Schauder Bases

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Introduction

Due to the restriction to (finite) linear combinations classical vector space bases are not always suitable for the analysis of infinite dimensional spaces. Therefore, it is natural in some way to consider generalized basis concepts.

Note, that all vector spaces in this paper are spaces over the field $\mathbb{F}$, where $\mathbb{F}$ denotes the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Whenever reference is made to some topological property, the norm topology is implied.

Definition 0.1 A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space $(X, \|.\|_X)$ is a called a *Schauder basis* of X if for every $x \in X$ there exists a unique sequence $(\alpha_n)_{n\in\mathbb{N}}$ of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$, i.e. such that $\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} \alpha_n x_n \right\|_X = 0$.

Throughout this paper we make the convention that a basis for a Banach space shall be a Schauder basis, unless explicit reference is made to a vector space basis.

Propostion 0.2 *Suppose that $(x_n)_{n\in\mathbb{N}}$ is a basis for a Banach space X. Then $(x_n)_{n\in\mathbb{N}}$ is linearly independent. In particular, every banach space with a basis is infinite dimensional.*

Proof: Suppose that an element x of X could be written in two different ways as a finite linear combination of the terms of $(x_n)_{n\in\mathbb{N}}$, i.e. for $n, m \in \mathbb{N}$, $(\alpha_i)_{i=1}^n \in \mathbb{F}^n$, $(\beta_i)_{i=1}^m \in \mathbb{F}^m$ satisfying $\alpha_n \neq 0 \neq \beta_m$ and $(\alpha)_{i=1}^n \neq (\beta_i)_{i=1}^m$ we had $x = \sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^m \beta_i x_i$. Then we had $\sum_{i=1}^{\infty} \tilde{\alpha}_i x_i = \sum_{i=1}^{\infty} \tilde{\beta}_i x_i$, for $(\tilde{\alpha}_i)_{i\in\mathbb{N}} := (\alpha_1, \dots, \alpha_n, 0, 0, \dots)$ and $(\tilde{\beta}_i)_{i\in\mathbb{N}} := (\beta_1, \dots, \beta_m, 0, 0, \dots)$. This is a contradiction to the uniqueness of expansions of vectors in terms of a basis in X.

$\square$

1 Coordinate Functionals

Our first aim is to show that the continuity of the coordinate functionals - a property that has been required in Schauder's original defintion - follows from the rest of Definition 0.1.

Definition 1.1 Let X be a Banach space with basis $(x_n)_{n\in\mathbb{N}}$. For each $m \in \mathbb{N}$ the maps $x_m^* : X \to \mathbb{F} : \sum_{n=1}^{\infty} \alpha_n x_n \mapsto \alpha_m$ and $P_m : X \to X : \sum_{n=1}^{\infty} \alpha_n x_n \mapsto \sum_{n=1}^{m} \alpha_n x_n$ are called the m^{th} *coordinate functional* and the m^{th} *natural projection* associated with $(x_n)_{n\in\mathbb{N}}$, respectively.

Remark 1.2 Due to the uniqueness of expansions of each vector in terms of a basis required in definition 0.1 it is instantly verified, that the coordinate functionals actually are linear and the natural projections are projections.

For the sake of convenience we will not work with the original norm of the underlying Banach space, but with the following one.

Lemma 1.3 *Let* $(X, \|.\|_X)$ *be a Banach space with basis* $(x_n)_{n\in\mathbb{N}}$*. Then the norm* $\|.\|$ *defined by the formula* $\|\sum_{n=1}^{\infty} \alpha_n x_n\| = \sup_{m\in\mathbb{N}} \|\sum_{n=1}^{m} \alpha_n x_n\|_X$ *is a Banach space norm equivalent to the norm* $\|.\|_X$ *(i.e. they induce the same topology) satisfying* $\|x\| \geq \|x\|_X$ *for all* $x \in X$*.*

Proof: We prove this lemma in four steps. In step one and step two we show, that $\|.\|$ actually is a norm and the claimed inequality, respectively. In step three we find a limit in X for a random Cauchy sequence with respect to $\|.\|$ in X. Finally, in step four we complete the proof by using the open mapping theorem to show the equivalence of the two norms.

Note that in this proof convergence of series in X is *always* meant to be with respect to $\|.\|_X$.

1. Since the other two requirements of the definition of a norm follow instantly from the fact that $\|.\|_X$ is a norm, we will confine ourselves to showing that the triangle inequality holds for $\|.\|$. For this purpose pick two vectors x and y in X having the expansions $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and $y = \sum_{n=1}^{\infty} \tilde{\alpha}_n x_n$. Then by the linearity of P_m

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \alpha_n x_n + \sum_{n=1}^{\infty} \tilde{\alpha}_n x_n \right\| &= \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \alpha_n x_n + \sum_{n=1}^{m} \tilde{\alpha}_n x_n \right\|_X \\ \leq \sup_{m\in\mathbb{N}} \left(\left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X + \left\| \sum_{n=1}^{m} \tilde{\alpha}_n x_n \right\|_X \right) & \\ \leq \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X + \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \tilde{\alpha}_n x_n \right\|_X & \\ = \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| + \left\| \sum_{n=1}^{\infty} \tilde{\alpha}_n x_n \right\| &= \|x\| + \|y\| \end{aligned}$$

2. For a vector x in X with the expansion $x = \sum_{n=1}^{\infty} \alpha_n x_n$ we obtain from the continuity of the norm $\|.\|_X$

$$\|x\| = \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| = \sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X \geq \lim_{n\to\infty} \left\| \sum_{n=1}^{m} \alpha_n x_n \right\|_X = \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|_X .$$

3. We want to show next, that an arbitrary Cauchy sequence $(b_i)_{i\in\mathbb{N}} = (\sum_{n=1}^{\infty} \beta_{n,i} x_n)_{i\in\mathbb{N}}$ with respect to $\|.\|$ in X converges towards $\sum_{n=1}^{\infty} \beta_n x_n$ in X, for $\beta_n := \lim_{i\to\infty} \beta_{n,i}$, $n \in \mathbb{N}$. In order to see that the sequence $(\beta_{n,i})_{i\in\mathbb{N}}$ is Cauchy and hence convergent in $\mathbb{F}$ for each $n \in \mathbb{N}$ let j, k, n be in $\mathbb{N}$ and $n \geq 2$. Then we have

$$\begin{aligned} |\beta_{1,j} - \beta_{1,k}| \, \|x_1\|_X &= \|(\beta_{1,j} - \beta_{1,k}) \cdot x_1\|_X \\ \leq \sup_{m\in\mathbb{N}} \left\| \sum_{l=1}^{m} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\|_X &= \left\| \sum_{l=1}^{\infty} (\beta_{l,j} - \beta_{l,k}) \cdot x_l \right\| \end{aligned}$$

and

$$\begin{aligned}
|\beta_{n,j}-\beta_{n,k}|\,\|x_n\|_X &= \|(\beta_{n,j}-\beta_{n,k})\cdot x_n\|_X \\
&= \left\|\sum_{l=1}^{n}(\beta_{l,j}-\beta_{l,k})\cdot x_l - \sum_{l=1}^{n-1}(\beta_{l,j}-\beta_{l,k})\cdot x_l\right\|_X \\
&\leq \left\|\sum_{l=1}^{n}(\beta_{l,j}-\beta_{l,k})\cdot x_l\right\|_X + \left\|\sum_{l=1}^{n-1}(\beta_{l,j}-\beta_{l,k})\cdot x_l\right\|_X \\
&\leq \sup_{m\in\mathbb{N}}\left\|\sum_{l=1}^{m}(\beta_{l,j}-\beta_{l,k})\cdot x_l\right\|_X + \sup_{m\in\mathbb{N}}\left\|\sum_{l=1}^{m}(\beta_{l,j}-\beta_{l,k})\cdot x_l\right\|_X \\
&= 2\cdot\left\|\sum_{l=1}^{\infty}(\beta_{l,j}-\beta_{l,k})\cdot x_l\right\|.
\end{aligned}$$

As $(\sum_{n=1}^{\infty}\beta_{n,i}x_n)_{i\in\mathbb{N}}$ is Cauchy so must be $(\beta_{n,i})_{i\in\mathbb{N}}$ for each $n\in\mathbb{N}$. Thus the sequence $(\beta_n)_{n\in\mathbb{N}}$ is well-defined.

Since $(\sum_{n=1}^{\infty}\beta_{n,i}x_n)_{i\in\mathbb{N}}$ is Cauchy with respect to $\|.\|$ we may choose $i(\epsilon)\in\mathbb{N}$ for each fixed $\epsilon>0$ such that for $i,j,M\in\mathbb{N},\ i,j\geq i(\epsilon)$

$$\begin{aligned}
\frac{\epsilon}{3} &> \left\|\sum_{n=1}^{\infty}\beta_{n,j}x_n - \sum_{n=1}^{\infty}\beta_{n,i}x_n\right\| = \sup_{m\in\mathbb{N}}\left\|\sum_{n=1}^{m}\beta_{n,j}x_n - \sum_{n=1}^{m}\beta_{n,i}x_n\right\|_X \\
&\geq \left\|\sum_{n=1}^{M}\beta_{n,j}x_n - \sum_{n=1}^{M}\beta_{n,i}x_n\right\|_X
\end{aligned}$$

The inequality holds true for $j\to\infty$. This way we obtain for all $i\geq i(\epsilon)$

$$\left\|\sum_{n=1}^{M}\beta_n x_n - \sum_{n=1}^{M}\beta_{n,i}x_n\right\|_X \leq \frac{\epsilon}{3}. \tag{1}$$

We will complete the proof of this step by showing, that $(\sum_{n=1}^{\infty}\beta_{n,i}x_n)_{i\in\mathbb{N}}$ converges towards $\sum_{n=1}^{\infty}\beta_n x_n$. Therefore, it is necessary to show, that $\sum_{n=1}^{\infty}\beta_n x_n$ exists. Due to the completeness of $(X,\|.\|_X)$ it is sufficient to proof that $\sum_{n=1}^{\infty}\beta_n x_n$ is Cauchy with respect to $\|.\|_X$.

Suppose that $m_1,m_2\in\mathbb{N},\ m_2\geq m_1>1$. Using (1) we have

$$\left\|\sum_{n=m_1}^{m_2}\beta_n x_n - \sum_{n=m_1}^{m_2}\beta_{n,i}x_n\right\|_X \tag{2}$$

$$\begin{aligned}
&= \left\|\sum_{n=m_1}^{m_2}\beta_n x_n - \sum_{n=m_1}^{m_2}\beta_{n,i}x_n \pm \sum_{n=1}^{m_1-1}\beta_n x_n \pm \sum_{n=1}^{m_1-1}\beta_{n,i}x_n\right\|_X \\
&= \left\|\sum_{n=1}^{m_2}\beta_n x_n - \sum_{n=1}^{m_2}\beta_{n,i}x_n - \sum_{n=1}^{m_1-1}\beta_n x_n + \sum_{n=1}^{m_1-1}\beta_{n,i}x_n\right\|_X \\
&\leq \left\|\sum_{n=1}^{m_2}\beta_n x_n - \sum_{n=1}^{m_2}\beta_{n,i}x_n\right\|_X + \left\|\sum_{n=1}^{m_1-1}\beta_n x_n - \sum_{n=1}^{m_1-1}\beta_{n,i}x_n\right\|_X \\
&\leq \frac{\epsilon}{3}+\frac{\epsilon}{3} = \frac{2\epsilon}{3}.
\end{aligned} \tag{3}$$

As the series $\sum_{n=1}^{\infty}\beta_{n,i}x_n$ is the expansion of a vector in X in terms of $(x_n)_{n\in\mathbb{N}}$, it must be convergent and so must be $\left(\left\|\sum_{n=1}^{N}\beta_{n,i}x_n\right\|_X\right)_{N\in\mathbb{N}}$. Therefore we can choose $m(\epsilon)\in\mathbb{N}$ such that

$$\left\|\sum_{n=m_1}^{m_2}\beta_{n,i}x_n\right\|_X < \frac{\epsilon}{3} \tag{4}$$

for $m_2, m_1 \in \mathbb{N},\ m_2 \geq m_1 > m(\epsilon)$. Finally (3) and (4) give us the Cauchy criterion for our series $\sum_{n=1}^{\infty}\beta_n x_n$:

$$\begin{aligned}\left\|\sum_{n=m_1}^{m_2}\beta_n x_n\right\|_X &= \left\|\sum_{n=m_1}^{m_2}\beta_n x_n \pm \sum_{n=m_1}^{m_2}\beta_{n,i}x_n\right\|_X \\ \leq \left\|\sum_{n=m_1}^{m_2}\beta_n x_n - \sum_{n=m_1}^{m_2}\beta_{n,i}x_n\right\|_X &+ \left\|\sum_{n=m_1}^{m_2}\beta_{n,i}x_n\right\|_X \leq \frac{2\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.\end{aligned}$$

and the convergence of $\sum_{n=1}^{\infty}\beta_n x_n$ is proven.

Since inequality (1) still holds when taking the supremum over all $M\in\mathbb{N}$ we see

$$\left\|\sum_{n=1}^{\infty}\beta_n x_n - \sum_{n=1}^{\infty}\beta_{n,i}x_n\right\| = \sup_{M\in\mathbb{N}}\left\|\sum_{n=1}^{M}\beta_n x_n - \sum_{n=1}^{M}\beta_{n,i}x_n\right\|_X \leq \frac{\epsilon}{3},$$

which completes the proof of this step.

4. The identity map $I:(X,\|.\|)\to(X,\|.\|_X)$ is a bijective, linear and due to the inequality, proofed in step 2, continuous operator. The open mapping theorem[1] ensures, that $I^{-1}:(X,\|.\|_X)\to(X,\|.\|)$ is continuous too.

□

Theorem 1.4 *Let $(X,\|.\|_X)$ be a Banach space with basis $(x_n)_{n\in\mathbb{N}}$. Then all natural projections and all coordinate functionals associated with $(x_n)_{n\in\mathbb{N}}$ are continuous.*

Proof: Fix m in $\mathbb{N}$ and a member of X having the expansion $\sum_{n=1}^{\infty}\alpha_n x_n$. Define a sequence $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ by

$$\tilde{\alpha}_n = \begin{cases}\alpha_n, & n\leq m\\ 0, & \text{else.}\end{cases}$$

Then the unique expansion of $P_m\left(\sum_{n=1}^{\infty}\alpha_n x_n\right)$ in terms of $(x_n)_{n\in\mathbb{N}}$ is given by $\sum_{n=1}^{\infty}\tilde{\alpha}_n x_n$. Now the continuity of the natural projection P_m for $(x_n)_{n\in\mathbb{N}}$ follows from

$$\begin{aligned}\left\|P_m\left(\sum_{n=1}^{\infty}\alpha_n x_n\right)\right\| &= \left\|\sum_{n=1}^{\infty}\tilde{\alpha}_n x_n\right\| = \sup_{M\in\mathbb{N}}\left\|\sum_{n=1}^{M}\tilde{\alpha}_n x_n\right\|_X = \sup_{M=1,\ldots,m}\left\|\sum_{n=1}^{M}\tilde{\alpha}_n x_n\right\|_X \\ &= \sup_{M=1,\ldots,m}\left\|\sum_{n=1}^{M}\alpha_n x_n\right\|_X \leq \sup_{M\in\mathbb{N}}\left\|\sum_{n=1}^{M}\alpha_n x_n\right\|_X = \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|.\end{aligned}$$

[1] see e.g. [1] theorem 12.1 and 12.5

For each $m > 1$ the coordinate functional x_m^* associated with $(x_n)_{n\in\mathbb{N}}$ is continuous as it is the composition of continuous maps

$$\sum_{n=1}^{\infty} \alpha_n x_n \mapsto (P_m - P_{m-1})\left(\sum_{n=1}^{\infty} \alpha_n x_n\right) = \alpha_m x_m \mapsto \alpha_m$$

and so it is for $m = 1$

$$\sum_{n=1}^{\infty} \alpha_n x_n \mapsto P_1\left(\sum_{n=1}^{\infty} \alpha_n x_n\right) = \alpha_1 x_1 \mapsto \alpha_1.$$

□

2 Banach's Basis Problem

In the following proposition we will see, that every Banach Space having a basis is separable. The question whether the converse is true, i.e. whether every infinite dimensional separable Banach space has a basis, is known as the classical *basis problem* for Banach spaces. It remained open for forty years until Per Enflo found a counterexample in 1973.

Propostion 2.1 *Every Banach space X with a basis $(x_n)_{n\in\mathbb{N}}$ is separable.*

Proof: We want to show, that the countable set

$$A := \left\{\sum_{n=1}^{m} \alpha_n x_n : \alpha_1, \dots, \alpha_n \in \tilde{\mathbb{Q}}, m \in N\right\},$$

where $\tilde{\mathbb{Q}}$ denotes the rational numbers or respectively the complex numbers with rational real and imaginary part, is dense in X. Since an arbitrary element of X having the expansion $\sum_{n=1}^{\infty} \alpha_n x_n$ in terms of $(x_n)_{n\in\mathbb{N}}$ can be written as $\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n$, it follows that $X = \overline{\text{span}\left(\{x_n : n \in \mathbb{N}\}\right)}$. Therefore, it suffices to show that A is a dense subset of $\text{span}(\{x_n : n \in \mathbb{N}\})$.

Fix $k \in \mathbb{N}$, $x_1, \dots, x_k \in \{x_n : n \in \mathbb{N}\}$ and $\alpha_1, \dots, \alpha_k \in \mathbb{F}$. Due to the fact, that $\tilde{\mathbb{Q}}$ is dense in $\mathbb{F}$, there is a sequence $(\alpha_{j,i})_{i\in\mathbb{N}} \in \tilde{\mathbb{Q}}^n$ converging towards α_j for each $j = 1, \dots, k$. From the continuity of the vector space operations it follows, that

$$\lim_{i\to\infty} \sum_{j=1}^{k} \alpha_{j,i} x_j = \sum_{j=1}^{k} \alpha_j x_j,$$

i.e. an arbitrary element of $\text{span}(\{x_n : n \in \mathbb{N}\})$ can be written as the limit of a sequence in A, which completes the proof of this proposition.

□

In fact Enflo showed how to construct a separable infinite dimensional Banach space lacking the *approximation property*. Therefore, it is of some interest, that every Banach space with a basis has the approximation property.

Definition 2.2 *Let X be a Banach space. X has the* approximation property *if the subset of operators of $B(Y, X)$ having finite rank is dense in $K(Y, X)$ for each Banach space Y, where $B(Y, X)$ and $K(Y, X)$ denote the set of all bounded linear operators and the set of all compact operators from Y into X, respectively.*

The following lemma gives us a sufficient condition for a Banach space to have the approximation property.

Lemma 2.3 *Suppose that $(X, \|.\|_X)$ is Banach space and that for each compact set $C \subset X$ and each $\epsilon > 0$ there exists a bounded linear Operator $A_{C,\epsilon}$ from X into X having finite rank such that $\|A_{C,\epsilon}x - x\|_X < \epsilon$ for each x in C. Then X has the approximation property.*

Proof: Let A be a compact operator from Y into X for an arbitrary Banach space Y. Since $\overline{A(U)}$ is a compact subset of X, when $\overline{U}$ denotes the closed unit ball in Y, there is a sequence $(B_n)_{n\in\mathbb{N}}$ of bounded linear operators from X into X satisfying $\|B_n x - x\|_X < \frac{1}{n}$ for each n in $\mathbb{N}$ and each x in $A(\overline{U}) \subset \overline{A(\overline{U})}$. For each n in $\mathbb{N}$ we have

$$\|B_n A - A\| = \sup\{\|B_n Ay - Ay\|_X : y \in Y, \|y\| \leq 1\} = \sup\left\{\|B_n x - x\|_X : x \in A\left(\overline{U}\right)\right\} \leq \frac{1}{n},$$

which completes the proof of this lemma.

□

Theorem 2.4 *Let $(X, \|.\|_X)$ be a Banach space with basis $(x_n)_{n\in\mathbb{N}}$. Then X has the approximation property.*

Proof: Let C be a compact subset of X and $\epsilon > 0$. By the preceding lemma it is sufficient to show that there is an N in $\mathbb{N}$ such that $\|P_N x - x\|_X < \epsilon$ for each x in C.

It follows readily from the uniform boundedness principle[2] that $\Pi := \sup_{n\in\mathbb{N}} \|P_n\|$ is finite. Due to the compactness of C we can pick finitely many $y_1, \ldots, y_l$ in C such that $\min_{i=1,\ldots,l} \|x - y_i\|_X \leq \frac{\epsilon}{2(1+\Pi)}$ for each x in C.

Let x_0 be in C. Then there is $j \in \mathbb{N}$ such that $\|x_0 - y_j\|_X \leq \frac{\epsilon}{2(1+C)}$ and, since it follows from definition 0.1 that $\lim_{n\to\infty} \|y_j - P_n y_j\|_X = 0$, there is $N_\epsilon \in \mathbb{N}$ such that $\|y_j - P_n y_j\|_X \leq \frac{\epsilon}{2}$ for each $n \geq N_\epsilon$. We conclude

$$\begin{aligned}
\|P_n x_0 - x_0\|_X &= \|P_n x_0 - x_0 \pm y_j \pm P_n y_j\|_X \\
&\leq \|y_j - x_0\|_X + \underbrace{\|P_n y_j - P_n x_0\|_X}_{\leq \|P_n\| \|y_j - x_0\|_X \leq \Pi \|y_j - x_0\|_X} + \underbrace{\|y_j - P_n y_j\|_X}_{\frac{\epsilon}{2}} \\
&\leq (1+\Pi)\|y_j - x_0\|_X + \frac{\epsilon}{2} \leq (1+\Pi)\frac{\epsilon}{2(1+\Pi)} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

□

References

[1] John B. Conway, *A course in functional analysis*, Springer, New York, 2nd edition, 1990.

[2] Robert E. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 1998.

[3] Ivan Singer, *Bases in Banach Spaces I*, Springer, Berlin, 1970.

[2] see e.g. [1] theorem 14.1