Complex scaled infinite elements for exterior Helmholtz problems

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1 Introduction

Please note, that the work at hand is a work-in-progress preprint.

Perfectly matched layers (PML) are a popular method for treating acoustic resonance and scattering problems in open domains (cf. [2, 13, 4]). The idea behind this method is the application of a complex coordinate stretching to the unbounded exterior domain to generate exponentially decaying outgoing solutions (complex scaling). Subsequently the exterior domain is truncated to obtain a bounded one. Then the finite element method is applied to the complex scaled equation on the now bounded domain.

PMLs are rather easy to implement in standard finite element codes but have the downside that there are many method parameters to choose: The scaling function, truncation of the exterior domain and the finite element discretization of the exterior. All this parameters have to be balanced to ensure efficiency of the method.

A different approach to such problems is the use of Hardy space infinite elements (cf. [9]). These infinite elements rely on a Laplace transformation in the unbounded spacial direction. Then the resulting equation is discretized using ansatz functions in a certain Hardy space. Hardy space infinite elements exhibit super-algebraic convergence in the number of unkowns. Moreover there are only two method parameters: the number of Hardy Space unknowns and one method parameter. On the downside Hardy space infinite elements are set in the unusual framework of the Laplace domain and allow no straightforward way of dealing with inhomogeneous exterior domains.

In this work we show how Hardy space infinite elements can be interpreted as infinite elements applied to the complex scaled equation. This interpretation omits the detour of the Laplace transform. It thus allows us to apply numerical integration, enabling us to treat inhomogeneous exterior domains as well. The remainder of the paper is organized as follows: In Section 2 we define the problems in question and give a brief explanation of the method of complex scaling. In Section 3 we explain the used tensor product exterior discretizations. The complex scaled infinite elements are defined in Section 4 and their connection to Hardy space infinite elements is explained in Subsection 4.5. In Section 5 we develop some results concerning the approximation by our ansatz functions. We conclude with a section consisting of numerical experiments to underline our theoretical findings.

2 Problem setting

Definition 2.1. For $d \in \{1, 2, 3\}$ let $\Omega \subset \mathbb{R}^d$ be an unbounded open domain such that Ω can be splitted into a bounded interior part Ω_{int} an unbounded exterior part Ω_{ext} and an interface Γ. $\Omega_{\text{int}}, \Omega_{\text{ext}}, \Gamma$ should fulfill the following assumptions:

(i) $\Omega = \Omega_{\text{int}} \dot\cup \Gamma \dot\cup \Omega_{\text{ext}}$,

(ii) there exists $R > 0$, such that $\Omega_{\text{int}} = \Omega \cap B_R(0)$, $\Omega_{\text{ext}} = \Omega \setminus \overline{\Omega_{\text{int}}}$ and $\Gamma = \{\mathbf{x} \in \Omega : \|x\| = R\}$, and

(iii) $\Omega_{\text{ext}} = \left\{ \left(1 + \frac{\xi}{R}\right) \hat{\mathbf{x}} : \hat{\mathbf{x}} \in \Gamma, \xi \in \mathbb{R}_{\geq 0} \right\}$.

Remark 2.2. Note, that these conditions imply that for each $\mathbf{x} \in \Omega_{\text{ext}} \cup \Gamma$ there exists a unique pair $(\xi, \hat{\mathbf{x}}) \in \mathbb{R}_{\geq 0} \times \Gamma$, such that

$$\mathbf{x} = \left(1 + \frac{\xi}{R}\right) \hat{\mathbf{x}}. \tag{1}$$

For the mapping defined by (1) we also write $\mathbf{x}(\xi, \hat{\mathbf{x}})$ and $\xi(\mathbf{x}), \hat{\mathbf{x}}(\mathbf{x})$ for the inverse mapping. In the case $d = 1$, we have $\hat{\mathbf{x}} \in \{-R, R\}$.

Definition 2.3 (Scattering and resonance problem)**.** Let $\Omega = \Omega_{\text{int}} \dot\cup \Gamma \dot\cup \Omega_{\text{ext}}$ be such that the conditions above hold. Moreover, let $p, f \in L_2(\Omega)$ such that $p|_{\Omega_{\text{ext}}} \equiv 1$ and $\operatorname{supp} f \subset \Omega_{\text{int}}$. For a fixed frequency $\omega \in \mathbb{C}$ we call the problem: Find $u \in C^2(\Omega)$ such that

$$\begin{aligned} -\Delta u(\mathbf{x}) - \omega^2 p(\mathbf{x})\, u(\mathbf{x}) = f(\mathbf{x})\,, \qquad & \mathbf{x} \in \Omega, \\ u \text{ fulfills some b.c.}, \qquad & \mathbf{x} \in \partial\Omega, \\ u \text{ is outgoing}, \qquad & \|\mathbf{x}\| \to \infty, \end{aligned} \tag{2}$$

the *Helmholtz scattering problem.* The problem: Find $\omega \in \mathbb{C}^- := \{z \in \mathbb{C} : \Im z \leq 0\}, u \in C^2(\Omega) \setminus \{0\}$, such that

$$\begin{aligned} -\Delta u(\mathbf{x}) = \omega^2 p(\mathbf{x})\, u(\mathbf{x})\,, \qquad & \mathbf{x} \in \Omega, \\ u \text{ fulfills some b.c.}, \qquad & \mathbf{x} \in \partial\Omega, \\ u \text{ is outgoing}, \qquad & \|\mathbf{x}\| \to \infty, \end{aligned} \tag{3}$$

is called the *Helmholtz resonance problem.*

In the following we will focus on the resonance problem.

2.1 Radiation condition

We call a solution u of (3) or (2) *outgoing* if it can be written in Ω_{ext} (i.e. for all $\xi \in \mathbb{R}_{\geq 0}, \hat{\mathbf{x}} \in \Gamma$) as

$$u(\mathbf{x}(\xi, \hat{\mathbf{x}})) = \begin{cases} \exp(i\omega \mathbf{x}(\xi, \hat{\mathbf{x}})), & d = 1, \\ \sum_{k=-\infty}^{\infty} \alpha_k H_{|k|}^{(1)}(\omega(R + \xi))\, \Phi_k\left(\frac{1}{R}\hat{\mathbf{x}}\right), & d = 2, \\ \sum_{k=0}^{\infty} \sum_{j=0}^{m_j} \beta_{k,j} h_k^{(1)}(\omega(R + \xi))\, Y_{k,j}\left(\frac{1}{R}\hat{\mathbf{x}}\right), & d = 3, \end{cases} \tag{4}$$

where $H_k^{(1)}$ are the Hankel functions of the first kind, $h_k^{(1)}$ the spherical Hankel functions of the first kind, Φ_k the cylindrical harmonics and $Y_{k,j}$ the spherical harmonics.

Remark 2.4. This also implies that an outgoing solution has an analytic continuation to $\mathbf{x}(\mathbb{C}, \Gamma)$. In the following we will use the symbol u for the analytic continuation as well.

2.2 Complex scaling

To incorporate (4) into our equation we use the technique of complex scaling. In this work we only consider linear complex scalings of the form

$$\tau(\xi) := \sigma\xi,$$
$$\gamma(\mathbf{x}(\xi, \hat{\mathbf{x}})) := \begin{cases} \mathbf{x}, & \mathbf{x} \in \Omega_{\text{int}}, \\ \mathbf{x}(\tau(\xi), \hat{\mathbf{x}}), & \mathbf{x} \in \Omega_{\text{ext}}, \end{cases}$$

for a given $\sigma \in \mathbb{C}$ with $\Im(\sigma) > 0$. We denote the Jacobian of the scaling by

$$J_\sigma(\mathbf{x}) = J_\sigma\big((x_1, \dots, x_d)^T\big) := \left(\frac{\partial \gamma_i\big((x_1, \dots, x_d)^T\big)}{\partial x_j} \right)_{i=1,\dots,d,j=1,\dots,d}.$$

Due to the fact, that the spherical Hankel functions are of the form $\exp(i\cdot)\,p(\cdot)$ for some polynomials p (cf. Definition 5.4) this gives for u of the form (4) that

$$\lim_{\xi\to\infty} u(\gamma(\mathbf{x}(\xi, \hat{\mathbf{x}}))) = 0.$$

2.3 Weak formulation

Since, that the complex scaled solution $u \circ \gamma$ decays exponentially for $\|\mathbf{x}\| \to \infty$, it is also square integrable and we can state a weak formulation of (2) using the following bilinear forms:

Definition 2.5. For $f, g \in H^1(\Omega_{\text{ext}})$ we define

$$\begin{aligned} m_{\text{int}}(f, g) &:= \int_{\Omega_{\text{int}}} p(\mathbf{x})\, f(\mathbf{x})\, g(\mathbf{x})\ d\mathbf{x}, \\ s_{\text{int}}(f, g) &:= \int_{\Omega_{\text{int}}} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x})\ d\mathbf{x}, \\ m^\sigma_{\text{ext}}(f, g) &:= \int_{\Omega_{\text{ext}}} f(\mathbf{x})\, g(\mathbf{x}) \det J_\sigma(\mathbf{x})\ d\mathbf{x}, \\ s^\sigma_{\text{ext}}(f, g) &:= \int_{\Omega_{\text{ext}}} \Big(J_\sigma(\mathbf{x})^{-T}\, \nabla f(\mathbf{x})\Big) \cdot \Big(J_\sigma(\mathbf{x})^{-T}\, \nabla g(\mathbf{x})\Big) \det J_\sigma(\mathbf{x})\ d\mathbf{x}. \end{aligned}$$

Problem 2.6. Find $u \in H^1(\Omega) \setminus \{0\}$, $\omega \in \mathbb{C}^-$, such that

$$s_{\text{int}}(u, v) + s^\sigma_{\text{ext}}(u, v) = \omega^2 \left(m_{\text{int}}(u, v) + m^\sigma_{\text{ext}}(u, v)\right), \tag{5}$$

for all $v \in H^1(\Omega)$.

Remark 2.7. The weakly formulated Problem 2.6 assumes homogeneous Neumann boundary conditions on $\partial\Omega$. For Dirichlet or mixed boundary conditions the problem has to be adapted accordingly.

2.4 Discrete formulation

Our goal is to discretize Problem 2.6. To this end we pick $\mathcal{N} \in \mathbb{N}$ and a family of functions $\mathcal{B}_\mathcal{N} := \{b_0, \dots, b_\mathcal{N}\} \subset H^1(\Omega)$ and define the discrete space $\mathcal{X}_\mathcal{N}$ by

$$\mathcal{X}_\mathcal{N} := \operatorname{span}(\mathcal{B}_\mathcal{N}) \subset H^1(\Omega)\,.$$

Defining the mass- and stiffness matrix by

$$\mathbf{M} := (m_{i,j})_{i,j=0,\dots,\mathcal{N}}, \quad \mathbf{S} := (s_{i,j})_{i,j=0,\dots,\mathcal{N}} \tag{6}$$

and

$$m_{i,j} = m_{\text{int}}(b_i, b_j) + m^\sigma_{\text{ext}}(b_i, b_j)\,, \quad s_{i,j} = s_{\text{int}}(b_i, b_j) + s^\sigma_{\text{ext}}(b_i, b_j) \tag{7}$$

respectively, we can formulate the discrete problem by

Problem 2.8. Find $(\omega, \mathbf{u}) \in \mathbb{C}^- \times \mathbb{C}^\mathcal{N} \setminus \{0\}$, such that

$$\mathbf{Su} = \omega^2 \mathbf{Mu}.$$

The discrete Problem 2.8 can be solved using standard eigenvalue solvers. The task for the remaining chapters will be to find a suitable basis $\mathcal{B}_\mathcal{N}$.

3 Tensor product exterior discretizations

In this section we will exploit the inherent structure of the exterior domain to find a simple way of discretizing it without having to mesh it explicitly. To simplify notation we will focus on the case $d = 3$ only.

For the remainder of this chapter we will assume that $\varphi : M \to \Gamma$ is a diffeomorphism for some open set $M \subset \mathbb{R}^2$.

Lemma 3.1. *We can calculate the Jacobian of the coordinate transformation*

$$\Psi_\varphi : \begin{cases} \mathbb{R}_{\geq 0} \times M & \to \Omega_{\text{ext}} \cup \Gamma, \\ (\xi, \eta) & \mapsto \left(1 + \frac{\xi}{R}\right) \varphi(\eta)\,, \end{cases}$$

its inverse, and its determinant by

$$\begin{aligned} D\Psi_\varphi(\xi, \eta) &= \left(\frac{1}{R} \varphi(\eta)\,, \left(1 + \frac{\xi}{R}\right) D\varphi(\eta) \right), \\ (D\Psi_\varphi(\xi, \eta))^{-1} &= \begin{pmatrix} \frac{1}{R} \varphi(\eta)^T \\ \frac{1}{1+\frac{\xi}{R}} D\varphi(\eta)^\dagger \end{pmatrix}, \\ |\det D\Psi_\varphi(\xi, \eta)\,| &= \left(1 + \frac{\xi}{R}\right)^2 \sqrt{\left|\det\left(D\varphi(\eta)^T D\varphi(\eta)\right)\right|}, \end{aligned}$$

where $A^\dagger := \left(A^T A\right)^{-1} A^T$ is the pseudo inverse of a matrix $A \in \mathbb{C}^{3\times 2}$ with full rank.

Proof. The Jacobian can be obtained by straightforward differentiation.

Its inverse can be easily verified using the facts that $\varphi(\eta)^T D\varphi(\eta) = 0$ and $D\varphi(\eta)^\dagger D\varphi(\eta) = I$.

For obtaining the determinant we calculate

$$\begin{aligned}
\det\bigl(D\Psi_\varphi^T D\Psi_\varphi\bigr) &= \det\left(\begin{pmatrix} \frac{1}{R}\varphi(\eta)^T \\ \left(1+\frac{\xi}{R}\right) D\varphi(\eta)^T \end{pmatrix} \left(\frac{1}{R}\varphi(\eta)\,, \left(1+\frac{\xi}{R}\right) D\varphi(\eta)\right)\right) \\
&= \det\begin{pmatrix} 1 & 0 \\ 0 & \left(1+\frac{\xi}{R}\right)^2 D\varphi(\eta)^T D\varphi(\eta) \end{pmatrix} \\
&= \left(1+\frac{\xi}{R}\right)^4 \det\Bigl(D\varphi(\eta)^T D\varphi(\eta)\Bigr)\,.
\end{aligned}$$

By taking the square root we obtain the desired result. □

Definition 3.2. Let $M \subset \mathbb{R}^2$ and $\varphi : M \to \Gamma$ be a diffeomorphism. Then we define the surface gradient on of a function $f : \Gamma \to \mathbb{C}$ by

$$\hat{\nabla} f(\varphi(\eta)) := \Bigl(D\varphi(\eta)^\dagger\Bigr)^T \nabla_\eta (f \circ \varphi)(\eta)\,.$$

Remark 3.3. It can be shown that the surface gradient $\hat{\nabla}$ defined above is independent of the specific embedding φ.

Theorem 3.4. *Let $f, g \in H^1(\Omega_{\text{ext}})$ and $\breve{f}(\xi, \hat{\mathbf{x}}) := f(\mathbf{x}(\xi, \hat{\mathbf{x}}))$, $\breve{g}(\xi, \hat{\mathbf{x}}) := g(\mathbf{x}(\xi, \hat{\mathbf{x}}))$. Then the exterior bilinear forms from Definition 2.5 can be rewritten in coordinates $\xi, \hat{\mathbf{x}}$ by*

$$\begin{aligned}
m_{\text{ext}}^\sigma(f,g) &= \sigma \int_{\mathbb{R}_{\geq 0}\times\Gamma} \breve{f}(\xi,\hat{\mathbf{x}})\, \breve{g}(\xi,\hat{\mathbf{x}}) \left(1+\frac{\sigma\xi}{R}\right)^2 d(\xi,\hat{\mathbf{x}}), \\
s_{\text{ext}}^\sigma(f,g) &= \frac{1}{\sigma} \int_{\mathbb{R}_{\geq 0}\times\Gamma} \frac{\partial \breve{f}}{\partial \xi}(\xi,\hat{\mathbf{x}})\, \frac{\partial \breve{g}}{\partial \xi}(\xi,\hat{\mathbf{x}}) \left(1+\frac{\sigma\xi}{R}\right)^2 d(\xi,\hat{\mathbf{x}}), \\
&\quad + \sigma \int_{\mathbb{R}_{\geq 0}\times\Gamma} \hat{\nabla}\breve{f}(\xi,\hat{\mathbf{x}})\, \hat{\nabla}\breve{g}(\xi,\hat{\mathbf{x}})\; d(\xi,\hat{\mathbf{x}}),
\end{aligned}$$

where integration over Γ of a function $h : \Gamma \to \mathbb{C}$ means integration by the surface measure i.e.

$$\int_{\varphi(M)} h(\hat{\mathbf{x}})\; d\hat{\mathbf{x}} := \int_M h(\varphi(\eta)) \sqrt{\left|\det\Bigl(D\varphi(\eta)^T D\varphi(\eta)\Bigr)\right|}\, d\eta.$$

Proof. Using the determinant calculated in Lemma 3.1 and the fact that

$$\gamma(\Psi_\varphi(\xi,\eta)) = \Psi_\varphi(\sigma\xi,\eta)\,,$$

we obtain

$$\begin{aligned}
D(\gamma\circ\Psi_\varphi)(\xi,\eta) &= D\Psi_\varphi(\sigma\xi,\eta)\begin{pmatrix} \sigma & 0 \\ 0 & \mathrm{Id}_2 \end{pmatrix}, \\
(D(\gamma\circ\Psi_\varphi)(\xi,\eta))^{-1} &= \begin{pmatrix} \frac{1}{\sigma} & 0 \\ 0 & \mathrm{Id}_2 \end{pmatrix} (D\Psi_\varphi(\sigma\xi,\eta))^{-1}\,, \\
\det D(\gamma\circ\Psi_\varphi)(\xi,\eta) &= \sigma \det D\Psi_\varphi(\sigma\xi,\eta)\,.
\end{aligned}$$

After applying the transformation rule we immediately obtain the formula for m^σ_{ext}.

For the formula for s^σ_{ext} we calculate

$$\begin{aligned}
\nabla f(\Psi_\varphi(\xi,\eta)) &= (D(\gamma\circ\Psi_\varphi)(\xi,\eta))^{-T}\,\nabla_{\xi,\eta}\,(f\circ\Psi_\varphi)(\xi,\eta)\\
&= \left(\frac{1}{\sigma R}\varphi(\eta)\,,\frac{1}{1+\frac{\xi\sigma}{R}}\left(D\varphi(\eta)^\dagger\right)^T\right)\begin{pmatrix}\frac{\partial f\circ\Psi_\varphi}{\partial\xi}(\xi,\eta)\\ \nabla_\eta(f\circ\Psi_\varphi)(\xi,\eta)\end{pmatrix}\\
&= \frac{1}{\sigma R}\varphi(\eta)\,\frac{\partial\breve{f}}{\partial\xi}(\xi,\varphi(\eta)) + \frac{1}{1+\frac{\sigma\xi}{R}}\hat{\nabla}\breve{f}(\xi,\varphi(\eta))\,.
\end{aligned}$$

Plugging this into the integral and applying the transformation rule leads to the desired result. □

3.1 Exterior spaces

Let

$$\begin{aligned}
\tilde{\mathcal{B}}_N &:= \{\phi_j : j=0,\dots,M\}\subset H^1(\mathbb{R}_{\geq 0})\,,\\
\hat{\mathcal{B}}_M &:= \{b_j : j=0,\dots,M\}\subset H^1(\Gamma)\,,
\end{aligned}$$

be families of linearly independent functions. Then we define discrete spaces on $\mathbb{R}_{\geq 0}$ and Γ respectively by

$$\tilde{\mathcal{X}}_N := \operatorname{span}\left(\tilde{\mathcal{B}}_N\right)\subset H^1(\mathbb{R}_{\geq 0})\,,$$

and

$$\hat{\mathcal{X}}_M := \operatorname{span}\left(\hat{\mathcal{B}}_M\right)\subset H^1(\Gamma)\,.$$

To discretize the exterior problem, we use a tensor product space of the form

$$\tilde{\mathcal{X}}_N\otimes\hat{\mathcal{X}}_M := \operatorname{span}\{\phi(\xi)\,b(\hat{\mathbf{x}}) : \phi\in\tilde{\mathcal{B}}_N, b\in\hat{\mathcal{B}}_M\}.$$

To obtain the entries of the mass and stiffness matrix defined in (6) and (7), we need to evaluate the exterior bilinear forms for all basis functions. Since our basis functions are composed of a radial and a tangential part, we can decompose the bilinear forms accordingly and obtain for $\tilde{f},\tilde{g}\in H^1(\mathbb{R}_{\geq 0})$ and $\hat{f},\hat{g}\in H^1(\Gamma)$

$$\begin{aligned}
m^\sigma_{\text{ext}}\left(\tilde{f}\hat{f},\tilde{g}\hat{g}\right) &= \tilde{m}^\sigma_1\left(\tilde{f},\tilde{g}\right)\hat{m}\left(\hat{f},\hat{g}\right),\\
s^\sigma_{\text{ext}}\left(\tilde{f}\hat{f},\tilde{g}\hat{g}\right) &= \tilde{s}^\sigma\left(\tilde{f},\tilde{g}\right)\hat{m}\left(\hat{f},\hat{g}\right) + \tilde{m}^\sigma_0\left(\tilde{f},\tilde{g}\right)\hat{s}\left(\hat{f},\hat{g}\right),
\end{aligned}$$

with

$$\begin{aligned}
\tilde{m}^\sigma_0\left(\tilde{f},\tilde{g}\right) &= \sigma\int_0^\infty \tilde{f}(\xi)\,\tilde{g}(\xi)\;d\xi,\\
\tilde{m}^\sigma_1\left(\tilde{f},\tilde{g}\right) &= \sigma\int_0^\infty \left(1+\frac{\sigma\xi}{R}\right)^2\tilde{f}(\xi)\,\tilde{g}(\xi)\;d\xi,\\
\tilde{s}^\sigma\left(\tilde{f},\tilde{g}\right) &= \frac{1}{\sigma}\int_0^\infty \left(1+\frac{\sigma\xi}{R}\right)^2\tilde{f}'(\xi)\,\tilde{g}'(\xi)\;d\xi,\\
\hat{m}\left(\hat{f},\hat{g}\right) &= \int_\Gamma \hat{f}(\hat{\mathbf{x}})\,\hat{g}(\hat{\mathbf{x}})\;d\hat{\mathbf{x}},\\
\hat{s}\left(\hat{f},\hat{g}\right) &= \int_\Gamma \hat{\nabla}\hat{f}(\hat{\mathbf{x}})\cdot\hat{\nabla}\hat{g}(\hat{\mathbf{x}})\;d\hat{\mathbf{x}}.
\end{aligned}$$

A usual PML approach in this tensor product setting would be to truncate the set $\mathbb{R}_{\geq 0}$ to some finite interval $[0,T]$ for $T>0$ and to use

$$\tilde{\mathcal{X}}_N \subset H_0^1([0,T]) := \{f \in H^1(\mathbb{R}_{\geq 0}) : f(x) = 0, x \geq T\}.$$

Differing from this approach, we will choose basis functions with infinite support to omit truncation and ensure faster convergence. Our requirements to the basis functions ϕ_j and the discrete space $\tilde{\mathcal{X}}_N$ are:

(R 1) The basis functions ϕ_j should be easy to evaluate numerically stable,

(R 2) the radial part of the solution should be well approximated by functions from $\tilde{\mathcal{X}}_N$,

(R 3) it should be easy to couple the interior to the exterior problem,

(R 4) the integrals $\int_{\mathbb{R}_{\geq 0}} p(\xi)\, \phi_i(\xi)\, \phi_j(\xi)\; d\xi$ and $\int_{\mathbb{R}_{\geq 0}} p(\xi)\, \phi_i'(\xi)\, \phi_j'(\xi)\; d\xi$ should be easy to compute (numerically), for polynomials p,

(R 5) the discretization matrices should be sparse, and

(R 6) the condition numbers of the discretization matrices should behave well for large N.

4 Infinite elements based on complex scaling

4.1 Interior and interface discretization

For discretizing the interior domain Ω_{int} basically any discrete space $\mathcal{X}_{\text{int}} = \text{span}\{b_j : j = 0, \dots, L\} \subset H^1(\Omega_{\text{int}})$ such that $\mathcal{X}_{\text{int}}|_\Gamma := \{f|_\Gamma : f \in \mathcal{X}_{\text{int}}\} \subset H^1(\Gamma)$ can be used. The trace space of this interior discrete space is then used for the interface discretization (cf. 3), i.e.

$$\hat{\mathcal{X}}_M := \mathcal{X}_{\text{int}}|_\Gamma = \text{span}\{b_j|_\Gamma : j = 0, \dots, L\} \subset H^1(\Gamma)\,.$$

Remark 4.1. In our examples we will choose $\mathcal{X}_{\text{int}}$ as a standard high order conforming finite element space. Since in this case all of the basis functions corresponding to inner nodes in Ω_{int} will be zero on the interface Γ, we expect the dimension of $\hat{\mathcal{X}}_M$ to be much smaller than the dimension of $\mathcal{X}_{\text{int}}$.

4.2 Radial discretization

For the radial discretization we use the space of generalized Laguerre functions which we define as follows:

Definition 4.2. For $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$, we define the generalized Laguerre polynomials by

$$L_{n,m}(x) := \sum_{k=0}^{n} \binom{n+m}{n-k} \frac{(-x)^k}{k!}.$$

Further we define the generalized Laguerre functions by

$$\phi_{n,m}(x) := \exp(-x)\, L_{n,m}(2x). \tag{8}$$

We will shorten notation by writing $\phi_n := \phi_{n,0}$ and $L_n := L_{n,0}$. Moreover we define the radial discrete space by

$$\tilde{\mathcal{X}}_N := \text{span}\,\{\phi_n : n = 0, \dots, N\}\,.$$

We proceed to study whether the basis functions defined in Definition 4.2 satisfy our requirements (R 1)-(R 6). To this end, we state a few properties of the generalized Laguerre functions.

Lemma 4.3 (properties of the generalized Laguerre functions)**.**

(i) *For* $k, n \in \mathbb{N}_0$, $m \in \mathbb{Z}$ *the functions* $\phi_{n,m} \in H^k(\mathbb{R}_{\geq 0})$ *and*

$$(\phi_n, \phi_k)_{L_2(\mathbb{R}_{\geq 0})} = \frac{1}{2}\delta_{n,k}.$$

The functions $\left\{\frac{1}{\sqrt{2}}\phi_k, k \in \mathbb{N}_0\right\}$ *form a complete orthonormal system of* $L_2(\mathbb{R}_{\geq 0})$.

(ii) *For* $n \in \mathbb{N}, m \in \mathbb{Z}$,

$$\phi_{n,m-1} = \phi_{n,m} - \phi_{n-1,m}.$$

(iii) *For* $k \in \mathbb{N}_0$, $x \in \mathbb{C}$

$$\frac{d^k}{dx^k}\phi_{n,m}(x) = (-1)^k \phi_{n-k,m+k}(x)\,.$$

(iv) *For* $n \in \mathbb{N}_0, m \in \mathbb{Z}$, $x \in \mathbb{C}$

$$L_{n,m}(x) = \frac{\exp(x)}{x^m n!}\frac{d^n}{dx^n}\left(\exp(-x)x^{n+m}\right).$$

(v) *For* $n, k, l \in \mathbb{N}$, $p \in \Pi_n$ *and* $|l-k| > n$

$$(p\phi_l, \phi_k)_{L_2(\mathbb{R}_{\geq 0})} = 0.$$

(vi) *For* $m \in \mathbb{Z}$, $N \in \mathbb{N}$,

$$\tilde{\mathcal{X}}_N = \operatorname{span}\{\phi_{n,m} : n = 0, \ldots, N\}$$

(vii) *For* $j \in \mathbb{N}_0$

$$\phi_j(0) = \delta_{0,j}.$$

(viii) *For* $k \in \mathbb{N}_{\geq 2}$, $x \in \mathbb{C}$

$$k\phi_k(x) = (2k - 1 - 2x)\phi_{k-1}(x) - (k-1)\phi_{k-2}(x). \tag{9}$$

(ix) For $t, x \in \mathbb{C}$, $|t| < 1$

$$\sum_{k=0}^{\infty} L_k(x) t^k = \frac{\exp\left(-\frac{tx}{1-t}\right)}{1-t}.$$

Proof. All of the statements are easily checked by the reader and can be found e.g. in [1, Chapter 22] $\square$

Remark 4.4. Item (vii) of Lemma 4.3 shows, that only the first radial basis function has to be coupled to an interior basis function i.e. (R 3) is fulfilled. Moreover items (i) and (v) together with (iii) and (ii) tell us that the resulting matrices will be sparse ((R 5))

4.3 Coupling the interior and exterior problems

Since we want to create a conforming discrete space for the whole problem we need to couple our interior and exterior discrete spaces in a manner such that the resulting space is a equivalent to a subspace of $H^1(\Omega)$. We achieve this by using

$$\mathcal{Y} := \left\{ (u_{\text{int}}, u_{\text{ext}}) : u_{\text{int}} \in \mathcal{X}_{\text{int}}, u_{\text{ext}} \in \mathcal{X}_{\mathcal{N}}, u_{\text{int}}|_{\Gamma} = u_{\text{ext}}(0, \cdot) \right\}.$$

With an embedding defined by

$$\iota : \begin{cases} \mathcal{Y} & \to H^1(\Omega) \\ \iota((u,v))(\mathbf{x}) & := \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega_{\text{int}} \\ v(\xi(\mathbf{x}), \hat{\mathbf{x}}(\mathbf{x})), & \mathbf{x} \in \Omega_{\text{ext}} \end{cases} \end{cases}$$

we have

$$\iota(\mathcal{Y}) \subset H^1(\Omega).$$

To obtain a basis of $\mathcal{Y}$ we have to couple basis functions, such that the resulting functions are continuous. This can be done by identifying an interior basis function b_j with non-vanishing trace on Γ with the exterior basis function $b_j \otimes \phi_0$.

Remark 4.5. Due to the tensor product structure of the exterior space the parts of $\mathbf{S}$ and $\mathbf{M}$ that correspond to the exterior domain can be assembled by calculating the radial and interface part seperately and tensorizing them appropriately.

4.4 Stable evaluation and numerical integration

The generalized Laguerre functions can be evaluated numerically stable by using the recursion given in Lemma 4.3(viii). We use Gauss rules for $(0, \infty)$ with weighting function $\exp(-\cdot)$ to obtain exact quadrature rules for the Laguerre functions (see [18, Chapter 7.1.2]).

Remark 4.6. This enables us to also deal with inhomogeneous potentials in the exterior domain which is not possible in a straightforward way using classical Hardy Space infinite elements.

4.5 Comparison to the Hardy Space infinite element method

The Hardy space infinite element method introduced in [9] uses the so called pole condition [17, 16, 11] as radiation condition. In its standard form, this pole condition is equivalent to the radiation condition underlying the complex scaling, which is equivalent to the radiation condition of Sec. 2.1 for certain domains of complex frequencies including positive frequencies (see [12] or more explicitly for waveguides in [10, 8]).

The Hardy space infinite element method is a tensor product method as introduced in Sec. 3. But since the pole condition characterizes radiating solutions of the Helmholtz equation by the poles or singularities of their Laplace transform, the discretization in the radial direction is done for the Laplace transformed function. The basis functions are elements of certain Hardy spaces such that they satisfy the pole condition. In order to use these basis functions, the Helmholtz equation has to be transformed into the Laplace domain leading to quite unusual variational formulations in unusual Hilbert spaces. Nevertheless, it is a pure Galerkin method.

For a comparison with the complex scaled infinite elements of this paper, Section 4.2 of [9] is of importance. In this section, the Hardy space variational formulation is related to a complex scaled variational formulation via a Fourier transform. If the isomorphism $\mathcal{Q}$ defined there is applied to the Hardy space basis functions from [9, Sec. 2.4], we arrive at the generalized Laguerre functions of the preceding subsections. Hence, the discretization matrices of the Hardy space infinite element method are exactly the same as those of the complex scaled infinite elements.

For the Helmholtz equation with homogeneous exterior domain the complex scaled infinite element method is therefore exactly identical to the standard Hardy space infinite element method. Only the functional setting and the theoretical justification is different. There are two situations, where the two methods differ. If the exterior domain is inhomogeneous with coefficient functions depending on the radius, the Hardy space infinite element method is complicated to use due to the involved Laplace transform. Nevertheless, inhomogeneous exterior problems with dependencies only on the surface variable can be solved with the pole condition framework as well (see [14]). On the other hand, the two pole Hardy space method introduced in [6] uses a more complicated form of the pole condition, which is not equivalent to a standard complex scaling radiation condition. So e.g. for elastic waveguide problems with different signs of group and phase velocity, the Hardy space infinite element method of [7] cannot be reinterpreted directly as a complex scaled infinite element method.

For problems, where the two methods are essentially identical, the convergence results in [9, 5] can be used for complex scaled infinite elements as well. Nevertheless, in the following section we present more detailed approximation results for the infinite element method, which have not been derived so far. They may help choosing appropriate method parameters in practice.

5 Convergence results

Since the error of a Galerkin approximation depends on the best approximation error of the solution, we will derive estimates for the best approximation error in this section. In [18, Chapter 7.3] it is shown that the error of interpolation by Laguerre functions is of order $N^{-\frac{1}{2}}$, where N is the order of Laguerre functions. Although this implies super algebraic convergence of our method, it does not help us in choosing optimal parameters. Therefore, in this section we will derive estimates depending on the method parameters σ and R and the frequency ω.

5.1 Best approximation in one dimension

Theorem 5.1. *For $b \in \mathbb{C}$, $\Re(b) > -1$ and $n \in \mathbb{N}_0$,*

$$\int_0^\infty \exp(-bx)\,\phi_n(x)\,dx = \frac{(b-1)^n}{(b+1)^{n+1}}.$$

For $\Re(b) > 0$ and $x \in \mathbb{R}_{\geq 0}$, we have

$$\exp(-bx) = \frac{2}{b+1}\sum_{k=0}^{\infty}\left(\frac{b-1}{b+1}\right)^k \phi_k(x).$$

Moreover, the $L^2(\mathbb{R}_{\geq 0})$-orthogonal projection onto $\tilde{\mathcal{X}}_N$ of $\exp(-b\cdot)$ *is given by*

$$\Pi_N \exp(-b\cdot) = \frac{2}{b+1}\sum_{k=0}^{N}\left(\frac{b-1}{b+1}\right)^k \phi_k(\cdot)\,.$$

Proof. It is easily shown by partial integration and induction over j, that for $j \in \mathbb{N}, j \leq n+1$

$$\int_0^\infty \exp(-bx)\,\phi_n(x)\,dx = \\ \frac{1}{b+1}\sum_{k=0}^{j-1}\left(\frac{2}{b+1}\right)^k L_n^{(k)}(0) + \left(\frac{2}{b+1}\right)^j \int_0^\infty \exp(-x(b+1))\,L_n^{(j)}(2x)\;dx.$$

For $j = n+1$ we obtain

$$\begin{aligned}
\int_0^\infty \exp(-bx)\,\phi_n(x)\,dx &= \frac{1}{b+1}\sum_{k=0}^{n}\left(\frac{2}{b+1}\right)^k L_n^{(k)}(0) \\
&\overset{4.3(viii)}{=} \frac{1}{b+1}\sum_{k=0}^{n}\left(-\frac{2}{b+1}\right)^k L_{n-k,k}(0) \\
&= \frac{1}{b+1}\sum_{k=0}^{n}\left(-\frac{2}{b+1}\right)^k \binom{n}{k} \\
&= \frac{1}{b+1}\left(1-\frac{2}{b+1}\right)^n \\
&= \frac{(b-1)^n}{(b+1)^{n+1}}.
\end{aligned}$$

If $\Re(b) > 0$ we have $\exp(b\cdot) \in L_2(\mathbb{R}_{\geq 0})$. Since $\{\phi_k, k \in \mathbb{N}_0\}$ is a complete orthogonal system of $L^2(\mathbb{R}_{\geq 0})$, we have

$$\exp(-bx) = \sum_{k=0}^{\infty}\frac{(\phi_k, \exp(-b\cdot))_{L_2(\mathbb{R}_{\geq 0})}}{(\phi_k, \phi_k)_{L_2(\mathbb{R}_{\geq 0})}}\phi_k(x) = \frac{2}{b+1}\sum_{k=0}^{\infty}\left(\frac{b-1}{b+1}\right)^k \phi_k(x)\,.$$

$\square$

Corollary 5.2. *For $b \in \mathbb{C}$, $\Re(b) > 0$ and $N \in \mathbb{N}_0$*

$$\inf_{u_N \in \tilde{\mathcal{X}}_N} \|\exp(-b\cdot) - u_N\|_{L_2(\mathbb{R}_{\geq 0})} \leq \|(I - \Pi_n)\exp(-b\cdot)\|_{L_2(\mathbb{R}_{\geq 0})} = \frac{1}{\sqrt{2\Re b}}\left|\frac{b-1}{b+1}\right|^{N+1}.$$

Proof.

$$\begin{aligned}
\|(I - \Pi_N)\exp(-b\cdot)\|^2_{L^2(\mathbb{R}_{\geq 0})} &= \left\|\frac{2}{b+1}\sum_{k=N+1}^{\infty}\left(\frac{b-1}{b+1}\right)^k \phi_k\right\|^2_{L^2(\mathbb{R}_{\geq 0})} \\
&= \left|\frac{2}{b+1}\right|^2 \sum_{k=N+1}^{\infty}\left|\frac{b-1}{b+1}\right|^{2k} \|\phi_k\|^2_{L^2(\mathbb{R}_{\geq 0})} \\
&= 2\left|\frac{(b-1)^{N+1}}{(b+1)^{N+2}}\right|^2 \sum_{k=0}^{\infty}\left|\frac{b-1}{b+1}\right|^{2k} \\
&= 2\left|\frac{(b-1)^{N+1}}{(b+1)^{N+2}}\right|^2 \frac{1}{1-\left|\frac{b-1}{b+1}\right|^2} \\
&= \left|\frac{b-1}{b+1}\right|^{2N+2}\frac{1}{2\Re b}.
\end{aligned}$$

□

Remark 5.3. Because of the representation of the solutions in the exterior

$$u_{\text{ext}}(\xi) = \exp(\pm i\omega R)\exp(i\omega\sigma\xi)$$

for $d = 1$ Theorem 5.1 and Corollary 5.2 (with $b = -i\sigma\omega$) state, that the approximation by Laguerre functions in the L^2-norm depends on the quantity $\left|\frac{1+i\omega\sigma}{1-i\omega\sigma}\right|$. It is exact if $\omega\sigma = i$. In particular we have for $\Im(\sigma\omega) > 0$

$$\inf_{u_h \in \tilde{\mathcal{X}}_N} \|\exp(i\sigma\omega\cdot) - u_h\|_{L^2(\mathbb{R}_{\geq 0})} \leq \frac{1}{\sqrt{2\Im(\sigma\omega)}}\left|\frac{1+i\sigma\omega}{1-i\sigma\omega}\right|^{N+1}.$$

5.2 Best approximation of the zeroth spherical Hankel function

Definition 5.4. For $n \in \mathbb{N}_0$, the spherical Hankel functions of the first kind $h_n^{(1)}$ can be defined by

$$h_n(\xi) := -\frac{i}{\xi}\exp(i\xi)\tilde{h}_n(\xi),$$

with

$$\tilde{h}_n(\xi) := (-i)^n \sum_{m=0}^{n} \frac{i^m}{m!(2\xi)^m}\frac{(n+m)!}{(n-m)!)}.$$

To simplify the notation we will omit the superscript (1) and simply write $h_n := h_n^{(1)}$.

Suppose we want to approximate

$$h_0(\omega R + i\xi) = \frac{-\exp(i\omega R)\exp(-\xi)}{-i\omega R + \xi},$$

using our basis functions ϕ_j. This would be the case if we applied a frequency dependent complex scaling $\sigma(\omega) = \frac{i}{\omega}$ (cf. [15]). Then the approximation error will be governed by the terms $\left(\frac{\exp(-\cdot)}{a+\cdot}, \phi_n\right)_{L_2\left(\mathbb{R}_{\geq 0}\right)}$, for $a = -i\omega R$. This motivates the following definition.

Definition 5.5. For $a \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $n, k \in \mathbb{N}_0$, we define

$$\alpha_{n,k}(a) := \int_0^\infty \frac{\exp(-\xi)}{(a+\xi)^k} \phi_n(\xi)\, d\xi,$$

The following lemma shows, that the numbers $\alpha_{n,1}(a)$ can be calculated by a simple integral.

Lemma 5.6. *For $a \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $n \in \mathbb{N}_0$, we have*

$$\alpha_{n,1}(a) = \int_0^\infty \frac{\xi^n \exp(-\xi)}{(2a+\xi)^{n+1}}\, d\xi$$

The numbers $\alpha_{n,1}(a)$ are the coefficients of the expansion of $\frac{\exp(-\cdot)}{a+\cdot}$ in the Laguerre functions ϕ_n and therefore

$$\frac{\exp(-\xi)}{a+\xi} = 2\sum_{k=0}^\infty \alpha_{k,1}(a)\phi_k(\xi).$$

Proof. It is easily shown by partial integration and induction in j, that for $j \leq n$

$$\int_0^\infty \frac{t^n \exp(-t)}{(2a+t)^{n+1}}\, dt = \frac{(n-j)!}{n!} \int_0^\infty \frac{1}{(2a+t)^{n+1-j}} \frac{d^j}{dt^j} \left(\exp(-t)\, t^n\right)\, dt.$$

For $j = n$ we obtain

$$\begin{aligned}
\int_0^\infty \frac{t^n \exp(-t)}{(2a+t)^{n+1}}\, dt &= \frac{1}{n!} \int_0^\infty \frac{1}{2a+t} \frac{d^n}{dt^n} \left(\exp(-t)\, t^n\right)\, dt \\
&\overset{4.3(iv)}{=} \int_0^\infty \frac{1}{2a+t} \exp(-t)\, L_n(t)\, dt \\
&= \int_0^\infty \frac{2\exp(-t)}{2a+2t} \phi_n(t)\, dt = \alpha_{n,1}(a)\,.
\end{aligned}$$

$\square$

The following theorem gives an asymptotic expansion of the terms $\alpha_{n,1}(a)$ with respect to n.

Theorem 5.7 (asymptotic behavior of $\alpha_{n,1}$)**.** *For $a \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$*

$$\alpha_{n,1}(a) = \exp\left(a - 2\sqrt{2a(n+1)}\right) \frac{\sqrt{\pi}}{(2a(n+1))^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n+1}}\right)\right), \qquad n \to \infty.$$

The symbols $\sqrt{z}$ and $z^{\frac{1}{4}}$ for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ assume their respective principal values (their image is symmetric with respect to the positive real axis).

Proof. Lemma 5.6 states that

$$\alpha_{n,1}(a) = n!\mathrm{U}(n+1,1,2a)\,,$$

where for $n \in \mathbb{N}_0$, $a \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$$U(n+1,m,a) = \frac{a^{1-m}}{n!} \int_0^\infty \frac{t^n \exp(-t)}{(a+t)^{n+2-m}}\,dt.$$

The function U is called *confluent hypergeometric function of the second kind.* Using (10.3.39) and (9.1.3) in [19] we obtain

$$n!U(n+1,1,2a) = 2\exp(a)\left(K_0\Big(2\sqrt{2a(n+1)}\Big) + \mathcal{O}\Big(\frac{1}{\sqrt{n+1}}\Big)\right),$$

for $n \to \infty$ and

$$K_n(z) = \sqrt{\frac{\pi}{2z}}\exp(-z)\left(1 + \mathcal{O}\Big(\frac{1}{z}\Big)\right),$$

for $|z| \to \infty$. All in all we obtain

$$\begin{aligned}\alpha_{n,1}(a) &= n!U(n+1,1,2a)\\ &= \sqrt{\pi}(2a(n+1))^{-\frac{1}{4}}\exp\Big(a - 2\sqrt{2a(n+1)}\Big)\left(1 + \mathcal{O}\Big(\frac{1}{\sqrt{n+1}}\Big)\right),\end{aligned}$$

for $n \to \infty$. □

Using the lemma above we can now bound the best approximation error of $h_0(\omega R + i\cdot)$ by Laguerre functions.

Lemma 5.8. *Let $R > 0$, $N \in \mathbb{N}$ and $\omega > 0$. Then there exists $c > 0$ such that*

$$\inf_{u_N \in \tilde{\mathcal{X}}_N} \|h_0(\omega R + i\cdot) - u_N\|_{L_2(\mathbb{R}_{\geq 0})} \leq \frac{c\sqrt{\pi}}{(2\omega R)^{\frac{1}{4}}}\exp\Big(-2\sqrt{\omega R(N+1)}\Big).$$

Proof.

$$\begin{aligned}\|(I-\Pi_N)h_0(\omega R + i\cdot)\|^2_{L_2(\mathbb{R}_{\geq 0})} &= \left\|(I-\Pi_N)(-1)\exp(i\omega R)\frac{\exp(-\cdot)}{-i\omega R + \cdot}\right\|^2_{L_2(\mathbb{R}_{\geq 0})}\\ &= \sum_{n=N+1}^{\infty} |2\alpha_{n,1}(-i\omega R)\,|^2\|\phi_n\|^2\\ &\leq 2c\sum_{n=N+1}^{\infty}\Bigg|\exp\Big(-i\omega R - 2\sqrt{-2i\omega R(n+1)}\Big)\\ &\quad\times\frac{\sqrt{\pi}}{(-2i\omega R(n+1))^{\frac{1}{4}}}\Bigg|^2\\ &= c\pi\sqrt{\frac{2}{\omega R}}\sum_{n=N+1}^{\infty}\frac{\exp\Big(-4\sqrt{\omega R(n+1)}\Big)}{\sqrt{n+1}}.\end{aligned}$$

Since the summand is a decreasing function in n we can replace the sum with an integral and obtain

$$\begin{aligned}
\|(I-\Pi_N)h_0(\omega R+i\cdot)\|^2_{L_2(\mathbb{R}_{\geq 0})} &\leq c\pi\sqrt{\frac{2}{\omega R}}\int_{N+1}^{\infty}\frac{\exp\left(-4\sqrt{\omega R t}\right)}{\sqrt{t}}dt \\
&= 2c\pi\sqrt{\frac{2}{\omega R}}\int_{\sqrt{N+1}}^{\infty}\exp\left(-4\sqrt{\omega R}s\right)ds \\
&= \frac{c\pi}{\sqrt{2\omega R}}\exp\left(-4\sqrt{\omega R(N+1)}\right).
\end{aligned}$$

□

Theorem 5.9. *Let $\omega,\sigma\in\mathbb{C}$ with $\Im(\omega\sigma)>0$, and $R>0$ and $N\in\mathbb{N}$. Then there exist constants $C_1,C_2>0$ such that the best approximation error of $h_0(\omega R+\omega\sigma\cdot)$ can be bounded by*

$$\inf_{u_h\in\tilde{\mathcal{X}}_N}\|h_0(\omega R+\omega\sigma\cdot)-u_N\|_{L_2(\mathbb{R}_{\geq 0})}\leq C_1\left|\frac{1+i\sigma\omega}{1-i\sigma\omega}\right|^{N+1}+C_2\varepsilon\left(N,\frac{R}{\sigma},-i\omega\sigma\right)$$

with

$$\varepsilon(N,a,b):=\|(I-\Pi_N)\frac{1}{a+\cdot}\Pi_N\exp(-b\cdot)\|_{L_2(\mathbb{R}_{\geq 0})}.$$

Proof.

$$\begin{aligned}
\|(I-\Pi_N)h_0(\omega R+\omega\sigma\cdot)\|_{L_2(\mathbb{R}_{\geq 0})} =&\|(I-\Pi_N)\frac{-i\exp(i\omega R)}{\omega\sigma}\frac{\exp(i\omega\sigma\cdot)}{\frac{R}{\sigma}+\cdot}\|_{L_2(\mathbb{R}_{\geq 0})} \\
\leq& C_2\|(I-\Pi_N)\frac{1}{\frac{R}{\sigma}+\cdot}(I-\Pi_N)\exp(i\omega\sigma\cdot)\|_{L_2(\mathbb{R}_{\geq 0})} \\
&+C_2\underbrace{\|(I-\Pi_N)\frac{1}{\frac{R}{\sigma}+\cdot}\Pi_N\exp(i\omega\sigma\cdot)\|_{L_2(\mathbb{R}_{\geq 0})}}_{\varepsilon\left(N,\frac{R}{\sigma},-i\omega\sigma\right)},
\end{aligned}$$

with $C_2=\left|\frac{\exp(i\omega R)}{\omega\sigma}\right|$. The first term can be bounded by

$$\begin{aligned}
\|(I-\Pi_N)\frac{1}{\frac{R}{\sigma}+\cdot}(I-\Pi_N)\exp(i\omega\sigma\cdot)\|_{L_2(\mathbb{R}_{\geq 0})} &\leq 2C\|(I-\Pi_N)\exp(i\omega\sigma\cdot)\|_{L_2(\mathbb{R}_{\geq 0})} \\
&\leq 2C\frac{1}{\sqrt{2\Im(\omega\sigma)}}\left|\frac{1+i\omega\sigma}{1-i\omega\sigma}\right|^{N+1},
\end{aligned}$$

for some constant $C>0$ (cf. Corollary 5.2). □

To obtain a bound for the best approximation error of h_0 in the space $\tilde{X}_N$ we need to find a bound for the expression ε. Since for $b\in\mathbb{C}$, $a\in\mathbb{C}\setminus\mathbb{R}_{\leq 0}$

$$(I-\Pi_N)\frac{1}{a+\cdot}\Pi_N\exp(-b\cdot)=\sum_{n=N+1}^{\infty}\int_0^{\infty}\frac{1}{a+\xi}\sum_{k=0}^{N}\frac{(1-b)^k}{(1+b)^{k+1}}\phi_k(\xi)\,\phi_n(\xi)\;d\xi\phi_n(\cdot),$$

we need to know the asymptotic behavior of the expressions

$$\beta_{n,k}(a) := \int_0^\infty \frac{\phi_n(x)\phi_k(x)}{a+x}\,dx, \tag{10}$$

for large $n \in \mathbb{N}$. Thus, we state the following lemma.

Lemma 5.10. *Let $a \in \mathbb{C} \setminus \mathbb{R}_{<0}$, and $\beta_{n,k}$ given by (10). Then for $n \geq k$ there holds*

$$\beta_{n,k}(a) = \alpha_{n,1}(a) L_k(-2a). \tag{11}$$

Proof. We prove by induction in k. For $k = 0$ we have

$$\beta_{n,0}(a) = \alpha_{n,1}(a) = \alpha_n(a) L_0(-2a),$$

and for $k = 1$, $n \geq 1$

$$\begin{aligned}
\beta_{n,1}(a) &= \int_0^\infty \frac{\exp(-x)(1-2x)}{a+x}\phi_n(x)\,dx \\
&= \alpha_{n,1}(a) - 2\int_0^\infty \frac{x+a-a}{a+x}\exp(-x)\phi_n(x)\,dx \\
&= \alpha_{n,1}(a) - \delta_{n,0} + 2a\alpha_{n,1}(a) \\
&= (2a+1)\alpha_{n,1}(a) = L_1(-2a)\alpha_{n,1}(a).
\end{aligned}$$

For $n \geq k$ we use the recursion (9) and write

$$\begin{aligned}
\beta_{n,k}(a) &= \int_0^\infty \frac{\phi_n(x)}{a+x}\left(\frac{2k-1-2x}{k}\phi_{k-1}(x) - \frac{k-1}{k}\phi_{k-2}(x)\right)dx = \\
&= \frac{2k-1}{k}\beta_{n,k-1}(a) - \frac{k-1}{k}\beta_{n,k-2}(a) - \frac{2}{k}\int_0^\infty \frac{x+a-a}{a+x}\phi_n(x)\phi_{k-1}(x)\,dx \\
&= \frac{2k-1}{k}\beta_{n,k-1}(a) - \frac{k-1}{k}\beta_{n,k-2}(a) + \frac{2a}{k}\beta_{n,k-1}(a) \\
&= \alpha_{n,1}(a)\left(\frac{2k-1+2a}{k}L_{k-1}(-2a) - \frac{k-1}{k}L_{k-2}(-2a)\right) \\
&= \alpha_{n,1}(a)L_k(-2a).
\end{aligned}$$

$\square$

Now we are able to bound the term $\varepsilon(N,a,b)$.

Lemma 5.11. *For $a \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $b \in \mathbb{C}, \Re(b) > 0$, there exists $C > 0$, such that*

$$\varepsilon(N,a,b) \leq C\exp\left(-2\Re\sqrt{2a(N+1)}\right).$$

Proof. Since

$$\Pi_N \exp(-bx) = \frac{2}{b+1}\sum_{k=0}^{N}\left(\frac{b-1}{b+1}\right)^k \phi_k(x),$$

we have

$$
\begin{aligned}
\varepsilon(N,a,b)^2 &= \left\| \sum_{n=N+1}^{\infty} \frac{4}{b+1} \sum_{k=0}^{N} \left(\frac{b-1}{b+1}\right)^k \left(\phi_n, \frac{\phi_k}{\cdot + a}\right)_{L^2(\mathbb{R}_{\geq 0})} \phi_n \right\|^2_{L^2(\mathbb{R}_{\geq 0})} \\
&= \left\| \sum_{n=N+1}^{\infty} \frac{4}{b+1} \sum_{k=0}^{N} \left(\frac{b-1}{b+1}\right)^k \beta_{n,k}(a)\phi_n \right\|^2_{L^2(\mathbb{R}_{\geq 0})} \\
&= \frac{1}{2} \sum_{n=N+1}^{\infty} \left| \frac{4}{b+1}\alpha_{n,1}(a) \sum_{k=0}^{N} \left(\frac{b-1}{b+1}\right)^k L_k(-2a) \right|^2 .
\end{aligned}
$$

Using the generating function of the Laguerre polynomials from Lemma 4.3 (ix) we have for some constant C

$$
\begin{aligned}
\sum_{k=0}^{N} \left(\frac{b-1}{b+1} L_k(-2a)\right)^k &\leq C \frac{1}{1-\frac{b-1}{b+1}} \exp\left(2a \frac{\frac{b-1}{b+1}}{1-\frac{b-1}{b+1}}\right) \\
&= C\frac{b+1}{2} \exp(a(b-1)) \,,
\end{aligned}
$$

and thus

$$
\varepsilon(N,a,b)^2 \leq 2C \sum_{n=N+1}^{\infty} |\alpha_{n,1}(a) \exp(a(b-1))|^2 \,.
$$

Substituting the asymptotic behavior of $\alpha_{n,1}$ and repeating the arguments of the proof of Lemma 5.8 we find for some constant $c \in \mathbb{R}$

$$
\sum_{n=N+1}^{\infty} |\alpha_{n,1}(a)|^2 \leq c \left| \exp(2a) \frac{\pi}{\sqrt{2a}} \right| \frac{1}{2\Re\sqrt{2a}} \exp\left(-4\Re\sqrt{2a(N+1)}\right) .
$$

All in all this gives

$$
\varepsilon(N,a,b) \leq \tilde{C} \frac{\exp(\Re(ab))\sqrt{\pi}}{\sqrt{|2a|\Re\sqrt{2a}}} \exp\left(-2\Re\sqrt{2a(N+1)}\right) .
$$

$\square$

Using Theorem 5.9 and Lemma 5.11 we have proven

Theorem 5.12. *For $R > 0$, $\omega, \sigma \in \mathbb{C}$, $\Im\sigma\omega > 0$, we can bound the approximation error of the complex scaled zeroth spherical Hankel function by*

$$
\inf_{u_N \in V_N} \|h_0(\omega(R+\sigma x)) - u_N\|_{\mathbb{R}_{\geq 0}} \leq c_1 \left| \frac{1+i\sigma\omega}{1-i\sigma\omega} \right|^{N+1} + c_2 \exp\left(-2\Re\sqrt{\frac{2R(N+1)}{\sigma}}\right) .
$$

5.3 Approximation of spherical Hankel functions with higher index

Theorem 5.13. *For $x \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $n \in \mathbb{N}$*

$$L_n(-2x) = \frac{\exp(-x)}{2\sqrt{\pi}} \frac{\exp\Big(2\sqrt{2x(n+1)}\Big)}{(2x(n+1))^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{(n+1)^{\frac{1}{2}}}\right)\right) \tag{12}$$

Proof. See [3], (4). □

Lemma 5.14.

$$|\alpha_{n,k}(x)| \leq C_{k,x} \left|\exp\Big(-2\sqrt{2x(n+1)}\Big)\right| (n+1)^{-\frac{3}{4}+\frac{k}{2}}, \quad n \to \infty$$

Proof. We prove only the cases $k = 1, 2$, for $k > 2$ one can proceed similarly. From Theorem 5.7 we have

$$\alpha_{n,1}(x) = \sqrt{\pi}(2x(n+1))^{-\frac{1}{4}} \exp\Big(x - 2\sqrt{2x(n+1)}\Big) \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n+1}}\right)\right)$$

Further we have

$$\begin{aligned}
\alpha_{n,k}(x) &= \int_0^\infty \frac{1}{x+t} \frac{\exp(-t)\,\phi_n(t)}{(x+t)^{k-1}}\,dt \\
&= \int_0^\infty \frac{1}{x+t} 2 \sum_{j=0}^{\infty} \alpha_{j,k-1}(x)\phi_j(t)\phi_n(t)\,dt \\
&= 2\sum_{j=0}^{\infty} \alpha_{j,k-1}(a)\beta_{j,n}(a) \\
&= 2\alpha_{n,1}(a) \underbrace{\sum_{j=0}^{n} \alpha_{j,k-1}(a) L_j(-2a)}_{*} + 2L_n(-2a) \underbrace{\sum_{j=n+1}^{\infty} \alpha_{j,k-1}(a)\alpha_{j,1}(a)}_{**}.
\end{aligned}$$

For $k = 2$ term $|(*)|$ can be bounded

$$\begin{aligned}
\left|\sum_{j=0}^{n} \alpha_{j,1}(a) L_j(-2a)\right| &\leq C \sum_{j=0}^{n} \left|(2x(j+1))^{-\frac{1}{2}}\right| \\
&\leq C \int_0^n (2|x|t)^{-\frac{1}{2}}\,dt \\
&= 2C(2|x|)^{\frac{1}{2}} n^{\frac{1}{2}} \\
&\leq C_{k,x}(n+1)^{\frac{1}{2}}.
\end{aligned}$$

For the term $(**)$ and $k = 2$ we have

$$\begin{aligned}
\left|\sum_{j=n+1}^{\infty} \alpha_{j,1}(a)^2\right| &\leq C \sum_{j=n+1}^{\infty} \left|\exp\Big(2x - 4\sqrt{2x(j+1)}\Big)\,(2x(j+1))^{-\frac{1}{2}}\right| \\
&\leq C \int_n^\infty \left|\exp\Big(2x - 4\sqrt{2x(t+1)}\Big)\,(2x(t+1))^{-\frac{1}{2}}\right|\,dt \\
&= \tilde{C} \left|\exp\Big(2x - 4\sqrt{2x(n+1)}\Big)\right|.
\end{aligned}$$

All in all this gives

$$|\alpha_{n,2}(x)| \leq C_{k,x}| \exp\Big(-2\sqrt{2x(n+1)}\Big)\,(n+1)^{\frac{1}{4}}.$$

□

Theorem 5.15. *The approximation error of the complex scaled spherical Hankel functions can be bounded by*

$$\inf_{u_N \in V_N} \|h_j(\omega(R+\sigma x)) - u_N\|_{\mathbb{R}_{\geq 0}} \leq c_1 \left|\frac{1+i\sigma\omega}{1-i\sigma\omega}\right|^{N+1} + c_2 \exp\left(-2\Re\sqrt{\frac{2R(N+1)}{\sigma}}\right)(N+1)^{\frac{j}{2}}.$$

Proof. We skip the technical details and give just a short sketch of the proof. Similar to Lemma 5.10 one can show that

$$\int_0^\infty \frac{1}{(x+t)^j}\phi_n(t)\phi_k(t)\,dt = \alpha_{n,1}(x)L_k(-2x) - \frac{2}{k}\alpha_{n,j-1}(x).$$

Using this, the asymptotic of $\alpha_{n,k}$ and similar ideas as in the proof of Theorem 5.9 and Lemma 5.11 lead to the desired result. □

6 Numerical Experiments

In the following we illustrate our theoretical findings from the previous sections by numerical examples. Figure 1 shows the approximation error of the complex scaled zeroth Hankel function h_0

$$\mathrm{error}(N) = \|(I - \Pi_N)h_0(\omega R + \omega\sigma\cdot)\|_{L_2(\mathbb{R}_{\geq 0})}.$$

The results coincide nicely with the theoretical results from Theorem 5.12. In Figure 1a we can observe the predicted exponential convergence until at $N \approx 35$ the super-algebraic part ε takes over, which is also illustrated in Figure 1b.

Figure 2 shows the approximation of Hankel functions with different indices and exhibits the predicted behavior from Theorem 5.15. Figure 3 shows the condition numbers of the discretization matrices of the bilinear forms

$$\tilde{s}^\sigma + \lambda_n \tilde{m}_0^\sigma - \omega^2 \tilde{m}_1^\sigma,$$

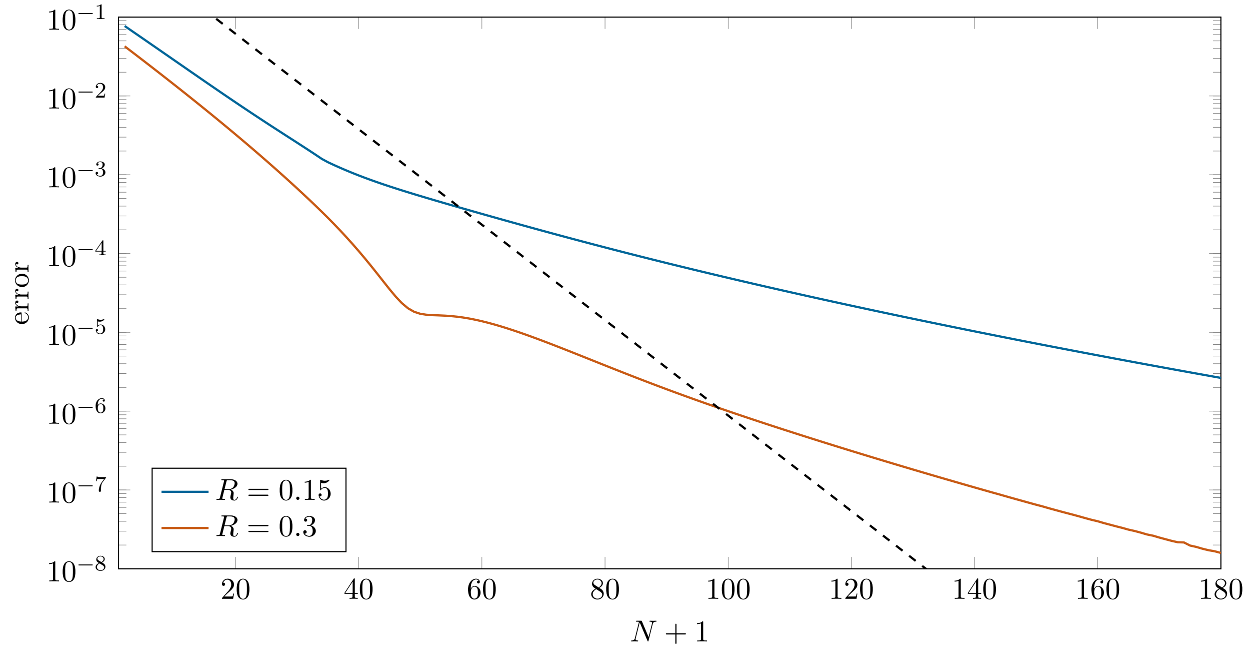
with respect to different infinite element orders N. These matrices correspond to discretizations of the spherical Bessel equations with index $\lambda_n = n(n+1)$ (cf. Section 6.1). The condition numbers grow polynomially in N.

Figure 4 shows the convergence in the number of unknowns for one selected eigenvalue of the separated problem (cf. Subsection 6.1). Again the convergence agrees with the predicted error from the approximation results (the approximation error squared cf. [4]).

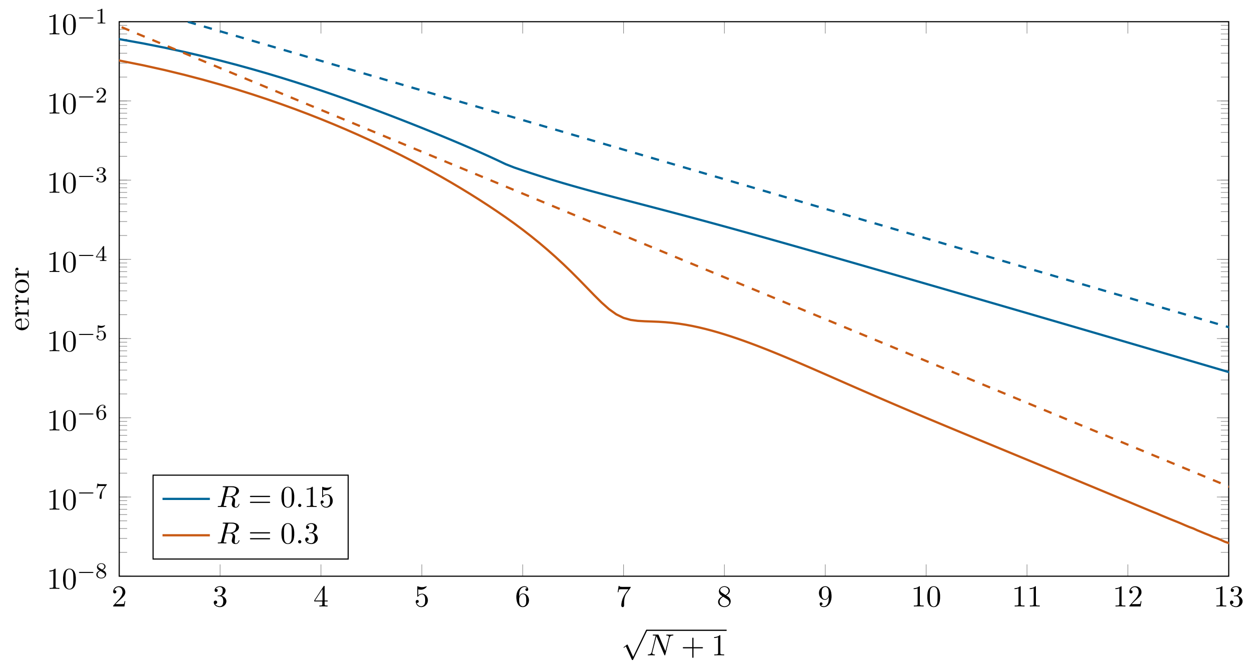
6.1 An example with inhomogeneous exterior

In this subsection we approximate the resonances of the Helmholtz equation on $\Omega_{\mathrm{ext}} := \Omega := \mathbb{R}^3 \setminus B_1(0) = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| > 1\}$ for a potential

$$p(\mathbf{x}(\xi, \hat{\mathbf{x}})) := (1 + \hat{\epsilon}\hat{p}(\hat{\mathbf{x}}))(1 + \tilde{\epsilon}\tilde{p}(\xi)),$$



(a) exponential convergence



(b) super-algebraic convergence

Figure 1: Approximation error of $h_0(\omega(R + \sigma\cdot))$ with $\sigma = 0.3 + 1i$, $\omega = 13 - 0.5i$ and different R. The dashed lines mark the predicted convergence rates from Section 5.

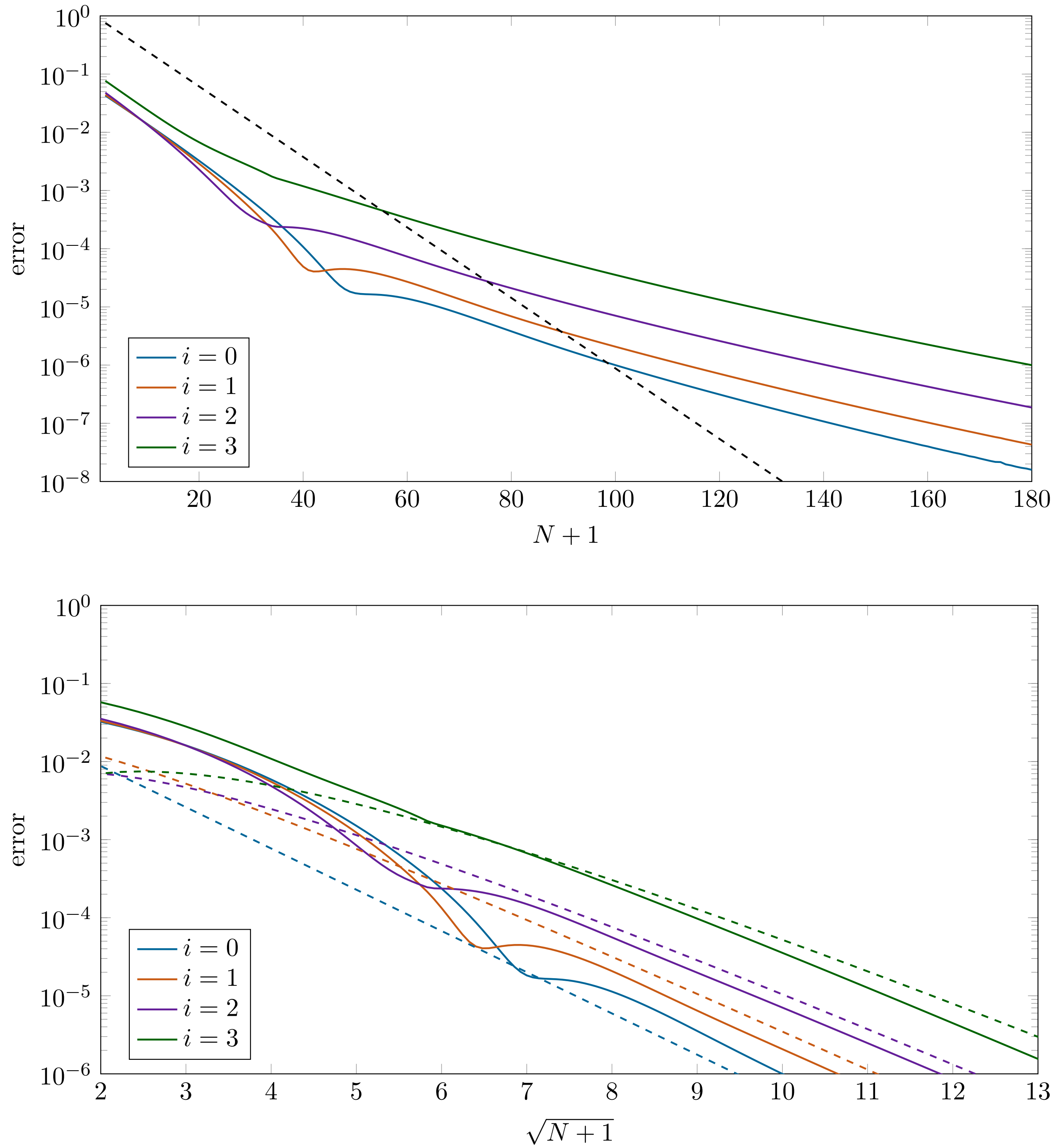


Figure 2: Approximation error of $h_j(\omega(R + \sigma \cdot))$ with $\sigma = 0.3 + 1i$, $\omega = 13 - 0.5i$, $R = 0.3$ and different indices i. The dashed lines mark the predicted convergence rates from Section 5.

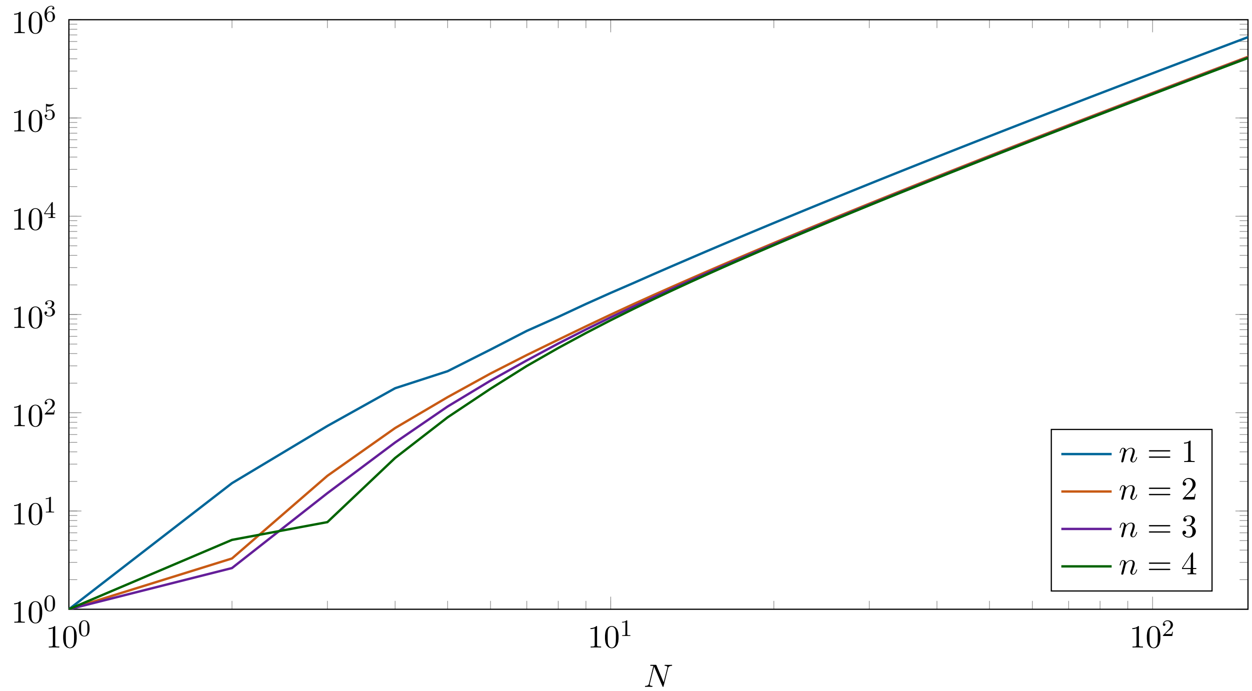


Figure 3: condition numbers of the discretization matrices of $\tilde{s}^\sigma + \lambda_n \tilde{m}_0^\sigma - \omega^2 \tilde{m}_1^\sigma$ with parameters $\sigma = 1 + 1i$, $\omega = 1 - 0.5i$, $R = 1$

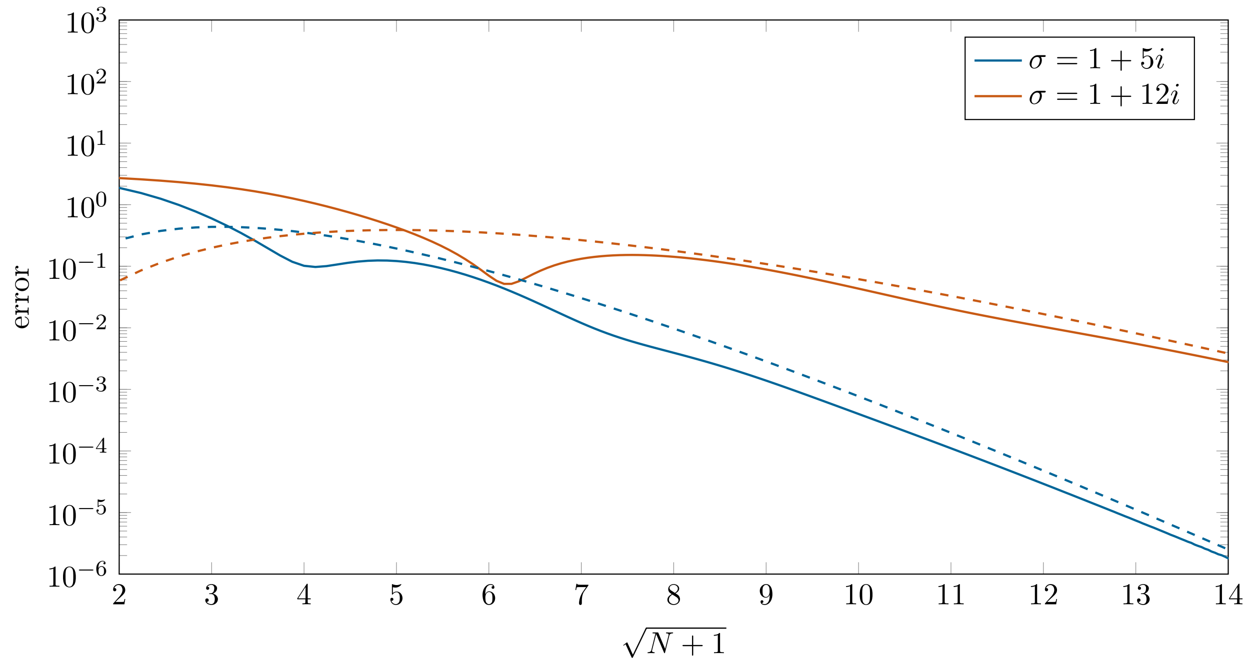


Figure 4: Errors of the eigenvalue $\omega \approx 2.903916 + 1.201866i$ obtained by solving the separated problem. The dashed lines mark the squared super-algebraic convergence rates from Section 5.

with functions $\hat{p} : \Gamma \to \mathbb{R}$, $\tilde{p} : \mathbb{R}_{\geq 0} \to \mathbb{R}$. If we assume $\hat{\epsilon}$ to be zero, the equation can be separated using an ansatz

$$u(\mathbf{x}(\xi, \hat{\mathbf{x}})) := \tilde{u}_n(\xi)\, Y_{n,j}(\hat{\mathbf{x}})\,.$$

Since the spherical harmonics $Y_{n,j}$ are eigenfunctions of the surface Laplacian and the according bilinear form $\hat{s}$ with the corresponding eigenvalues $n(n+1)$, this ansatz, combined with complex scaling as before leads to the set of one dimensional eigenvalue problems

$$\tilde{s}^{\sigma}(\tilde{u}, \tilde{v}) + n(n+1)\tilde{m}_0^{\sigma}(\tilde{u}, \tilde{v}) = \omega^2 \tilde{m}_1^{\sigma}((1+\tilde{\epsilon})\tilde{p}\tilde{u}, \tilde{v})\,, \quad n \in \mathbb{N}_0.$$

All three-dimensional experiments use the parameters $N = 50$, finite element mesh size $h = 0.3$ and polynomial order $p = 4$. For the one-dimensional expamples we used $N = 200$.

Figure 5 shows the eigenvalues of discretized separated problem for

$$\tilde{p}(\xi) := \frac{\xi^2}{0.1 + \xi^4}$$

$n = 0, \ldots, 5$ and different choices of $\tilde{\epsilon}$, as well as the eigenvalues of the full three-dimensional simulation.

Figure 6 shows resonances of the same problem with an additional potential

$$\hat{p}(x, y, z) = z,$$

and varying values of $\hat{\epsilon}$. Due to the disturbed symmetry, the multiple eigenvalues fan out. The resonances located close to the negative imaginary axis in Figures 5 and 6 are part of the discretization of the essential spectrum (cf. [4]).

Figures 7 and 8 show selected eigenfunctions corresponding to the previously approximated resonances. To visualize the eigenfunctions $\Omega_{\text{ext}} = B_2(0)$ was chosen here.

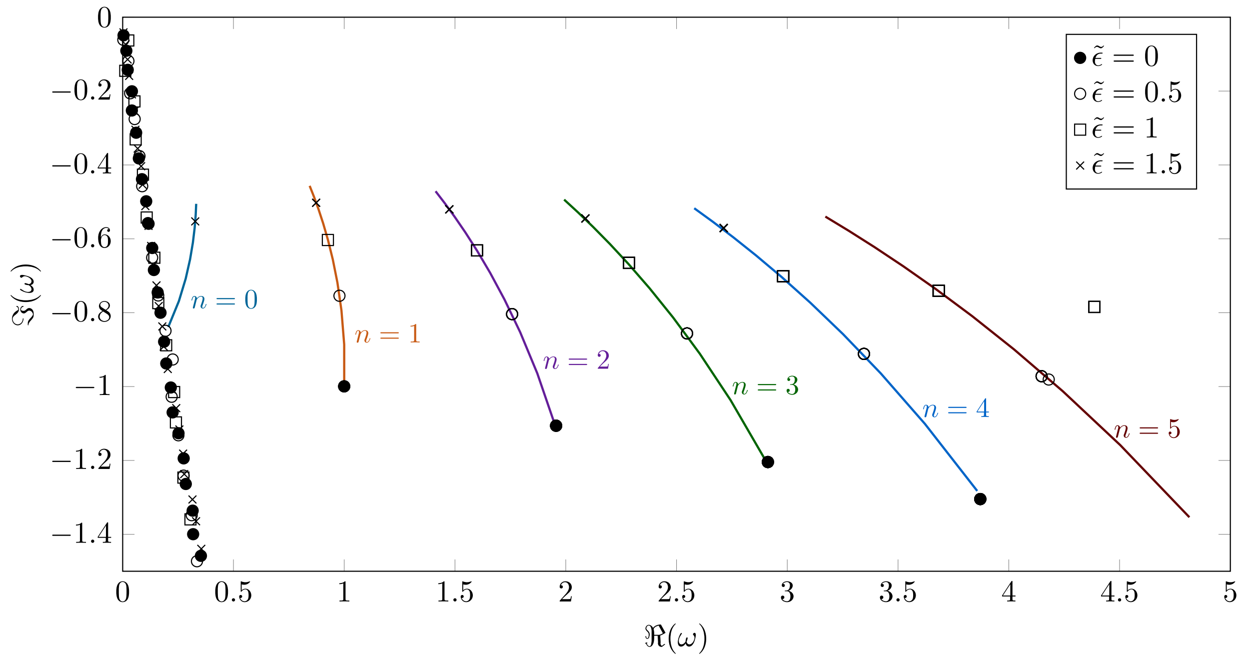


Figure 5: Inhomogeneous exterior problem with radial inhomogeneity. The lines mark the locations of resonances for a given Bessel index n and varying $\tilde{\epsilon}$, obtained by solving the separated problem.

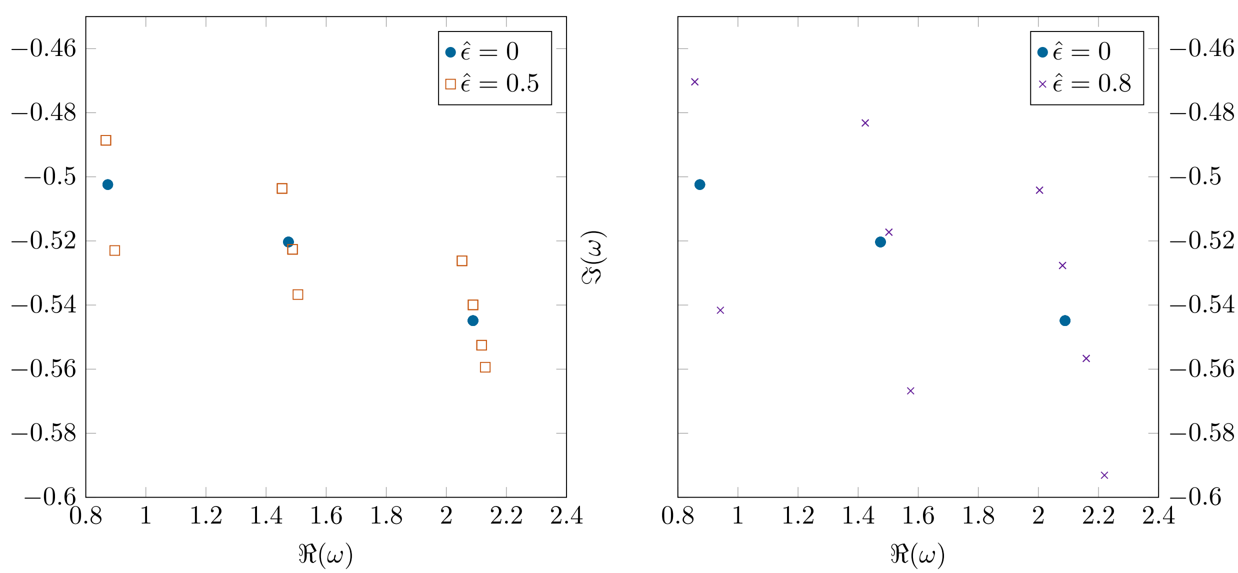
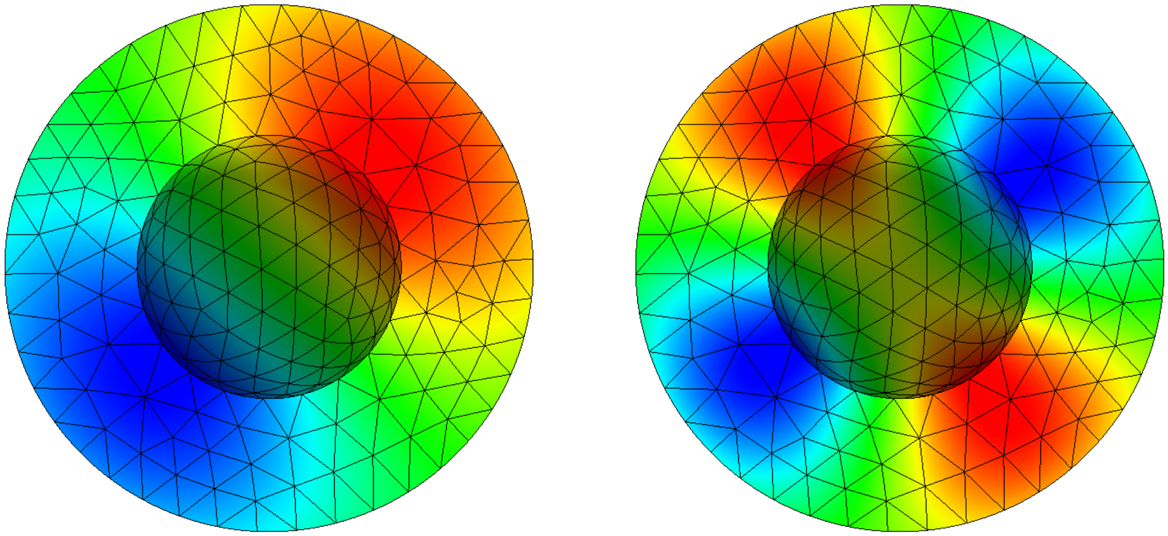
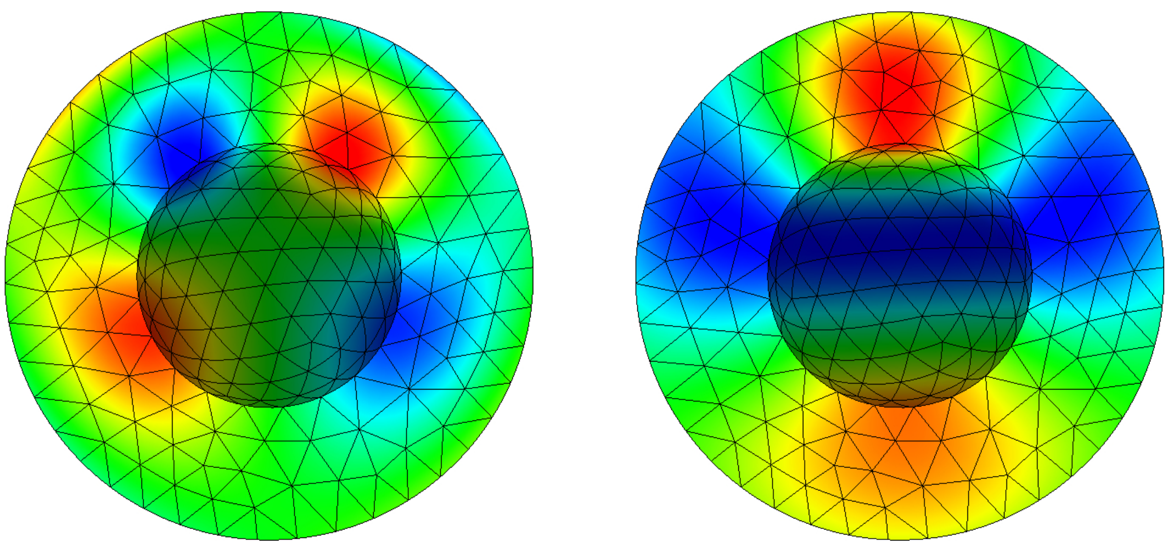


Figure 6: Inhomogeneous exterior problem with $\tilde{\epsilon} = 1.5$ and variable $\hat{\epsilon}$.



(a) $\omega \approx 0.87 - 0.50i$ (b) $\omega \approx 1.47 - 0.52i$

Figure 7: Resonance functions corresponding to eigenvalues from Figure 5 with $\tilde{\varepsilon} = 0.5$.



(a) $\omega \approx 1.49 - 0.52i$ (b) $\omega \approx 1.50 - 0.53i$

Figure 8: Resonance functions corresponding to eigenvalues from Figure 6 with $\hat{\varepsilon} = 0.5$.

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