it.

□

We therefore see that $(pt(L), \{\Sigma_L(a) | a \in L\})$ is a topological space.[5]

If $L \xrightarrow{f} M$ is a frame homomorphism, then $pt(f)[F] := f^{-1}[F]$ is a well-defined map $pt(M) \xrightarrow{pt(f)} pt(L)$ by 3.1. We will show it is continuous as well:

[5] This construction mimics the Zariski-topology on the spectrum of a ring. The set $pt(L)$ is sometimes also called spectrum of the lattice L.

Proposition 3.4. *For a frame homomorphism* $L \xrightarrow{f} M$, $pt(f)^{-1}[\Sigma_L(a)] = \Sigma_M(f(a))$.

Proof.

$$pt(f)^{-1}[\Sigma_L(a)] = \{F \mid pt(f)[F] \in \Sigma_L(a)\} = \left\{F \mid f^{-1}[F] \in \Sigma_L(a)\right\} = \tag{11}$$
$$\left\{F \mid a \in f^{-1}[F]\right\} = \{F \mid f(a) \in F\} = \Sigma_M(f(a)) \tag{12}$$

□

Since forming preimages reverses composition, we now know that pt is a contravarient functor:

$$pt : \mathbf{Frm} \rightarrow \mathbf{Top} \tag{13}$$
$$L \mapsto pt(L) \tag{14}$$
$$f : L \rightarrow M \mapsto pt(M) \xrightarrow{pt(f)} pt(L),\ pt(f)[F] := f^{-1}[F] \tag{15}$$

The transformation Σ, which sends the lattice L to the mapping $L \xrightarrow{\Sigma_L} \Omega \circ pt(L)$ has an important role as well. Proposition 3.3 not only showed that $pt(L)$ is a topological space, it also showed that Σ_L is a frame homomorphism. 3.4 not only showed that for $L \xrightarrow{f} M$, $pt(f)$ is continuous, it also showed the equation:

$$(\Omega \circ pt(f)) \circ \Sigma_L(a) = \Sigma_M \circ f(a) \tag{16}$$

Σ is therefore a natural transformation:

$$id_{\mathbf{Frm}} \xrightarrow{\Sigma} \Omega \circ pt. \tag{17}$$

3.3 The neighbourhood-filter U as a natural transformation

We already observed in the previous section that for a topological space $(X, \mathcal{T})$ and an element $x \in X$ the neighbourhood filter $U_X(x)$ is a c.p.filter in $\mathcal{T}$.

Proposition 3.5. $U_X : X \rightarrow pt(\Omega(X))$ *is continuous.*

Proof. Let $U \subset X$ be open.

$$U_X^{-1}[\Sigma_U] = \{x \in X \mid U_X(x) \in \Sigma_U\} = \{x \in X \mid U \in U_X(x)\} = U. \tag{18}$$

□

Proposition 3.6. $id_{\mathbf{Top}} \xrightarrow{U} \Omega \circ pt$ *is a natural transformation.*

Proof. Let $X \xrightarrow{f} Y$ be a continuous map, then:

$$pt\Omega(f) \circ U_X(x) = \Omega(f)^{-1}[U_X(x)] = \{O \text{ open in } Y \mid f^{-1}[O] \in U_X(x)\} = \tag{19}$$
$$= \{O \text{ open in } Y \mid f(x) \in O\} = U_Y(f(x)). \tag{20}$$

□

3.4 The main theorem

By going to the category **Loc**, we can interpret Ω and pt as *co*variant functors. We have to remember to reverse the direction of $\Sigma : \Omega pt \to id_{\mathbf{Loc}}$ as well.

Theorem 3.7. *Ω and pt form an adjoint situation between **Top** and **Loc** $=$ **Frm**^{op}, $\Omega \dashv pt$ with unit U and counit Σ.*

Proof. We still have to show the counit and unit triangles.
1. Let L be a frame and F a c.p.filter/point in L:

$$pt(\Sigma_L)(U_{pt(L)}(F)) = \Sigma_L^{-1}[U_{pt(L)}(F)] = \big\{a \in L \mid \Sigma_L(a) \in U_{pt(L)}(F)\big\} = \tag{21}$$

$$= \{a \in L \mid F \in \Sigma_L(a)\} = \{a \in L \mid a \in F\} = F \tag{22}$$

2. Let X be a topological space, and O an open subset of X:

$$\Omega(U_X)(\Sigma_{\Omega(X)}(O)) = U_X^{-1}[\Sigma_{\Omega(X)}(O)] = \big\{x \in X | U_X(x) \in \Sigma_{\Omega(X)}(O)\big\} = \tag{23}$$

$$= \{x \in X | O \in U_X(x)\} = \{x \in X | x \in O\} = O \tag{24}$$

□

Without showing the proof here, using this adjunction, it's not hard to show that a locale comes from a topological space (i.e. the locale is *spatial*) iff it is isomorphic with the locale coming from it's space of points. One can restrict the adjunction to the subcategories of sober spaces and spatial locales to obtain an equivalence. [15] Further restrictions give the other known stone dualities, including Stone's representation theorem.

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