TECHNISCHE UNIVERSITÄT WIEN
Vienna | Austria

INSTITUTE FOR ANALYSIS
AND SCIENTIFIC COMPUTING
WIEDNER HAUPTSTRASSE 8-10
1040 WIEN, AUSTRIA

DR.
PHILIP LEDERER
email: philip.lederer@tuwien.ac.at

Finite Element Methods in Computational Fluid Dynamics

Exercise 1 – 2022

Example 1.1

Prove that the stabilized $\mathbb{P}2\mathbb{P}2$ method is inf-sup stable (set $\beta = 0$). To this end we define $B((u_h, p_h), (v_h, q_h))$ as the "big" stabilized bilinear form. Further, note that there holds (see proof of Theorem 13)

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1} \geq c_1 \|q_h\|_0 - c_2 \|q_h\|_{0,h} \quad \forall q_h \in Q_h. \tag{1}$$

with two constants $c_1, c_2 > 0$. Now follow these steps:

1. First prove by scaling arguments that there exists a constant c_3 such that
$$c_3 \sum_{T \in \mathcal{T}_h} h^2 \| \operatorname{div}(\varepsilon(u_h))\|_T^2 \leq \|\nabla v_h\|_0^2 \quad \forall v_h \in V_h.$$
2. Now let v_h, q_h be arbitrary but fixed.
3. Choose u_h^1 to be the supremum of (1) and scale it such that $\|u_h^1\|_1 = \|q_h\|_0$. Further choose $p_h^1 = 0$. Bound $B((u_h^1, p_h^1), (v_h, q_h))$ from below. This step should give you a bound from below with a positive sign in front of $\|q_h\|_0$ and some other negative parts...
4. Choose $u_h^2 = v_h$ and $p_h^2 = ...$ such that the off-diagonal constraints vanish and that you get something "positive" that could be used to compensate the term from the previous step. Here you should see that $0 < \alpha < c_3$ **needs to be small enough**.
5. Combine the previous two steps (you might have to add a scaling parameter that needs to be adjusted at the end).

For above proofs you will need the inverse inequality for polynomials and the Young inequality (and Friedrich and Korn).

Example 1.2

Derive an (improved $\mathcal{O}(h^{k_V+1})$) error estimate of the L^2-norm error of the velocity by a Aubin-Nitsche duality argument. For this we define the dual problem: Find $(w, \lambda) \in V \times Q$ such that

$$a(v, w) + b(w, q) + b(v, \lambda) = (f, v) \quad \forall (v, q) \in V \times Q.$$

and assume that the solution fulfills the regularity $(w, \lambda) \in H^2 \times H^1$. Then follow similar steps as in the derivation of the improved estimates for the Poisson equation.

Example 1.3

Implement two arbitrary inf-sup stable finite element methods for the Stokes problem in Netgen/NGSolve (with a cont. and disc. pressure space). Plot the expected order of convergence of the H^1-semi norm and the L^2-norm error of the velocity and the L^2-norm error of the pressure. Choose the example from the lecture and set $\nu = 1$. You can go to `www.ngsolve.org` to find a documentation. There, you should also find an entry regarding the Stokes equations (for the simplified version $-\Delta u + \nabla p = f$, thus you need to adapt it). For the mean value constraint either use a Lagrange parameter in $\mathbb{R}$ (in NGSolve this is the `Numberspace(mesh)`) or use a pertubed problem by adding the bilinearform

$$-\varepsilon \int_\Omega p_h q_h \,\mathrm{d}x,$$

with $\varepsilon \ll 1$. (Try both versions)

Example 1.4

Implement the stabilized $\mathbb{P}2\mathbb{P}2$ method and plot the expected order of convergence of the H^1-semi norm and the L^2-norm error of the velocity and the L^2-norm error of the pressure. If you do not see the proper order try a smaller α. A different stabilization is given by the bilinear form

$$c((u_h, p_h), (v_h, q_h)) = \alpha \sum_{T \in \mathcal{T}_h} h^2 \int_T (-\nu \operatorname{div}(\varepsilon(u_h)) + \nabla p_h) \cdot (\nu \operatorname{div}(\varepsilon(v_h)) + \nabla q_h) \,\mathrm{d}x\,.$$

Change the right hand side appropriately and redo the calculations. How big can you choose α? What do you observe in the L^2-norm error of the velocity (see Example 1.2)? You can access second order derivatives (on each element) in NGSolve by `u.Operator("hesse")`, where `u` is a trial or test function.