SOLVING RESONANCE PROBLEMS ON UNBOUNDED DOMAINS USING FINITE ELEMENTS AND COMPLEX SCALING

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INTRODUCTION

A variety of waves, for example acoustic, elastic or electromagnetic waves can be modelled using partial differential equations. In general wave phenomena can be considered non-local, in a sense, that a local perturbation can lead to changes in distant regions. Therefore, a genuinely non-local problem cannot straightaway be reduced to a bounded one without altering it fundamentally. The Finite Element Method is a popular choice for solving Partial Differential Equations numerically. However, since this method bases on the decomposition of the computational domain into finitely many bounded subdomains it is not right away suitable for solving non-local problems on unbounded domains. To overcome these difficulties, we use the method of complex scaling or perfectly matched layers [1], [2]. This method introduces an artificial non-reflecting damping layer which is cut-off after a sufficiently large thickness. In this work we are concerned with using some varieties of this method to solve resonance problems.

PROBLEM SETTING

As a model problem we consider the Helmholtz equation on unbounded domains. The real part of an eigenfunction of the associated resonance problem can be interpreted as the amplitude of a time-harmonic acoustic wave. If the domain in question is not the whole space suitable boundary conditions have to be imposed. To ensure physical meaningful resonances an additional radiation condition, the pole condition is stated.

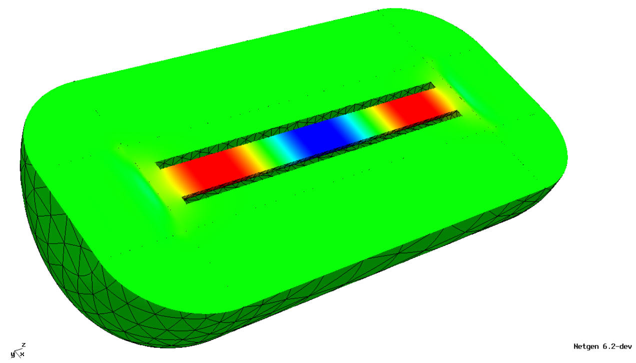

Figure 1: Cross section of an approximated resonance function of a pipe shaped domain using a cylindrical scaling

COMPLEX SCALING

To overcome the problem of the given infinite domain and realize the aforementioned radiation condition, we introduce complex scaling. This technique can be described in three steps. At first the initial domain is split up into a bounded *interior* and an unbounded *exterior* domain, such that all inhomogeneities of the geometry and (if given) the potential are located in the interior domain. Next a transformation of the initial equation in the exterior domain is introduced, such that the solution of the transformed equation coincides with the sought for solution in the interior domain. Moreover, only solutions that satisfy the radiation condition are exponentially decreasing in the exterior domain. Because of this decrease, the error arising from truncating the exterior domain in the last step can be shown to be small for a sufficiently thick exterior domain.

Since the truncated complex scaled equation is now stated on a bounded domain, the finite element method can be applied for discretization leading to a finite dimensional generalized eigenvalue problem, which can be solved using standard numerical methods. This is done using the high order finite element library Netgen/NGSolve.

RESULTS AND DISCUSSION

Theoretical analysis of the method shows, that the complex scaling has several effects on the spectrum of the initial problem including the appearance of an essential spectrum and spurious resonances. Numerical experiments show a heavy dependence of the essential spectrum and the spurious resonances on the numerous parameters of the complex scaling such as the shape of the scaling layer and the damping profile. Moreover the theoretical convergence rates [4], [5] are only achieved for sufficiently small mesh sizes $h \leq h_0$, where h_0 also depends on the parameters of the complex scaling and the resonance itself. In our work discuss these dependencies and how they can be reduced using frequency dependent complex scalings [3].

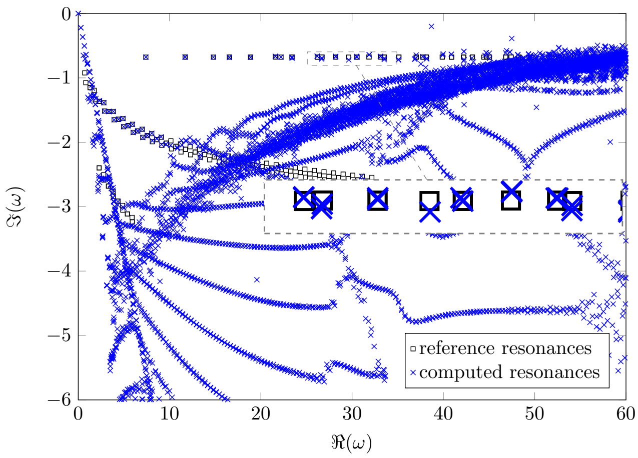

Figure 2: Spectrum of the discretized problem including correct resonances, the essential spectrum and spurious resonances

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