Discrete resonances of the complex scaled Helmholtz equation

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Abstract

Complex scaling is a popular method for treating scattering and resonance problems in open domains. For solving scattering problems it is common to use frequency dependent scaling parameters. Using similar ideas for resonance problems leads to non-linear eigenvalue problems. In this talk we analyze the discrete resonances of both, the frequency independent and the frequency dependent complex scaled Helmholtz equation.

Keywords: Helmholtz resonance problem, perfectly matched layer, spurious solution

Introduction

We are concerned with the approximation of eigenpairs $(\omega, u) \in \mathbb{C} \times H^1_{loc}(\mathbb{R}^n) \setminus \{0\}$ of the Helmholtz equation

$$-\Delta u(x) - \omega^2 (1 + p(x))^2 u(x) = 0, \qquad (1)$$

in $\mathbb{R}^n$. The function p is a given potential function such that there exists an open and simply connected interior domain $\Omega_{int} \subset \mathbb{R}^n$, with $\operatorname{supp} p \subset \Omega_{int}$. As radiation condition, we demand, that u satisfies

$$u(x) = \sum_{\nu=0}^{\infty} \sum_{k=0}^{l_\nu} c_{\nu,k} \mathcal{H}^{(1)}_\nu \left(\omega \|x\|\right) \Phi^n_{\nu,k} \left(\frac{x}{\|x\|}\right) \qquad (2)$$

for $x \in \mathbb{R}^n \setminus \overline{\Omega_{int}}$, where $\mathcal{H}^{(1)}_\nu$ are the (spherical) Hankel functions of the first kind, and $\Phi^n_{\nu,k}$ are the circular (spherical) harmonics.

To realize the radiation condition we use complex scalings of the form

$$\hat{x}(x) = x - d(x) + \tau_\omega \left(\|d(x)\|\right) d(x), \qquad (3)$$

for some continuous and piecewise smooth distance function $d : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$d(x) = 0, \qquad x \in \overline{\Omega_{int}}, \qquad (4)$$

$$x - d(x) \in \partial\Omega_{int}, \qquad x \in \mathbb{R}^n \setminus \Omega_{int}, \qquad (5)$$

and some continuous and piecewise smooth scaling function $\tau_w : \mathbb{R}^+ \cup \{0\} \to \mathbb{C}$, such that

$$\lim_{t\to\infty} \operatorname{Im} \tau_\omega(t) > 0, \quad \lim_{t\to\infty} \frac{\operatorname{Im} \tau_\omega(t)}{\operatorname{Re} \tau_\omega(t)} > 0. \qquad (6)$$

It can be shown, that for any given eigenpair (ω, u) of (1), the pair $(\omega, \hat{u})$, with the scaled function $\hat{u}(x) := u(\hat{x}(x)) \in H^1(\mathbb{R}^n)$ is an eigenpair of the complex scaled equation

$$-\hat{\Delta} \hat{u}(x) - \omega^2 (1 + p(x))^2 \hat{u}(x) = 0, \qquad (7)$$

where $\hat{\Delta} := J^T_{\hat{x}} \nabla \cdot J^T_{\hat{x}} \nabla$ and $J_{\hat{x}}(x)$ is the Jacobian of the scaling $\hat{x}(x)$.

The essential Spectrum of the unbounded problem

The simplest scaling is an affine, linear and frequency independent radial complex scaling, i.e. $\Omega_{int} := B_{R_0}(0)$ for some $R_0 > 0$ and

$$\hat{x}(x) := x + \chi_{\Omega^c_{int}}(x)(\sigma - 1)(\|x\| - R_0)\frac{x}{\|x\|} \qquad (8)$$

for $\sigma \in \mathbb{C}$, with $\operatorname{Im} \sigma > 0$. It can be shown, that the set

$$\Sigma_{ess} := \left\{\omega \in \mathbb{C} : \omega\sigma \in \mathbb{R}^+\right\}, \qquad (9)$$

is the essential spectrum of the complex scaled problem (7). Moreover the eigenvalues of the initial problem (1) and the complex scaled problem (7) coincide if they are located in the sector bounded by the positive real axis and Σ_{ess} (cf. [1]). For large $\|x\|$, the complex scaled resonance functions are approximately given by (2) where the argument of the Hankel functions is replaced by $\omega\sigma\|x\|$, indicating an exponential decay for $\|x\| \to \infty$.

Choosing a frequency dependent complex slope $\sigma(\omega)$ in (8) changes the shape of the essential spectrum of the resulting complex scaled problem. Particularly notable is the fact that for

$$\sigma(\omega) := \sigma_0 \omega^{-1},\ \sigma_0 \in \mathbb{C},\ \operatorname{Im} \sigma_0 > 0 \qquad (10)$$

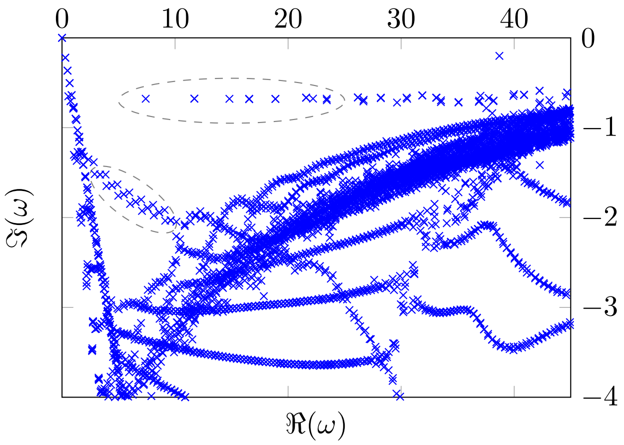

Figure 1: Computed resonances of a two dimensional resonance problem using $\sigma(\omega) \equiv 1 + 3i$.

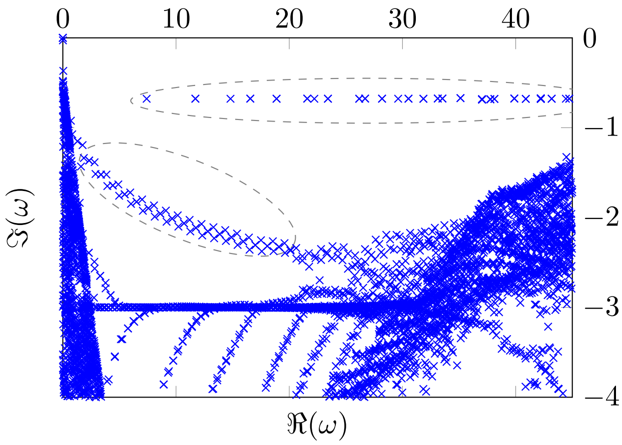

Figure 2: Computed resonances of a two dimensional resonance problem using $\sigma(\omega) := \frac{9+9i}{\omega}$.

the argument of the Hankel functions becomes asymptotically $\sigma_0\|x\|$. The problem has no essential spectrum in the lower complex half plane and the decay of the resonance functions is almost independent of the resonance ω.

In general, choosing a frequency dependent scaling function τ_ω leads to a non-linear eigenvalue problem. In case of a frequency scaled affine linear scaling however, the problem can be reduced to a polynomial eigenvalue problem.

Asymptotic results

The complex scaled problem (7) is discretized on a sufficiently large computational domain $\Omega_T := B_T(0)$ using high order finite element spaces $V_{h,p,T}$ of order p and meshsize h. Spectral convergence can be shown using results from [3] and [4] for sufficiently good discretizations. Hence, for all $h < h_\omega$ and $T > T_\omega$, there are no spurious eigenvalues in a small neighborhood of a resonance ω, and the error of the approximated eigenvalues decays with h^{2p} and exponentially with T. Nevertheless solving one eigenvalue problem on a too coarse discrete space might result in discrete eigenvalues, which are neither an approximation to the essential spectrum, nor approximations to the desired eigenvalues of (1), since the approximation quality of the discrete spaces to the complex scaled eigenfunctions $\hat{u}$ depends heavily on the frequency w.

Discretization resonances

Figures 1 and 2 show resonances of a discretized two dimensional transmission problem with piecewise constant potential function $p(x) := p_0\chi_{[-r,r]\times[-r,r]}(x)$, affine, linear complex scaling (8).

The set of discrete resonances can be separated into the approximated sought for outgoing resonances of (1) (marked by the dashed ellipses), the approximation of the essential spectrum and the aforementioned spurious resonances. Even though the unbounded problem in the frequency dependent case does not have an essential spectrum in the lower complex half-plane, due to truncation a corresponding set of discrete resonances appears, located on a horizontal line. An analysis of some easy accessible examples and numerical experiments in [2] show, that these spurious resonances, can be categorized into interior and exterior spurious resonances induced by the discretization errors of the interior and exterior domain respectively. The exterior spurious resonances depend heavily on the choice of the parameters of the complex scaling. Choosing a frequency dependent scaling (10) however, reduces this dependency as mentioned above. As Figures 1 and 2 indicate a clear improvement in the number of well approximated resonances it might pay off to invest into solving the polynomial eigenvalue problem arising from a frequency dependent scaling for more evolved examples, e.g. by using the contour integration method.

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