A Hierarchical Error Estimator for the MSFEM for the Eddy Current Problem in 3D

Abstract

Purpose - The multiscale finite element method is used to solve the linear eddy current problem on a stack of iron sheets. This allows for a calculation of the solution without having to resolve each sheet in the finite element mesh. The aim of the paper is to develop an a posteriori hierarchical error estimator for the multiscale solution.

Design/methodology/approach - A hierarchical local error estimator for nodal elements and edge elements is adapted to the multiscale setting.

Findings - The estimator allows for adaptive p-refinement on the multiscale mesh. Numerical examples show an increased order of convergence compared to uniform refinement.

Originality/value - The proposed error estimator increases the efficiency of the multiscale finite element method, requiring even less degrees of freedom.

Keywords: Eddy currents, multiscale FEM, hierarchical error estimator

1. Introduction

The simulation of eddy currents in a laminated iron core is a challenging task, as resolving each sheet in the finite element mesh leads to an unfeasible number of unknowns. Several ways to solve this problem have been proposed, including homogenization techniques for the core (see for example [1] or [2]) or the 2D/1D method to reduce the dimensionality of the problem [3], [4].

This paper focuses on the multiscale finite element method (MSFEM), which has been presented in [2] for the eddy current problem in 3D, using the A formulation based on the magnetic vector potential [5]. The aim is to provide an a posteriori error estimator for the MSFEM solution.

The developed estimator is based on a hierarchical error estimator that has been proposed for the $H^1$ in [6] and for the $H(\text{curl})$ in [7] and included in the analysis presented in [8]. It is adapted to fit into the setting of the
MSFEM. If the used finite elements are based on hierarchical basis functions, the estimator can be calculated efficiently and parallel over all elements.

Two numerical examples are carried out, which use the proposed estimator for adaptive p-refinement. It is demonstrated, that the estimator correctly identifies the local behavior of the solution and that it increases the rate of convergence of the numerical error with respect to the used degrees of freedom.

2. The Multiscale Finite Element Method

Consider the linear eddy current problem in the frequency domain. Its weak formulation is given as: Find the magnetic vector potential $A \in H(\text{curl})$, so that

$$
\int_{\Omega} \mu^{-1} \text{curl} A \cdot \text{curl} v + i\omega \sigma A \cdot v \, d\Omega = \int_{\Omega} J \cdot v \, d\Omega \quad \text{(1)}
$$

for all $v \in H(\text{curl})$, including boundary conditions. In (1) $\mu$ denotes the magnetic permeability, $\sigma$ the electric conductivity, $\omega$ the angular frequency and $i$ the imaginary unit. For the analysis it will be advantageous to refer to the problem in terms of the bilinear form $a$ and the linear form $f$.

The material parameters $\mu$ and $\sigma$ are assumed to be constant inside the sheets, but the finite element mesh still needs to resolve each of the sheets to account for the insulation layers between them. This results in an unfeasible amount of finite elements if there are many sheets.

The multiscale method solves this problem by choosing an expansion for the magnetic vector potential that resolves the local behavior in each sheet using predefined micro-shape functions. The simplest first order expansion has the form

$$
A_{MS} = A_0 + \phi_1 A_1 + \nabla (\phi_1 w_1),
$$

where $\phi_1$ is a piecewise linear function, see Figure 1. The unknown functions $A_0, A_1 \in H(\text{curl})$ and $w_1 \in H^1$ model the global behavior of the magnetic vector potential and vary on a scale that is much larger than the lamination thickness. Therefore they can be defined on a very coarse mesh that does not resolve the iron sheets. Simply speaking, $A_0$ resolves the global behavior of the potential, $A_1$ controls the local variations due to the sheets and $w_1$ allows for the inclusion of the edge effect [2].
To use (2) in (1), the similar expansion

\[ v_{MS} = v_0 + \phi_1 v_1 + \nabla (\phi_1 q_1) \]  

is used for the test function. Both \( A \) and \( v \) in (1) are substituted for \( A_{MS} \) and \( v_{MS} \), respectively. The final problem is given as: Find \( A_0, A_1 \in H(\text{curl}) \) and \( w_1 \in H^1 \), so that

\[
\int_\Omega \mu^{-1} \text{curl} A_0 \cdot \text{curl} v_0 + \mu^{-1} \phi_1^2 \text{curl} A_1 \cdot \text{curl} v_1 \\
+ \mu^{-1} \phi_{1,z} \left( \text{curl} A_0 \cdot \begin{pmatrix} -v_{1y} \\ v_{1x} \\ 0 \end{pmatrix} + \begin{pmatrix} -A_{1y} \\ A_{1x} \\ 0 \end{pmatrix} \cdot \text{curl} v_0 \right) \\
+ \mu^{-1} \phi_{1,z} \left( \begin{pmatrix} -A_{1y} \\ A_{1x} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -v_{1y} \\ v_{1x} \\ 0 \end{pmatrix} + \nu_0 A_0 \cdot v_0 + \nu_0 A_1 \cdot v_1 \right) \\
+ i \omega \sigma \phi_{1,z} \left( A_{0z} \cdot q_1 + w_1 v_{0z} \right) \\
+ i \omega \sigma \phi_{1,z}^2 \left( \nabla w_1 \cdot v_1 + A_1 \cdot \nabla q_1 + \nabla w_1 \cdot \nabla q_1 \right) \, d\Omega = \int_\Omega J \cdot v_0 \, d\Omega
\]

for all \( v_0, v_1 \in H(\text{curl}) \) and \( q_1 \in H^1 \). The symbol \( \phi_{1,z} \) denotes the derivative of \( \phi_1 \) with respect to \( z \), while the coordinates in the indices of vector valued functions denote the respective vector components. The bars indicate the average over one sheet plus insulation layer, for example

\[
\mu^{-1} \phi_{1,z} := \int_{d+1/2}^{d+1/2} \mu^{-1} \phi_{1,z} \, dz. \tag{5}
\]
A detailed explanation of (4) can be found in [2]. Note that the averaging of the coefficients removes the direct dependencies on the fine scale and all integrals can be computed on the coarse scale.

Similar to (1), the expression (4) will be shortened using the notation

$$\bar{a}(A_{MS}, v_{MS}) = \bar{f}(v_{MS})$$

(6)

for the rest of the paper.

3. The Error Estimator

The developed estimator is based on solving an auxiliary problem on a higher order space with the residuum as the right hand side in order to get an estimate of how much a refinement would affect the solution. For a detailed explanation see [6] for the $H^1$ and [7] for the $H(\text{curl})$. In the following a given, fixed mesh on $\Omega$ is assumed. Denote by $I$ the index set of the mesh elements.

Let $\mathcal{P}^{n_I} \subset H^1$ be defined as the space of scalar functions which are polynomials of order $n_i$ on the $i$th mesh element and continuous across the element boundaries. Likewise, denote by $\mathcal{N}^{n_I} \subset H(\text{curl})$ the Nedelec space, consisting of tangentially continuous vector valued functions of polynomial order depending on the mesh element.

Denote the finite element solution of (6) as $A^n_{MS} \in \mathcal{V}^n := \mathcal{N}^n \times \mathcal{N}^n \times \mathcal{P}^{n+1}$. Similar to the literature, the finite element error in the energy norm can be estimated by

$$\|A^n_{MS} - A_{MS}\|_{\bar{a}} \leq C\|z\|_{\bar{a}}$$

(7)

with $z$ being the solution of: Find $z \in \mathcal{V}^{n+1}$ so that

$$\bar{a}(z, \chi) = \bar{a}(A^n_{MS}, \chi) - \bar{f}(\chi)$$

(8)

for all $\chi \in \mathcal{V}^{n+1}$.

In this form, (8) would require the solution of a higher order problem on the entire domain, which is not feasible for an error estimator. If $\mathcal{P}^{n_I}$ and $\mathcal{N}^{n_I} \subset H(\text{curl})$ are implemented using hierarchical basis functions, (6) can be utilized to simplify (8) to: Find $z \in \mathcal{B}^n$ so that

$$\bar{a}(z, \hat{\chi}) = \bar{a}(A^n_{MS}, \hat{\chi}) - \bar{f}(\hat{\chi})$$

(9)

for all $\hat{\chi} \in \mathcal{B}^n$ with $\mathcal{B}^n := \mathcal{V}^{n+1} \setminus \mathcal{V}^n$ denoting the space of higher order bubble functions.
Lastly, by norm equivalence, $z$ may be split into local components $z_i$, solving (9) on each element individually. This reduces the computational expense of the error estimation to the solution of many small local problems, which can be done in parallel.

The localization also allows for an adaptive $p$-refinement. In the following numerical examples, in each iteration the finite element order on the $i^{th}$ element is increased by one if $\|z_i\|_{\bar{a}} \geq \frac{1}{4} \max_j \|z_j\|_{\bar{a}}$. Instead of $\frac{1}{4}$ any factor $0 < C < 1$ can be used to control the granularity of the refinements. The case $C = 0$ leads to uniform refinement.

4. Numerical Example

Consider an eighth of a stack of iron sheets, as illustrated in Figure 2. The magnetic permeability in iron is $\mu = 1,000\mu_0$, the electric conductivity is given as $\sigma = 2 \cdot 10^6 S/m$ and the frequency is chosen as $50Hz$. In air and the insulation layers $\mu = \mu_0$ and a small conductivity $0 < \sigma_0 \ll \sigma$ is chosen for regularity of the system. In the first example, a stack with only three iron sheets is considered and the problem is exited by a surface current density $K$. All calculations were done using the software Netgen/NGSolve.

![Figure 2: The domain $\Omega$, split into the laminated core $\Omega_C$ and the outer air region $\Omega_0$. All dimensions are in millimeters.](image)

The error estimator and refinement techniques described in section 3 are applied to this example. The left figure in Figure 3 shows the absolute value of $J$ on a cross section of the sheets. As one would expect, the solution has its greatest variations in the corner. The right figure illustrates the number of refinements in each finite element after 9 iterations. It can be seen, that the error estimator recognizes the behavior of the solution and focuses its refinements in the vicinity of the corner.
Figure 3: The absolute value of the current density $|J|$ in a cross section of the domain (left) and the distribution of refinements in each element (right, adaptive orders ranging from 0 (dark blue) to 9 (dark red)).

The effectiveness of the estimator is illustrated in Figure 4, where the total estimated error is compared for the adaptive approach and a uniform refinement across all elements. As expected, the adaptive refinement yields an increased order of convergence.

Figure 4: The estimated total error with respect to the degrees of freedom $\text{ndof}$ for the first example.

As a second example a similar setup as in Figure 2 is chosen, but with a stack consisting of 100 sheets. Since the sheets do not have to be resolved in the MSFEM, this only has a small impact on the number of degrees of freedom. In this example, the problem is exited by prescribing the Biot-Savart field of a filamentary current, representing a cylindrical coil.

It can be seen in Figure 5 that the estimator gives similar results to that of the first example.
5. Conclusion

A hierarchical error estimator has been applied to the MSFEM for the three dimensional linear eddy current problem. It is based on solving local problems on the space of the higher order bubble functions with the residuum on the right hand side. Adaptive p-refinement using this estimator results in a higher order convergence with respect to the degrees of freedom, as has been demonstrated by the numerical examples.

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References


