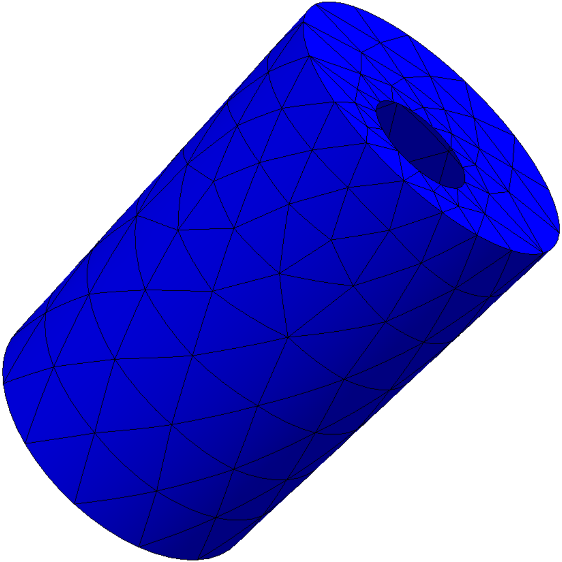
Nonlinear Shells in NGSolve

Michael Neunteufel
Institute of Analysis and Scientific Computing

NGSolve Seminar, June 18, 2020

Motivation



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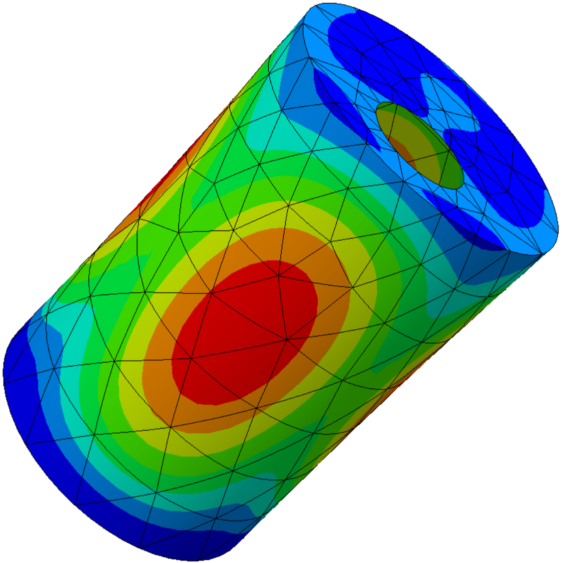

Motivation

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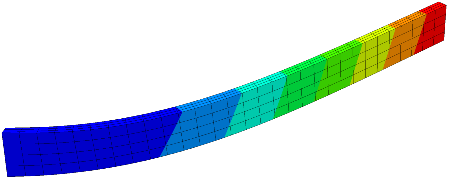
Notation

Deformation $\Phi : \Omega \to \mathbb{R}^3$



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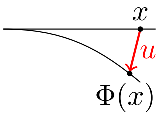
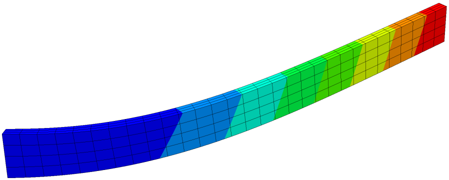
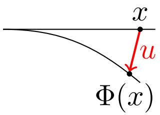
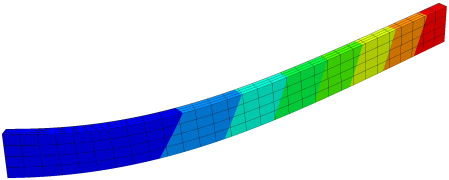
Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$





Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \nabla \Phi$





Deformation	$\Phi : \Omega \to \mathbb{R}^3$
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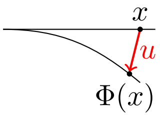
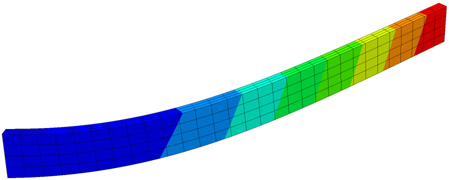
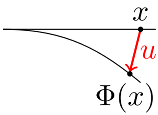




Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^\top \boldsymbol{F}$



$$\frac{||\Phi(x+\Delta x) - \Phi(x)||^2}{||\Delta x||^2} = \frac{\Delta x^\top F^\top F \Delta x}{||\Delta x||^2} + \mathcal{O}(||\Delta x||)$$

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
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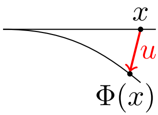
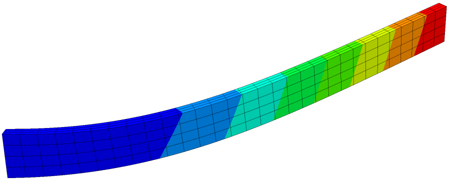
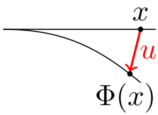
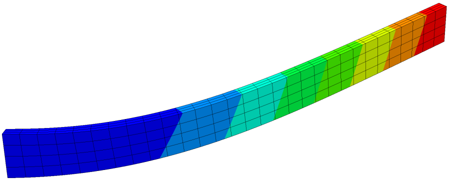




Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^\top \boldsymbol{F}$
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$
Linearized strain tensor	$\epsilon(u) := \frac{1}{2}(\nabla u^\top + \nabla u)$





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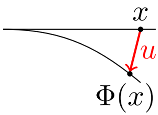
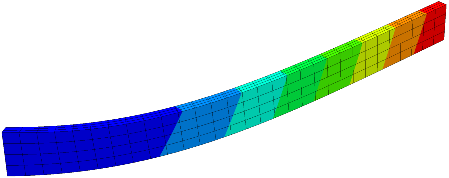
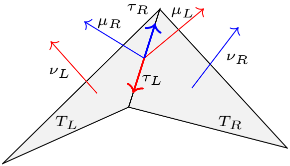


Elasticity

$$\mathcal{W}(u) = \frac{1}{2}\|\boldsymbol{E}\|^2_{\boldsymbol{M}} - \langle f, u\rangle$$



- Normal vector ν
  Tangent vector τ
  Element normal vector $\mu = \nu \times \tau$



- Normal vector $\hat{\nu}$

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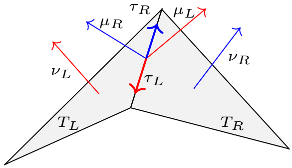

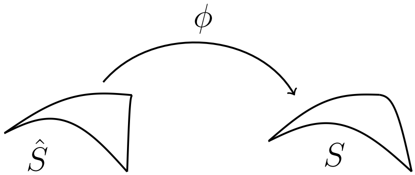

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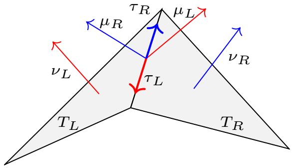
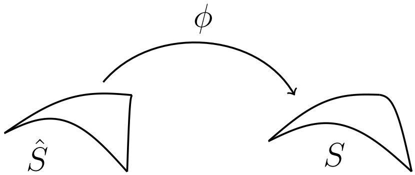


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$



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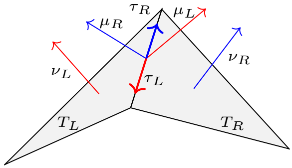
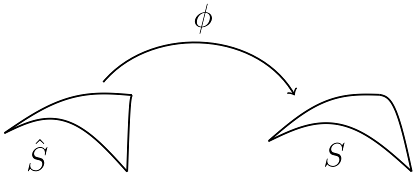


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



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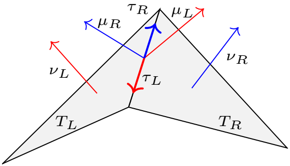
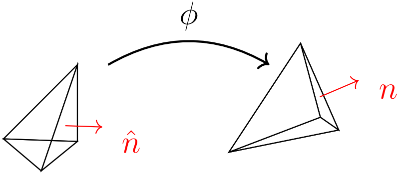


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi, \; J = \| \operatorname{cof}(\boldsymbol{F}) \|_F$
- $\nu \circ \phi = \frac{1}{J} \operatorname{cof}(\boldsymbol{F}) \hat{\nu}$

  $\tau \circ \phi = \frac{1}{J_B} \boldsymbol{F} \hat{\tau}$

  $\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi$



$$n \circ \phi = \frac{J \boldsymbol{F}^{-\top} \hat{n}}{\| J \boldsymbol{F}^{-\top} \hat{n} \|_2} = \frac{\operatorname{cof}(\boldsymbol{F}) \hat{n}}{\| \operatorname{cof}(\boldsymbol{F}) \hat{n} \|_2}$$

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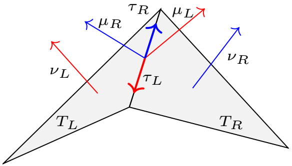

- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi,\ J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
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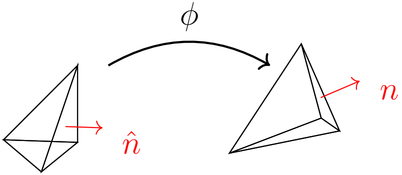

$$\boldsymbol{n} \circ \phi = \frac{J\boldsymbol{F}^{-\top}\hat{\boldsymbol{n}}}{\|J\boldsymbol{F}^{-\top}\hat{\boldsymbol{n}}\|_2} = \frac{\operatorname{cof}(\boldsymbol{F})\hat{\boldsymbol{n}}}{\|\operatorname{cof}(\boldsymbol{F})\|_F}$$

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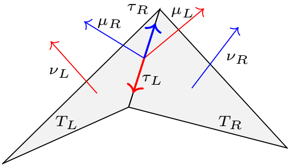

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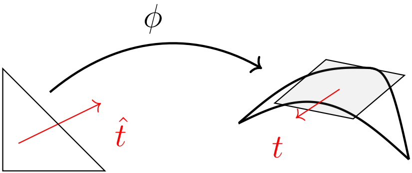

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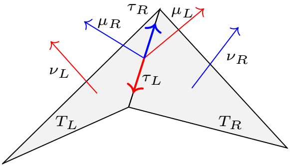

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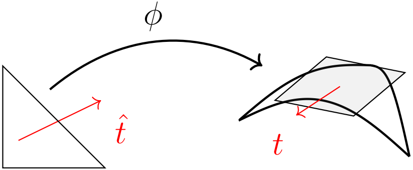
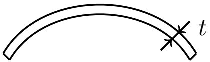
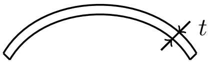
  $\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi$
  $= \frac{(\boldsymbol{F}^{\dagger})^{\top}\hat{\mu}}{\|(\boldsymbol{F}^{\dagger})^{\top}\hat{\mu}\|}$





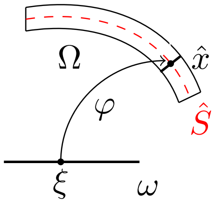
- Model of reduced dimensions

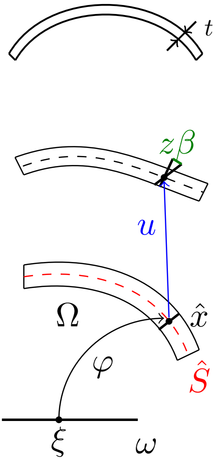
Thin-walled structures



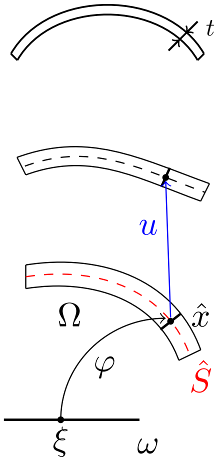
- Model of reduced dimensions

- $\Omega = \left\{ \varphi(\xi) + z\hat{\nu}(\xi) : \; \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2} \right] \right\}$





- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\ (\nu + \beta) \circ \phi(\hat{x})$



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- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\, \nu \circ \phi(\hat{x})$

Shell energy (Kirchhoff–Love)

$$\mathcal{W}(u) = \frac{t}{4}\|I - \overline{I}\|_{M}^{2} + \frac{t^3}{24}\|II - \overline{II}\|_{M}^{2}$$

C. Weischedel: A discrete geometric view on shear-deformable shell models, *PhD thesis, Universität Göttingen* (2012).

Shell energy (Kirchhoff–Love)

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$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

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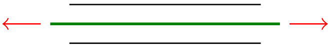
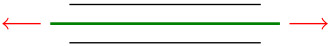
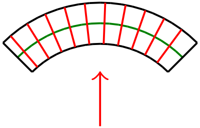


- Membrane energy

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|_{\boldsymbol{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\boldsymbol{M}}^2$$





- Membrane energy
- Bending energy

Shell energy (Kirchhoff–Love)

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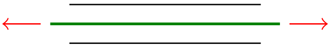
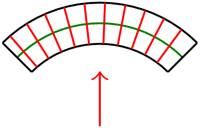
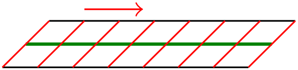




- Membrane energy
- Bending energy
- Shearing energy



Examples

First (linear) shells in NGSolve!

New NGSolve WebGUI!

Method and Shell Element

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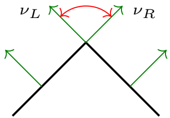
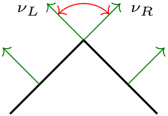
$$\mathcal{W}(u) := \|\nabla \nu\|_{\boldsymbol{M}}^2 = \sum_{T \in \mathcal{T}_h} \int_T \|\nabla \nu\|_{\boldsymbol{M}}^2 \, dx$$



Grinspun, Gingold, Reisman and Zorin: Computing discrete shape operators on general meshes, *Computer Graphics Forum 25*, 3 (2006), pp. 547–556.

$$\mathcal{W}(u) := \|\nabla \nu\|_{\boldsymbol{M}}^2 = \sum_{T\in\mathcal{T}_h} \int_T \|\nabla \nu\|_{\boldsymbol{M}}^2 \, dx$$



$$\langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathcal{T}} = \sum_{T\in\mathcal{T}_h} \int_T \nabla \nu|_T : \boldsymbol{\sigma} \, ds + \sum_{E\in\mathcal{E}_h} \int_E \sphericalangle(\nu_L, \nu_R) \boldsymbol{\sigma}_{\mu\mu} \, d\gamma$$

- Measure jump of normal vector

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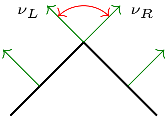

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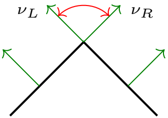
- Measure jump of normal vector

$$\Rightarrow \mathcal{L}(u, \boldsymbol{\sigma}) := -\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\nabla\nu, \boldsymbol{\sigma}\rangle_{\mathcal{T}}$$

- Obtain Lagrangian

Moment tensor

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



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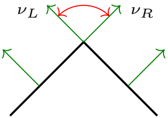

- Measure change of angles

Moment tensor

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$
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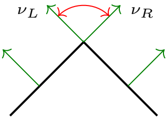

- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu},\boldsymbol{\sigma}\rangle$$
$$+ \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R),\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

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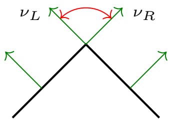

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- $\boldsymbol{\sigma}$ has physical meaning of moment

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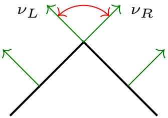

- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu},\boldsymbol{\sigma}\rangle + \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R),\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

- $\boldsymbol{\sigma}$ has physical meaning of moment
- Fourth order problem $\to$ second order problem

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Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u, \boldsymbol{\sigma}) - \langle f, u\rangle,$$

with

$$G(u, \boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1 - \hat{\nu}\cdot\nu)\nabla\hat{\nu})\, d\hat{x}$$

$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

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with

$$G(u, \boldsymbol{\sigma}) = \sum_{\hat{T} \in \hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu \qquad) \, d\hat{x}$$
$$- \sum_{\hat{E} \in \hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) \qquad) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i) \nu_i$

Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\text{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u, \boldsymbol{\sigma}) - \langle f, u\rangle,$$

with

$$G(u, \boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : u_{3|\alpha\beta}\, d\hat{x} - \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} [\![u_{3|\hat{\mu}}]\!] \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s}.$$

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Shell problem (Hybridization)

Find $u \in [H^1(\hat{S})]^3$, $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})^{dc}$ and $\alpha \in \Gamma(\hat{S})$ for

$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u, \boldsymbol{\sigma}, \alpha) - \langle f, u\rangle,$$

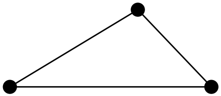
with

$$G(u, \boldsymbol{\sigma}, \alpha) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1 - \hat{\nu}\cdot\nu)\nabla\hat{\nu})\, d\hat{x}$$
$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\frac{1}{2}(\boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L} + \boldsymbol{\sigma}_{\hat{\mu}_R\hat{\mu}_R})\, d\hat{s}$$
$$+ \int_{\hat{E}} \alpha_{\hat{\mu}} [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\, d\hat{s}.$$

$$H^1(\Omega) := \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$

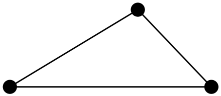
The space H(divdiv)

$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}$$
$$V_k := \Pi^k(\mathcal{T}_h) \cap C(\Omega)$$



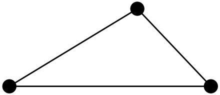
The space H(divdiv)

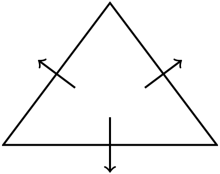
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}(\sigma) \in L^2(\Omega)\}$$



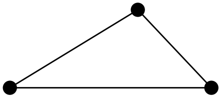
$$H(\text{div}) := \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}(\sigma) \in L^2(\Omega)\}$$

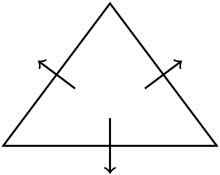
$$BDM_k := \{\sigma \in [\Pi^k(\mathcal{T}_h)]^d \,|\, \sigma_n \text{ is continuous over elements}\}$$





$$H(\mathrm{divdiv}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) \in H^{-1}(\Omega)\}$$





The space H(divdiv)

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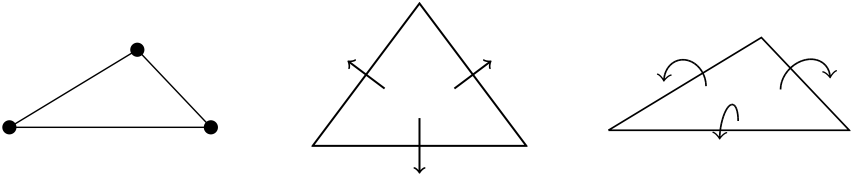
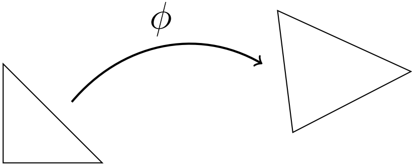


A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math.* (2017) 137, pp. 713–740.

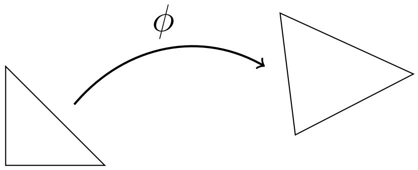
Mapping to the surface



Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \det(\boldsymbol{F})$$



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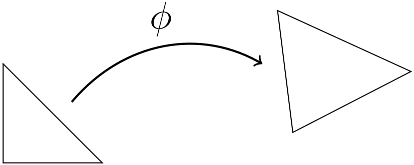
- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^\top$$



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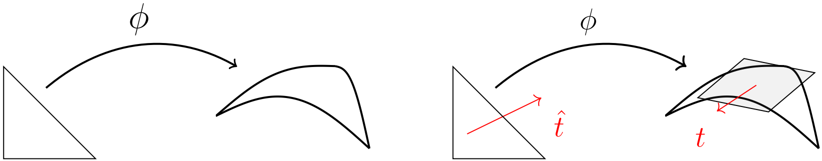

Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$$

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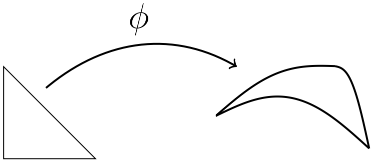

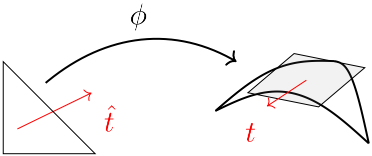

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \|\operatorname{cof}(\boldsymbol{F})\|$$

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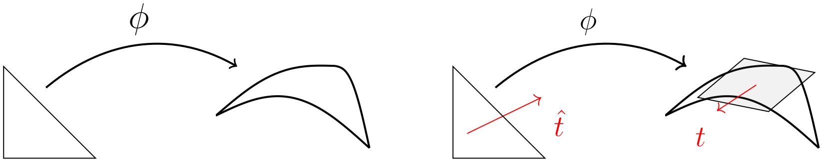

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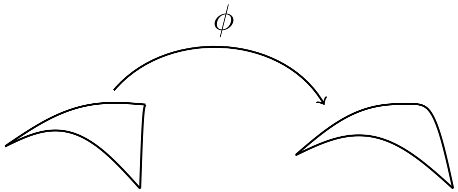
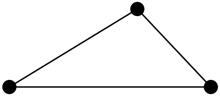
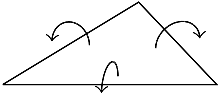
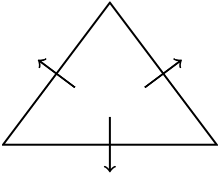


Shell element







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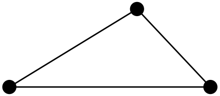
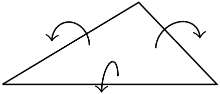
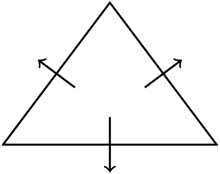
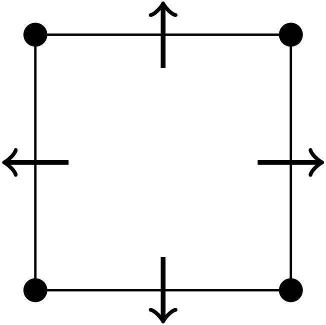
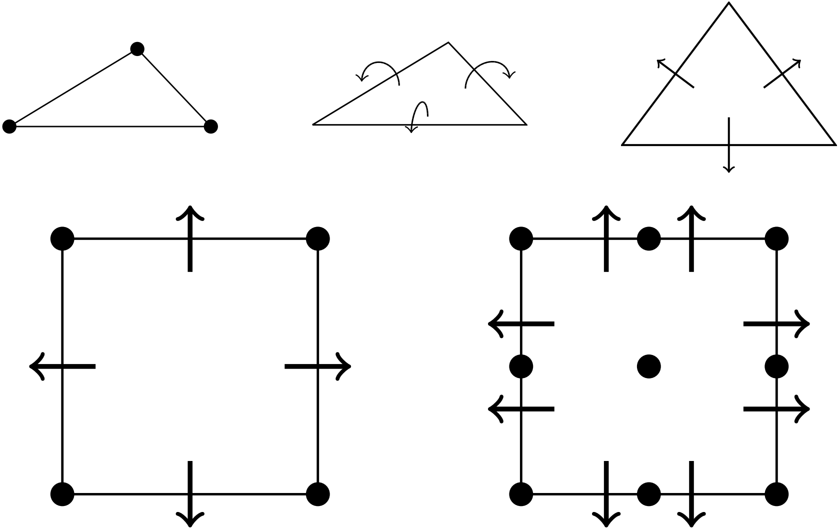


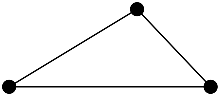


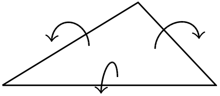


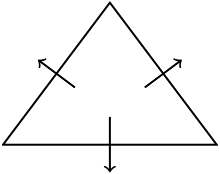


Shell element

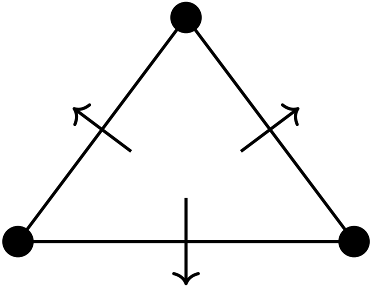








Morley triangle:

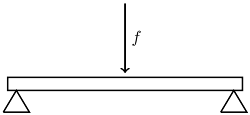


Relation to HHJ

Hellan–Herrmann–Johnson method

- Discretization method for 4th order elliptic problems

$$\operatorname{div}(\operatorname{div}(\nabla^2 w)) = f$$



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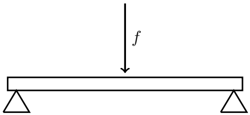
- Discretization method for 4th order elliptic problems

$$\mathrm{div}(\mathrm{div}(\nabla^2 w)) = f \quad \Rightarrow w \in H^2(\Omega)$$



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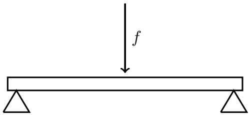

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 w, \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, \end{aligned}$$

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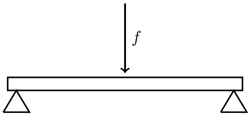

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 w, & \Rightarrow w \in H^1(\Omega) \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, & \Rightarrow \boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega) \end{aligned}$$

Hellan–Herrmann–Johnson method

Hellan–Herrmann–Johnson

Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

$$\mathcal{L}(w, \boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2 - \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \mathrm{div}(\boldsymbol{\sigma})\, dx + \int_{\partial T} (\nabla w)_\tau \boldsymbol{\sigma}_{\mu\tau}\, ds - \langle f, w \rangle.$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

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Hellan–Herrmann–Johnson

Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

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M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

Hellan–Herrmann–Johnson

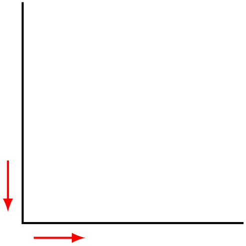
Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

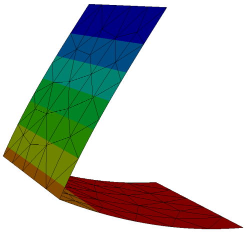
$$\mathcal{L}(w, \boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2 + \sum_{T\in\mathcal{T}_h} \int_T w_{|\alpha\beta} : \boldsymbol{\sigma}\, dx - \int_{\partial T} w_{|\mu} \boldsymbol{\sigma}_{\mu\mu}\, ds - \langle f, w \rangle.$$

Linearization

If the undeformed configuration is a flat plane and f works orthogonal on it, the HHJ method is the linearization of the bending energy of our method.

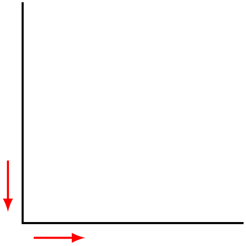
Structures with kinks

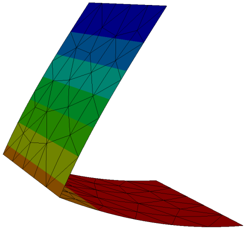




Structures with kinks

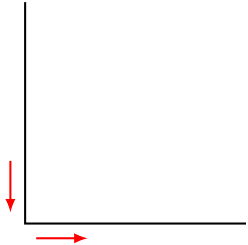
- Normal-normal continuous moment $\boldsymbol{\sigma}$

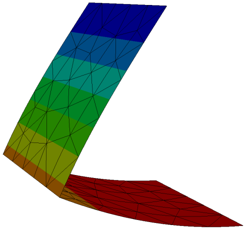


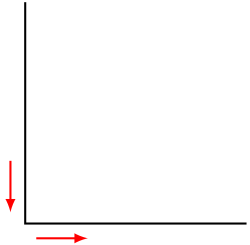


Structures with kinks

- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks



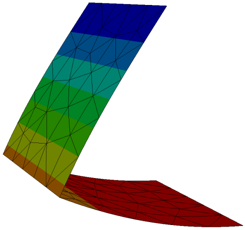




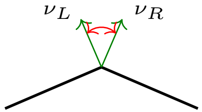
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s} \stackrel{!}{=} 0$$

$$\Rightarrow \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$



$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



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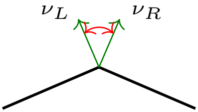
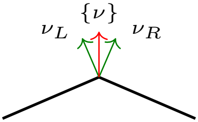
$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

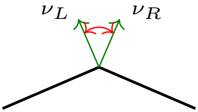


$$\int_{\partial \hat{T}} (\sphericalangle(\{\nu\}, \nu) - \sphericalangle(\{\hat{\nu}\}, \hat{\nu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

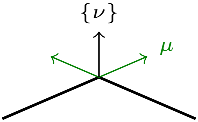


$$\{\nu\} := \frac{1}{\|\nu_L + \nu_R\|}(\nu_L + \nu_R)$$

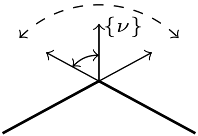
$$\int_{\hat{E}} \left(\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R)\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



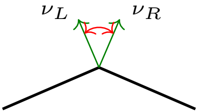
$$\int_{\partial \hat{T}} \left(\sphericalangle(\{\nu\}, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu})\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



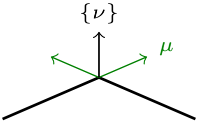
$$\{\nu\} := \frac{1}{\|\nu_L + \nu_R\|}(\nu_L + \nu_R)$$



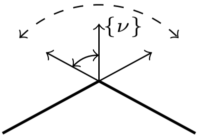
$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



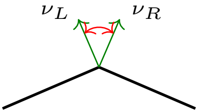
$$\int_{\partial \hat{T}} (\sphericalangle(\{\nu\}\ , \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



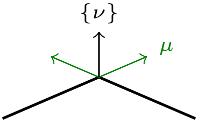
$$\{\nu\} \ := \frac{\operatorname{cof}(\boldsymbol{F}_L)\hat{\nu}_L + \operatorname{cof}(\boldsymbol{F}_R)\hat{\nu}_R}{\|\operatorname{cof}(\boldsymbol{F}_L)\hat{\nu}_L + \operatorname{cof}(\boldsymbol{F}_R)\hat{\nu}_R\|}$$



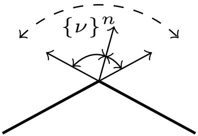
$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



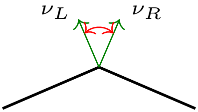
$$\int_{\partial \hat{T}} (\sphericalangle(\{\nu\}^n, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



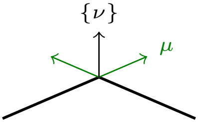
$$\{\nu\}^n := \frac{1}{\|\nu_L{}^n + \nu_R{}^n\|}(\nu_L{}^n + \nu_R{}^n)$$



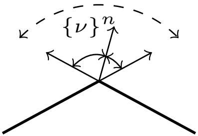
$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



$$\int_{\partial\hat{T}} (\sphericalangle(\boldsymbol{P}_{\tau}^{\perp}\{\nu\}^n, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



$$\{\nu\}^n := \frac{1}{\|\nu_L{}^n + \nu_R{}^n\|}(\nu_L{}^n + \nu_R{}^n)$$



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Final algorithm

For given u^n compute

$$\{\nu\}^n = Av(u^n).$$

Then find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\operatorname{divdiv}, \hat{S})$ for

$$\mathcal{L}_{\{\nu\}^n}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G_{\{\nu\}^n}(u, \boldsymbol{\sigma}) - \langle f, u\rangle,$$

with

$$G_{\{\nu\}^n}(u, \boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1 - \hat{\nu}\cdot\nu)\nabla\hat{\nu})\, d\hat{x}$$
$$- \int_{\partial\hat{T}} (\sphericalangle(\boldsymbol{P}_\tau^\perp\{\nu\}^n, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s}.$$

Examples

Hellan–Herrmann–Johnson method for plates and nonlinear shells in NGSolve!

NGSolve

Membrane Locking

Membrane locking

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} - f\cdot u$$

Membrane locking

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$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u$$

Membrane locking

$$\mathcal{W}(u) = \frac{1}{t^2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u$$

- Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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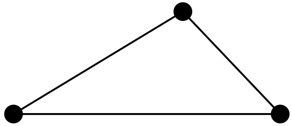

$V_h = \Pi(\mathcal{T}_h) \cap C(\Omega) \subset H^1(\Omega)$

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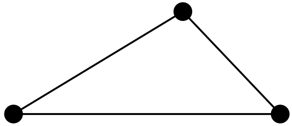
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



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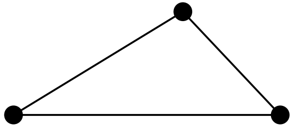

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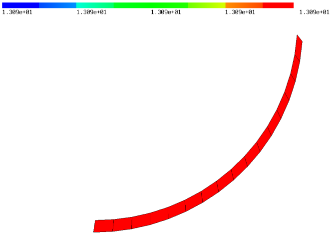
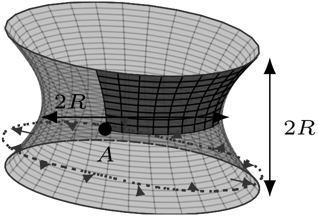
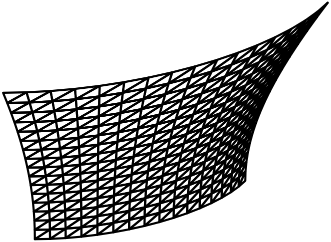
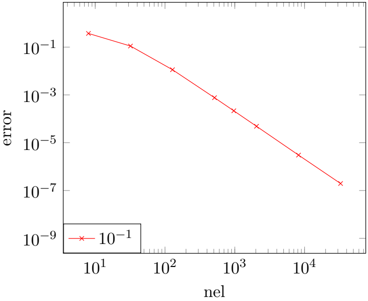
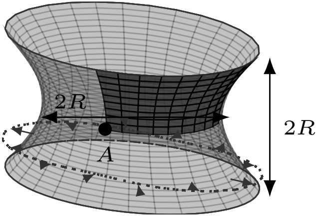


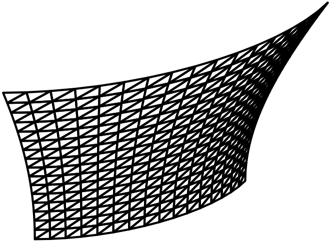
Hyperboloid with free ends

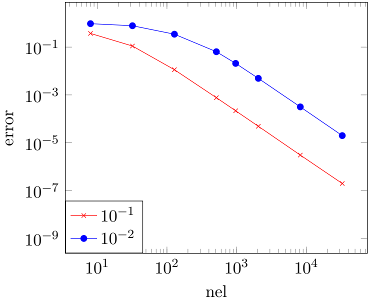






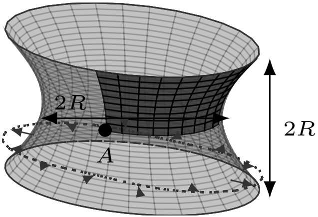


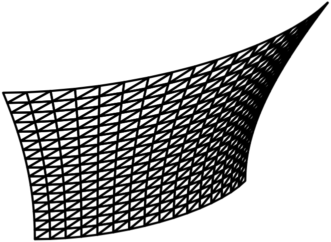


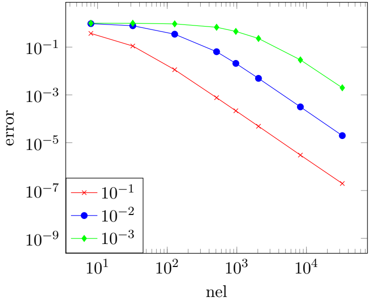


- Pre-asymptotic regime

Hyperboloid with free ends

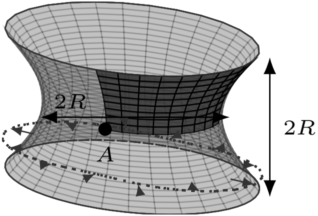


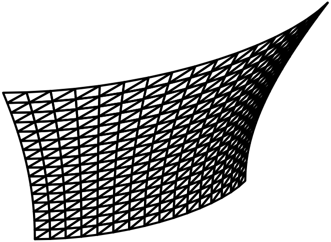


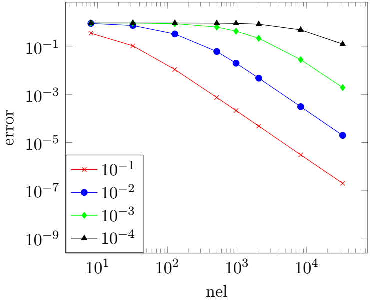


- Pre-asymptotic regime

Hyperboloid with free ends







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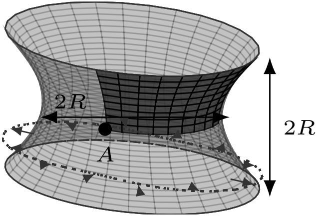
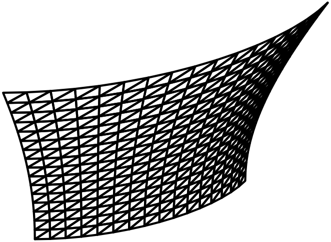
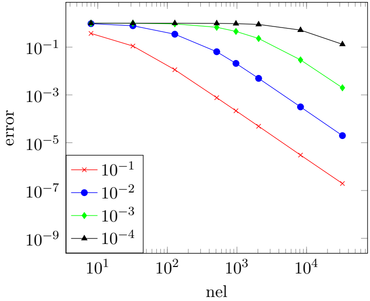
Hyperboloid with free ends

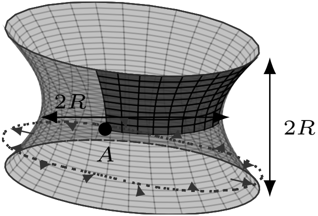


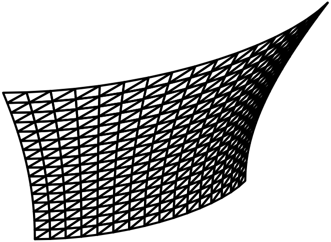


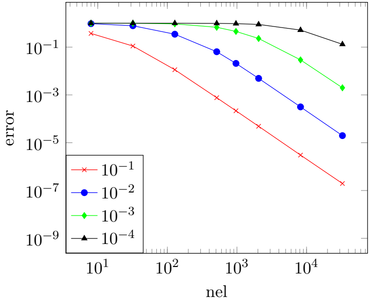


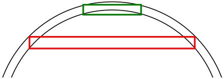
- Pre-asymptotic regime
- $h \prec \sqrt{Rt}$

Hyperboloid with free ends



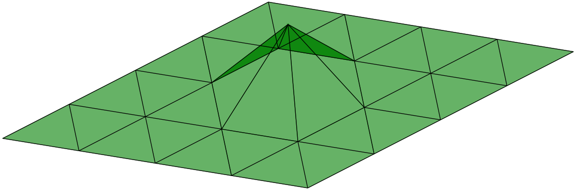


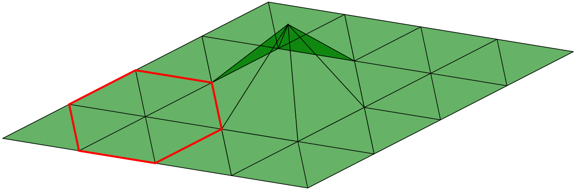


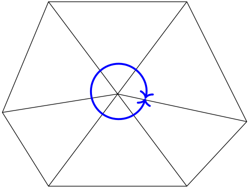


Regge Elements

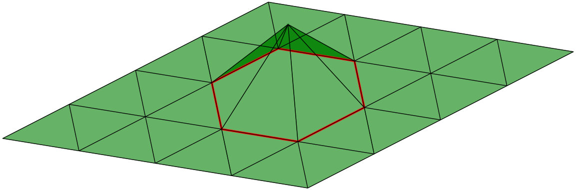
Regge calculus

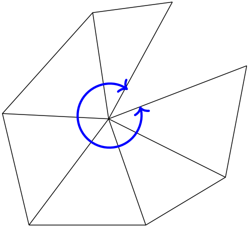




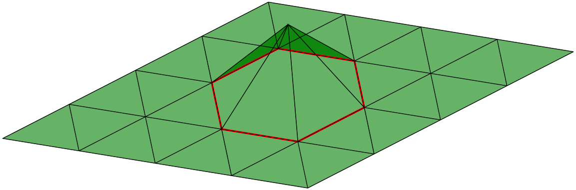


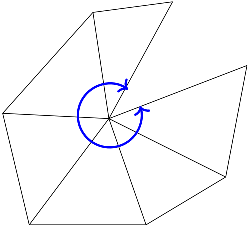
Regge calculus



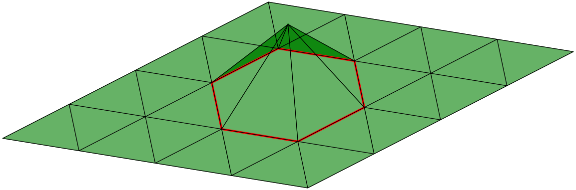


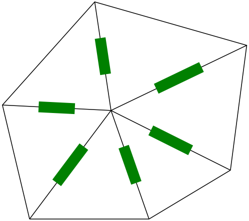
Regge calculus





T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571.*





- Metric tensor

T. REGGE: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571.*

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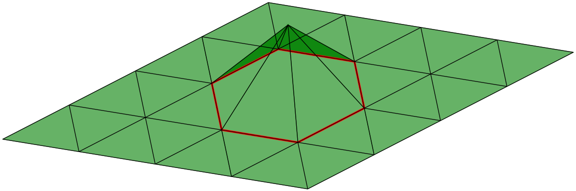

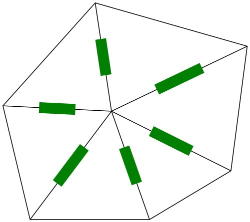

- Metric tensor
- tangential-tangential continuous

T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571*.

Regge elements

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]_{sym}^{d\times d} \mid t^\top \boldsymbol{\sigma} t \text{ is continuous over elements}\}$$

S. H. Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]_{sym}^{d\times d} \mid t^\top \boldsymbol{\sigma}\, t \text{ is continuous over elements}\}$$

L. Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota* (2018).

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]_{sym}^{d\times d} \,|\, t^\top \boldsymbol{\sigma} t \text{ is continuous over elements}\}$$

$$H(\mathrm{curlcurl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]_{sym}^{d\times d} \,|\, \mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{\sigma})^\top \in [H^{-1}(\Omega)]^{d^\star \times d^\star}\}$$

Membrane locking

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$$\frac{1}{t^2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}}$$

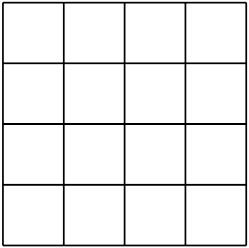
$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|^2_{\boldsymbol{M}} = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|^2_{\boldsymbol{M}}$$

Membrane locking

$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}}$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau\nabla_\tau u)\|^2_{\boldsymbol{M}} = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3\|^2_{\boldsymbol{M}}$$

- Reduced integration for quadrilateral meshes



Membrane locking

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}(u)\|_{\boldsymbol{M}}^2$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|_{\boldsymbol{M}}^2 = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|_{\boldsymbol{M}}^2$$

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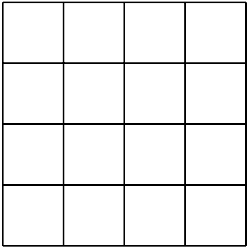

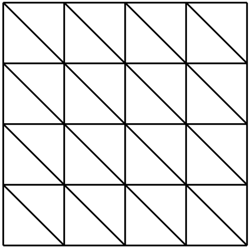

Membrane locking

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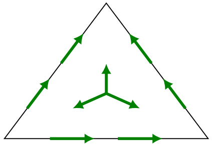
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- $\boldsymbol{R} \in \mathrm{Reg}_h^k$, $\boldsymbol{Q} \in [\mathrm{Reg}_h^k]^*$

$$\frac{1}{t^2}\|\boldsymbol{R}\|_{\boldsymbol{M}}^2 + \langle \boldsymbol{Q}, \boldsymbol{R} - \boldsymbol{E}_{\tau\tau}\rangle$$

Dual space

$$\langle \cdot, \cdot \rangle : [\mathrm{Reg}_h^k]^* \times \mathrm{Reg}_h^k \to \mathbb{R}$$
$$(\Psi, \varphi) \mapsto \Psi(\varphi)$$



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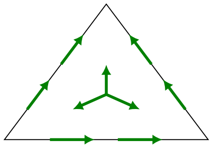

- Edge functionals

$$\Psi_{E_{\alpha\beta},i} : \boldsymbol{\sigma} \mapsto \int_{E_{\alpha\beta}} \boldsymbol{\sigma}_{\tau_E \tau_E} q_{E,i}\, ds, \qquad \{q_{E,i}\} \text{ basis of } \Pi^k(E_{\alpha\beta})$$

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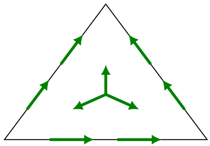

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$$\Psi_{E_{\alpha\beta},i} : \boldsymbol{\sigma} \mapsto \int_{E_{\alpha\beta}} \boldsymbol{\sigma}_{\tau_E \tau_E} q_{E,i}\, ds, \qquad \{q_{E,i}\} \text{ basis of } \Pi^k(E_{\alpha\beta})$$

- Element functionals

$$\Psi_{T,i} : \boldsymbol{\sigma} \mapsto \int_T \boldsymbol{\sigma} : \boldsymbol{q}_{T,i}\, dx, \qquad \{\boldsymbol{q}_{T,i}\} \text{ basis of } [\Pi^{k-1}(T)]^{2\times 2}_{sym}$$

Dual space

$$\langle \cdot, \cdot \rangle : [\mathrm{Reg}_h^k]^* \times \mathrm{Reg}_h^k \to \mathbb{R}$$
$$(\Psi, \varphi) \mapsto \Psi(\varphi)$$

$$\mathcal{I}_{\mathcal{R}}^k : [C^\infty(\Omega)]^{2\times 2} \to \mathrm{Reg}_h^k$$
$$\boldsymbol{\sigma} \mapsto \sum_{i=0}^{N_k} \Psi_i(\boldsymbol{\sigma})\varphi_i$$

- Edge functionals

$$\Psi_{E_{\alpha\beta},i} : \boldsymbol{\sigma} \mapsto \int_{E_{\alpha\beta}} \boldsymbol{\sigma}_{\tau_E \tau_E} q_{E,i}\, ds, \qquad \{q_{E,i}\} \text{ basis of } \Pi^k(E_{\alpha\beta})$$

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$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}\|^2_{\boldsymbol{M}}$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|^2_{\boldsymbol{M}} = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|^2_{\boldsymbol{M}}$$

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$$\frac{1}{t^2}\|\boldsymbol{R}\|^2_{\boldsymbol{M}} + \langle \boldsymbol{Q}, \boldsymbol{R} - \boldsymbol{E}_{\tau\tau}\rangle$$

u continuous (pw smooth), $\boldsymbol{F}_\tau$ t-continuous, $\boldsymbol{E}_{\tau\tau}$ tt-continuous

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$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}\|_{\boldsymbol{M}}^2$$

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New feature: InterpolationCF.

Membrane locking

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$$\frac{1}{t^2}\|\boldsymbol{R}\|_{\boldsymbol{M}}^2 + \langle \boldsymbol{Q}, \boldsymbol{R} - \boldsymbol{E}_{\tau\tau}\rangle$$

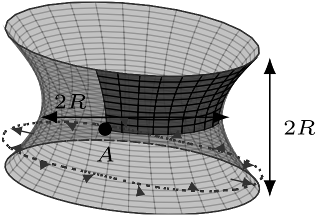
New feature: InterpolationCF. Lukas Kogler

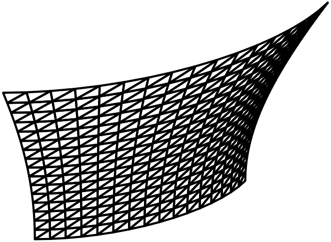
Examples

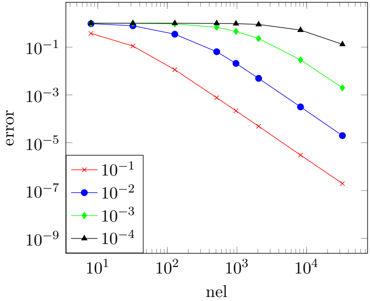
Application of interpolation operator: MITC elements!

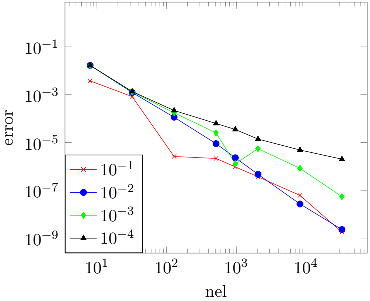
NGSolve

Hyperboloid with free ends

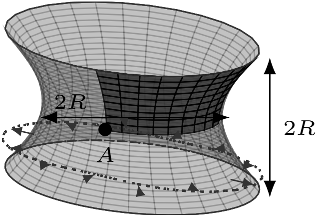


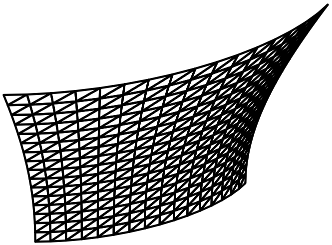


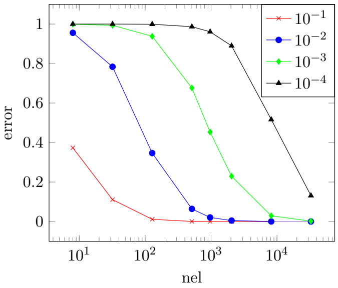


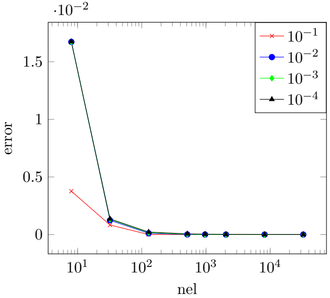


Hyperboloid with free ends

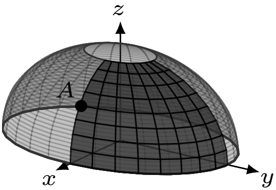


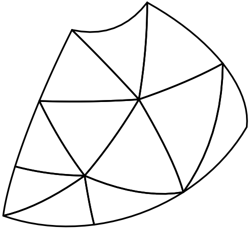


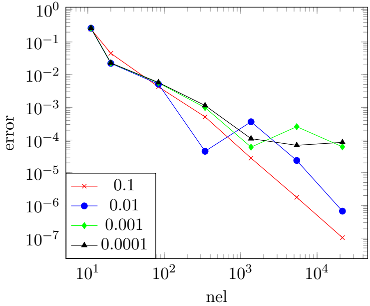


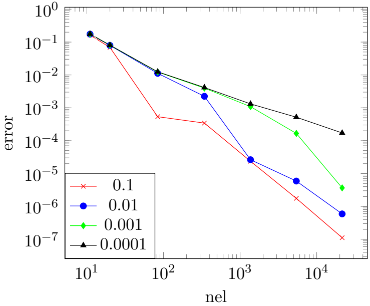


Open hemisphere with clamped ends









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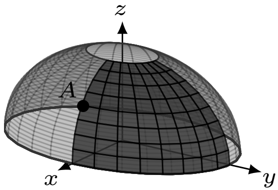
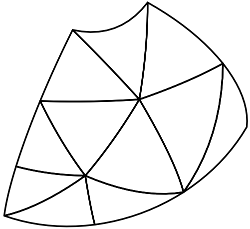
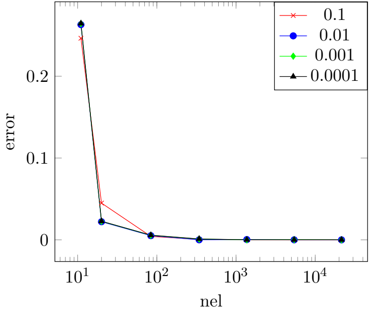
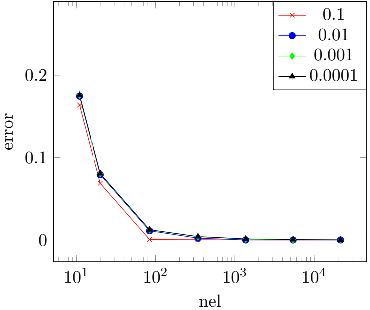


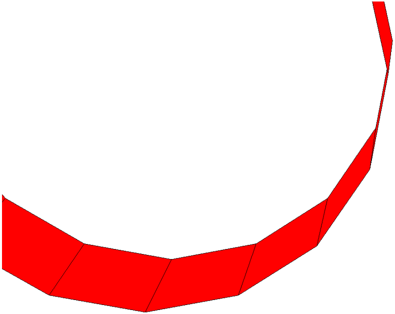


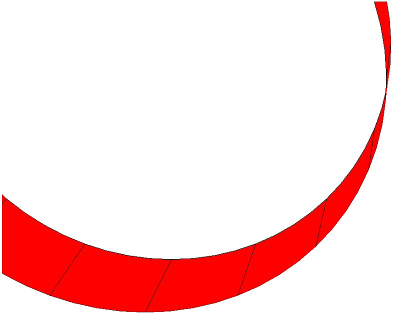




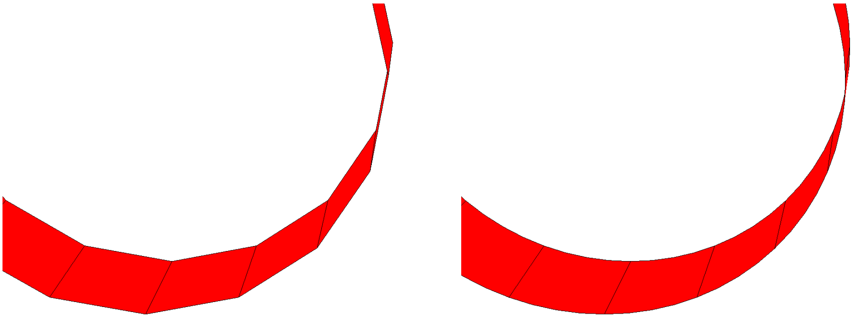
Numerical Examples

Cantilever subjected to end moment





Cantilever subjected to end moment



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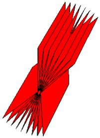
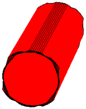
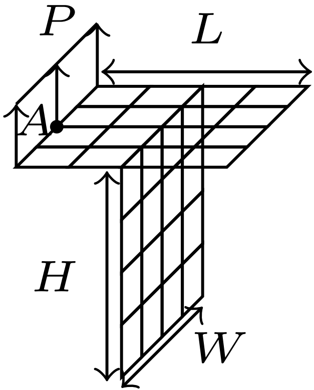


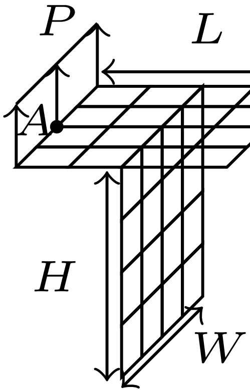


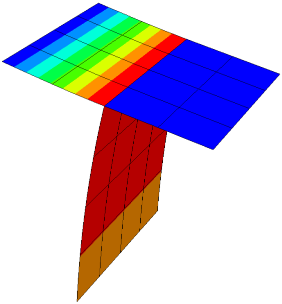
T-Section Cantilever



- $P = 2 \times 10^3$
  $E = 6 \times 10^6$
  $\nu = 0$
  $t = 0.1$
  $L = 1$
  $W = 1$
  $H = 1$

T-Section Cantilever





Summary

- Koiter shell element

Summary

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- Koiter shell element
- Moment tensor

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Thank You for Your attention!

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M. Neunteufel and J. Schöberl: The Hellan–Herrmann–Johnson Method for Nonlinear Shells, *Computers & Structures (2019) 225, 106109.*

M. Neunteufel and J. Schöberl: Avoiding Membrane Locking with Regge Interpolation, `http://arxiv.org/abs/1907.06232`