A Hierarchical Error Estimator for the MSFEM for the Eddy Current Problem in 3D

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Overview

- Problem Formulation
- Multi-Scale Finite Element Method MSFEM
- Hierarchical Error Estimator HEE
- Numerical Results
- Conclusions
The $A - V$ Formulation

We introduce the magnetic vector potential $\mathbf{A}$ with $\text{curl} \, \mathbf{A} = \mathbf{B}$ and the electric scalar potential $V$, satisfying

$$\text{curl} \, \mu^{-1} \text{curl} \, \mathbf{A} + i \omega \sigma (\mathbf{A} - \nabla V) = 0$$

$$\text{div} \, i \omega \sigma (\mathbf{A} - \nabla V) = 0,$$

with the parameters

$\mu$ . . . magnetic permeability,
$\sigma$ . . . electric conductivity,
$\omega$ . . . angular frequency.
The $A - V$ Formulation

\[
\text{curl } \mu^{-1} \text{ curl } A + i\omega\sigma (A - \nabla V) = 0 \\
\text{div } i\omega\sigma (A - \nabla V) = 0
\]  

(1)  

(2)

Multiply (1) with $v \in H(\text{curl})$, (2) with $q \in H^1$.

This results in the weak formulation:

Find $A \in H(\text{curl})$, $V \in H^1$, so that

\[
\int_{\Omega} \mu^{-1} \text{ curl } A \cdot \text{ curl } v + i\omega\sigma (A - \nabla V) \cdot (v - \nabla q) \, d\Omega = 0
\]

for all $v \in H(\text{curl})$, $q \in H^1$.

Plus boundary conditions.
Motivation for the Multi-Scale Finite Element Method

Problems of the classical FEM in a layered medium:

- Each laminate has to be resolved individually.
- High number of elements makes it computationally expensive.

The solution $\mathbf{A} - \nabla V$ has the following properties:

- It varies only little from sheet to sheet.
- It is an odd function in each laminate.
- It is an even function along the sheet edges.
We propose the following MSFEM expansion:

$$\mathbf{A} - \nabla V \approx \mathbf{A}_0 + \mathbf{A}_1 \phi_1 + \nabla (w_1 \phi_1) + \ldots$$

with $\mathbf{A}_0, \mathbf{A}_1, \ldots \in H(\text{curl}), w_1, \ldots \in H^1$.

This expansion is used in the weak formulation for the trial function and the test function.

- The shape functions $\phi_i$ are splines.
- Lobatto polynomials are used in iron.
Properties of the MSFEM expansion:

- The new unknowns are defined on a coarse mesh which does not resolve each single laminate.
- $A_0$ describes the overall field distribution.
- $A_1, A_3, \ldots$ model the main magnetic flux in the lamination.
- $w_1, w_3, \ldots$ consider the edge effect.
- Any number of terms can be added to improve the accuracy.

Coefficients containing the shape functions $\phi_i$ are averaged analytically.
Consider the general problem:

Find \( u \in V \), such that

\[
a(u, v) = f(v)
\]

for all \( v \in V \) with a bilinear form \( a \) and a linear form \( f \).

Galerkin method: Choose a subspace \( V_h \subset V \) of finite dimension, find \( u_h \in V_h \) such that

\[
a(u_h, v_h) = f(v_h)
\]

for all \( v_h \in V_h \).
The Residuum

If $V^1_h \subset V^2_h \subseteq V$, then

$$a(u^1_h, v^1_h) = f(v^1_h) \quad \forall v^1_h \in V^1_h$$
$$a(u^1_h, v^2_h) \neq f(v^2_h) \quad \forall v^2_h \in V^2_h.$$ 

This defines the residuum:

$$r(v^2_h) = a(u^1_h, v^2_h) - f(v^2_h) \quad \forall v^2_h \in V^2_h$$

Define $\chi$ as the solution of

$$a(\chi, \tilde{v}) = r(\tilde{v}) \quad \forall \tilde{v} \in V^2_h \setminus V^1_h.$$ 

For specific $V^1_h, V^2_h$ the error $\|u^1_h - u\|$ can be estimated by $\|\chi\|$ on each element.
The HEE for MSFEM

For a mesh $\Omega_h$ we define

$T_i$ ... the $i^{th}$ mesh element of $\Omega_h$.

$P^{p_i}(\Omega_h)$ ... the space of element-wise polynomials of order $p_i$ on $T_i$.

$N^{p_i}(\Omega_h)$ ... the Nedelec space of order $p_i$ on $T_i$.

$V_{MS}^{p_i} = N^{p_i}(\Omega_h) \times N^{p_i}(\Omega_h) \times P^{p_i+1}(\Omega_h)$ ... the finite element space for the first order MSFEM.

Note: All used finite element spaces need to be hierarchical. This way $V^2_h \setminus V^1_h$ can be constructed efficiently.
The HEE for MSFEM

The basic algorithm:

- Build the space $V_{MS}^{p_i}$. In the beginning, set $p_i = 1$ for all $T_i$.

- Calculate the MSFEM solution $u_{MS} \in V_{MS}^{p_i}$.

- Build the space of higher order bubble functions $V_b = V_{MS}^{p_i+1} \setminus V_{MS}^{p_i}$.

- Calculate $\chi \in V_b$ using the residuum of $u_{MS}$ in $V_b$.

- If $\|\chi\|_{T_j} \geq \frac{1}{4} \max_{i \in I} \|\chi\|_{T_i}$, set $p_j = p_j + 1$ on $T_j$.

- Repeat until $\|\chi\| \leq tol$. 
Numerical Example

A stack of laminates exited by a surface current density.

- All dimensions in mm.
- $\mu_r = 1,000$, $\sigma = 2\text{MS/m}$
Numerical Example

∥B∥ in the cross-section.

Finite element orders (blue= 1, red= 9)

The algorithm correctly refines in areas of great variations.
Estimated Errors

- Using the proposed hierarchical error estimator yields an increased rate of convergence.
- The results are very similar for the 3\textsuperscript{rd} order MSFEM.
Conclusions

- A MSFEM formulation for the eddy current problem in 3D has been presented.

- A hierarchical error estimator has been applied to the MSFEM.

- Numerical examples show a significant speed-up using the HEE.

- Future work: Development of a HEE for MSFEM, which takes account of $\|u - u_{MS}\|$. 
Thank you for your attention!