A Hierarchical Error Estimator for the MSFEM for the Eddy Current Problem in 3D

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Overview

- Problem Formulation
- Multi-Scale Finite Element Method MSFEM
- Hierarchical Error Estimator HEE
- Numerical Results
- Conclusions

The $A - V$ Formulation

We introduce the magnetic vector potential $\mathbf{A}$ with $\operatorname{curl} \mathbf{A} = \mathbf{B}$ and the electric scalar potential V, satisfying

$$\begin{aligned} \operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{A} + i\omega\sigma(\mathbf{A} - \nabla V) &= \mathbf{0} \\ \operatorname{div} i\omega\sigma(\mathbf{A} - \nabla V) &= 0, \end{aligned}$$

with the parameters

$\mu \ldots$ magnetic permeability,
$\sigma \ldots$ electric conductivity,
$\omega \ldots$ angular frequency.

The $A - V$ Formulation

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{A} + i\omega\sigma(\mathbf{A} - \nabla V) = \mathbf{0} \tag{1}$$

$$\operatorname{div} i\omega\sigma(\mathbf{A} - \nabla V) = \mathbf{0} \tag{2}$$

Multiply (1) with $\mathbf{v} \in H(\mathrm{curl})$, (2) with $q \in H^1$.

This results in the weak formulation:

Find $\mathbf{A} \in H(\mathrm{curl})$, $V \in H^1$, so that

$$\int_\Omega \mu^{-1} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \mathbf{v} + i\omega\sigma(\mathbf{A} - \nabla V) \cdot (\mathbf{v} - \nabla q)\, d\Omega = 0$$

for all $\mathbf{v} \in H(\mathrm{curl})$, $q \in H^1$.

Plus boundary conditions.

Motivation for the Multi-Scale Finite Element Method

Problems of the classical FEM in a layered medium:

- Each laminate has to be resolved individually.
- High number of elements makes it computationally expensive.

The solution $\mathbf{A} - \nabla V$ has the following properties:

- It varies only little from sheet to sheet.
- It is an odd function in each laminate.
- It is an even function along the sheet edges.

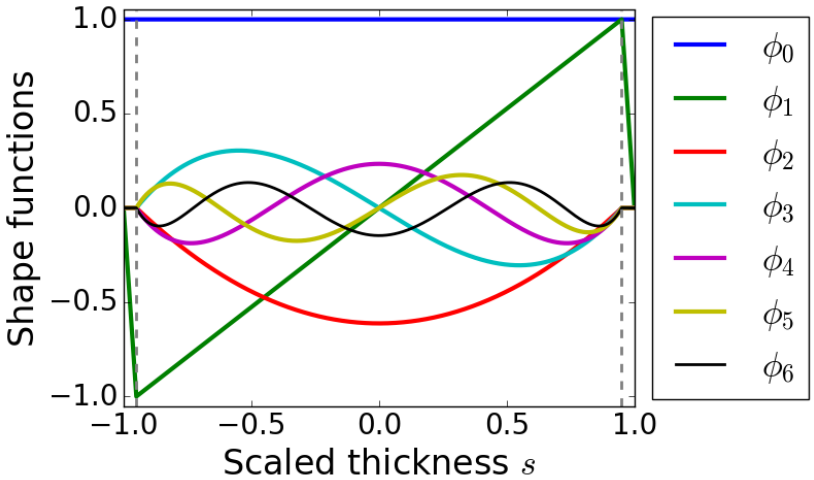
The Multi-Scale Finite Element Method

We propose the following MSFEM expansion:

$$\mathbf{A} - \nabla V \approx \mathbf{A}_0 + \mathbf{A}_1 \phi_1 + \nabla(w_1 \phi_1) + \dots$$

with $\mathbf{A}_0, \mathbf{A}_1, \dots \in H(\text{curl}), w_1, \dots \in H^1$.

This expansion is used in the weak formulation for the trial function and the test function.



- The shape functions ϕ_i are splines.
- Lobatto polynomials are used in iron.

The Multi-Scale Finite Element Method

Properties of the MSFEM expansion:

- The new unknowns are defined on a coarse mesh which does not resolve each single laminate.
- $\mathbf{A}_0$ describes the overall field distribution.
- $\mathbf{A}_1, \mathbf{A}_3, \ldots$ model the main magnetic flux in the lamination.
- $w_1, w_3, \ldots$ consider the edge effect.
- Any number of terms can be added to improve the accuracy.

Coefficients containing the shape functions ϕ_i are averaged analytically.

The Hierarchical Error Estimator HEE

Consider the general problem:

Find $u \in V$, such that

$$a(u, v) = f(v)$$

for all $v \in V$ with a bilinear form a and a linear form f.

Galerkin method: Choose a subspace $V_h \subset V$ of finite dimension, find $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h)$$

for all $v_h \in V_h$.

The Residuum

If $V_h^1 \subset V_h^2 \subseteq V$, then

$$a(u_h^1, v_h^1) = f(v_h^1) \quad \forall v_h^1 \in V_h^1$$
$$a(u_h^1, v_h^2) \neq f(v_h^2) \quad \forall v_h^2 \in V_h^2.$$

This defines the residuum:

$$r(v_h^2) = a(u_h^1, v_h^2) - f(v_h^2) \quad \forall v_h^2 \in V_h^2$$

Define χ as the solution of

$$a(\chi, \tilde{v}) = r(\tilde{v}) \quad \forall \tilde{v} \in V_h^2 \setminus V_h^1.$$

For specific V_h^1, V_h^2 the error $\|u_h^1 - u\|$ can be estimated by $\|\chi\|$ on each element.

The HEE for MSFEM

For a mesh Ω_h we define

$\mathcal{T}_i \ldots$ the ith mesh element of Ω_h.

$\mathcal{P}^{p_i}(\Omega_h) \ldots$ the space of element-wise polynomials of order p_i on $\mathcal{T}_i$.

$\mathcal{N}^{p_i}(\Omega_h) \ldots$ the Nedelec space of order p_i on $\mathcal{T}_i$.

$V_{MS}^{p_i} = \mathcal{N}^{p_i}(\Omega_h) \times \mathcal{N}^{p_i}(\Omega_h) \times \mathcal{P}^{p_i+1}(\Omega_h) \ldots$ the finite element space for the first order MSFEM.

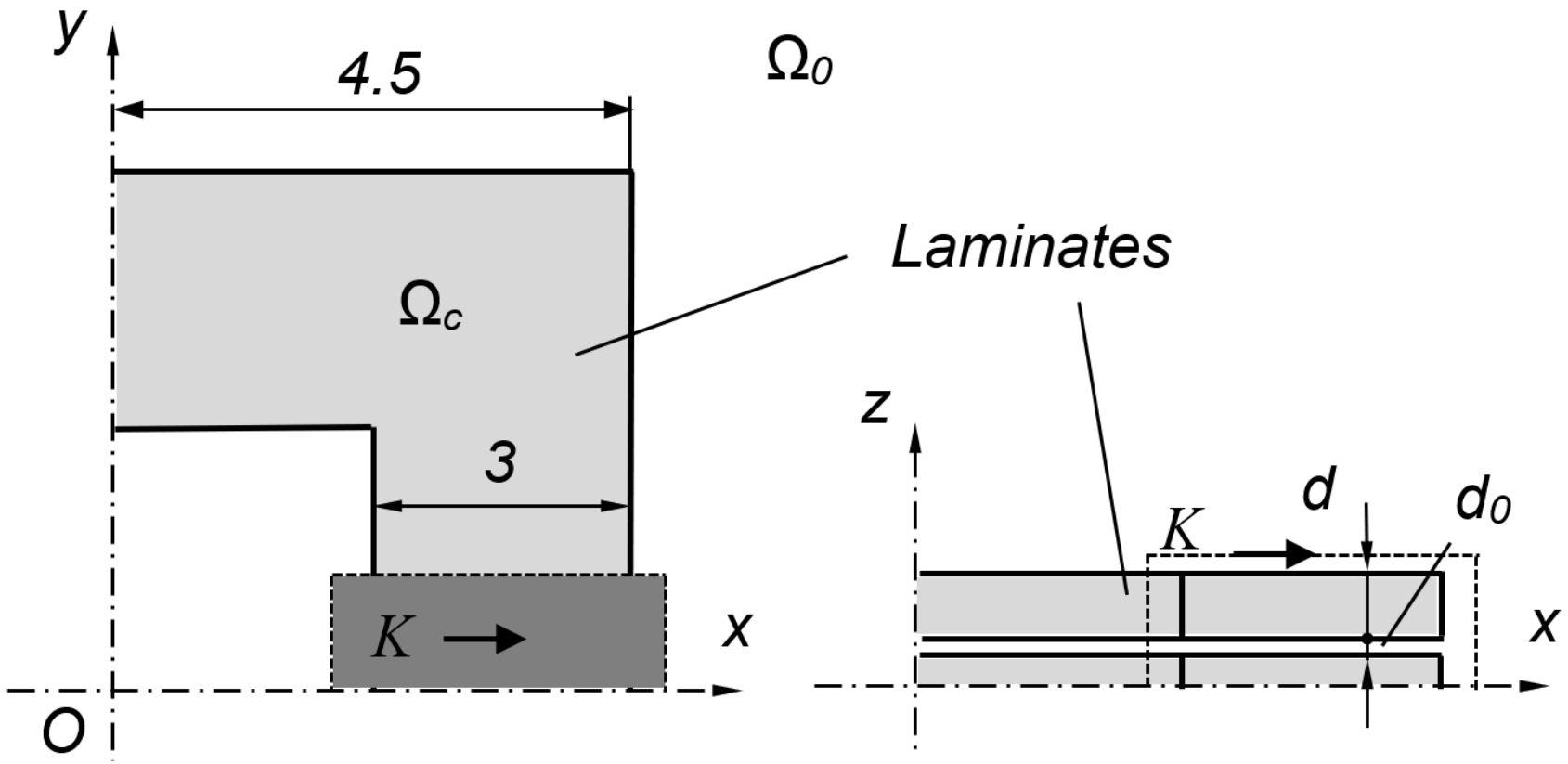
Note: All used finite element spaces need to be hierarchical. This way $V_h^2 \setminus V_h^1$ can be constructed efficiently.

The HEE for MSFEM

The basic algorithm:

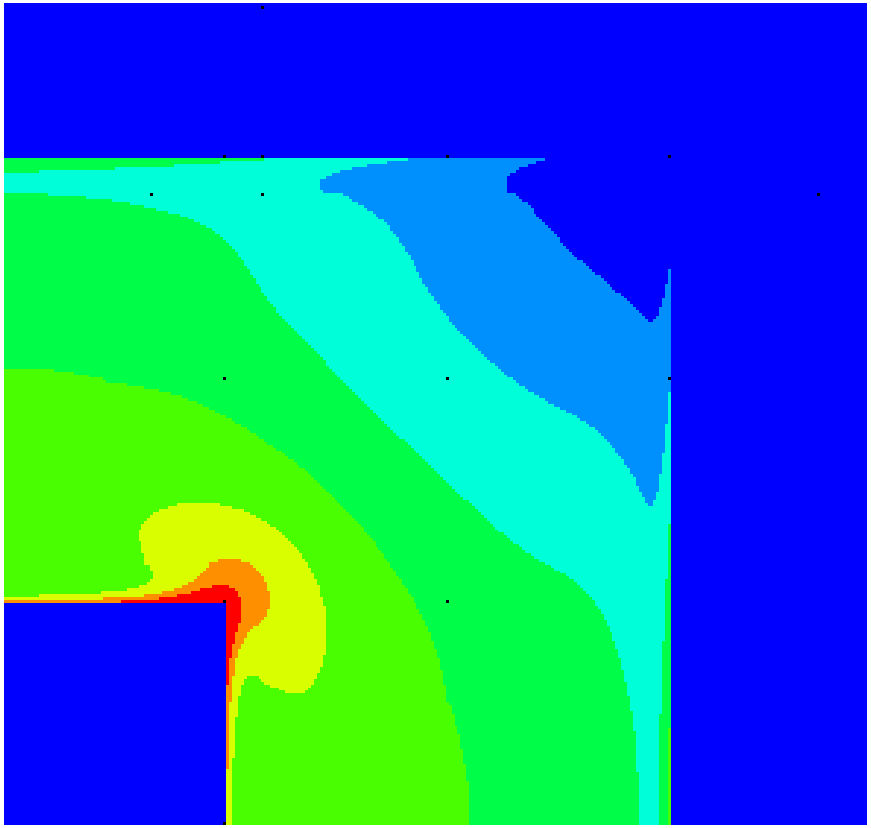
- Build the space $V_{MS}^{p_i}$. In the beginning, set $p_i = 1$ for all $\mathcal{T}_i$.
- Calculate the MSFEM solution $u_{MS} \in V_{MS}^{p_i}$.
- Build the space of higher order bubble functions $V_b = V_{MS}^{p_i+1} \setminus V_{MS}^{p_i}$.
- Calculate $\chi \in V_b$ using the residuum of u_{MS} in V_b.
- If $\|\chi\|_{\mathcal{T}_j} \geq \frac{1}{4} \max\limits_{i \in I} \|\chi\|_{\mathcal{T}_i}$, set $p_j = p_j + 1$ on $\mathcal{T}_j$.
- Repeat until $\|\chi\| \leq tol$.

Numerical Example

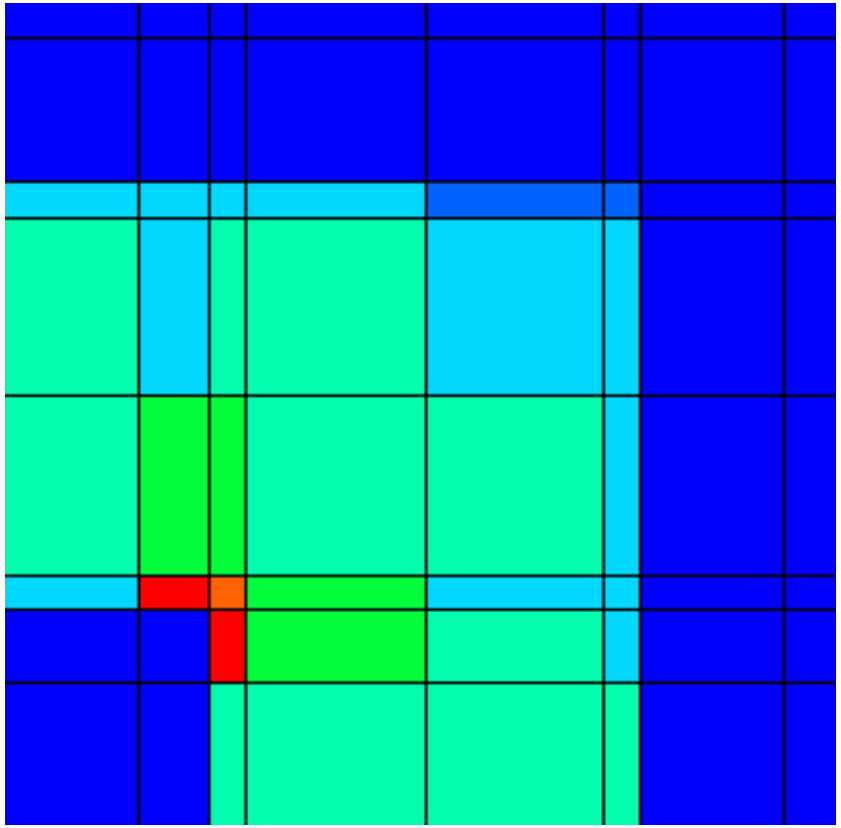


- A stack of laminates exited by a surface current density.
- All dimensions in *mm*.
- $\mu_r = 1,000,\ \sigma = 2MS/m$

Numerical Example



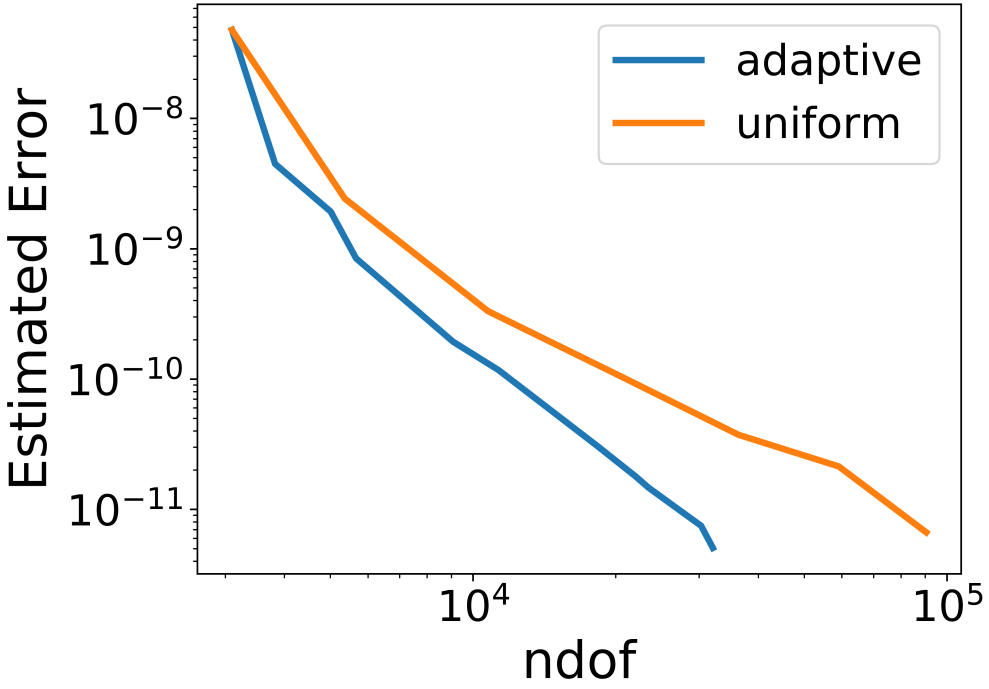
$\|\mathbf{B}\|$ in the cross-section.



Finite element orders
(blue= 1, red= 9)

The algorithm correctly refines in areas of great variations.

Estimated Errors



- Using the proposed hierarchical error estimator yields an increased rate of convergence.
- The results are very similar for the 3rd order MSFEM.

Conclusions

- A MSFEM formulation for the eddy current problem in 3D has been presented.
- A hierarchical error estimator has been applied to the MSFEM.
- Numerical examples show a significant speed-up using the HEE.
- Future work: Development of a HEE for MSFEM, which takes account of $\|u - u_{MS}\|$.

Thank you for your attention!