

Finite Element Methods in Computational Fluid Dynamics

Exercise 2 – Jan. 2023

Example 2.1

1. Implement the Nitsche penalty method for a weak incorporation of arbitrary but non-zero Dirichlet values on the unit square $\Omega = (0, 1)^2$ and polynomial order $k = 2$ and meshsize $h = 0.1$. For a first test use a direct sparse Cholesky solver. For the integrals on the boundary use the symbol `...*ds(skeleton = True)`. Further use

```
h = specialcf.mesh_size #meshsize
n = specialcf.normal(2) #normal vector
```

2. Calculate the condition number of the resulting stiffness matrix `a.mat` (from your bilinear form). For this either use `scipy` or the eigenvalue solver from `ngsolve` via

```
from ngsolve.la import EigenValues_Preconditioner
lams = EigenValues_Preconditioner(mat=a.mat, pre=I, tol=1e-10)
```

where `I = IdentityMatrix(a.mat.height)`. How does the condition number scale with the stabilization parameter α in the Nitsche method? Try different values in the range $(10^{-2}, 10^3)$.

3. Solve your system with a CG solver:

```
inv = CGSolver(a.mat, c, printrates = True, precision = 1e-8, maxsteps = 10000)
gfu.vec.data += inv * ...
```

where the preconditioner is either the identity matrix or a Jacobi preconditioner

```
c = Preconditioner(a, type = "local")
a.assemble()
```

What do you observe?

Example 2.2

Solve the problem from the section “2.5.6 Pressure robustness” and try to reproduce Figure 2.3 where you compare

1. The \mathbb{P}_2 -bubble element. (Use a `VectorH1` space for the velocity instead of two individual `H1` spaces)
2. The pressure robust version by including the reconstruction operator via

```
v.Operator("divfree_reconstruction")
```

where v is a test function of the velocity space.

Redo this example but now choose the exact pressure by

$$p = x + y - 1.$$

What do you observe and why is this the case (check the error estimates from the lecture).

Example 2.3

Prove Lemma 12 and Lemma 13 (continuity and coercivity of the scalar HDG method).

Example 2.4

Implement the **fully discontinuous** HDG method for the Stokes problem from section 2.6.2 on the unit square for arbitrary orders and homogeneous Dirichlet boundary conditions. Choose the right hand side of the second example. Is the method pressure robust? For the implementation use

```
V1 = L2(mesh, order = ...) #no Dirichlet bnd cond!
#scalar facet space WITH bc
V2 = FacetFESpace(mesh, order = ..., dirichlet = ...)
Q = L2(mesh, order = ...)

X = V1**2 * V2**2 * Q

#u,v and uhat,vhat have two components! Be careful with the gradients!
(u,uhat,p),(v,vhat,q) = X.TnT()

...

a = BilinearForm(x)
a += ... * dx
a += ... * dx(element_boundary = True) #this is a sum over all element boundaries!
```

Example 2.5

Redo the previous example but now with the $H(\text{div})$ -conforming method. For this choose the spaces

```
V1 = HDiv(mesh, order = ..., dirichlet = ) #WITH Dirichlet bnd cond!
V2 = TangentialFacetFESpace(mesh, order = ..., dirichlet = ...)
#V2 is a vector valued space!
Q = L2(mesh, order = ...)

X = V1 * V2 * Q

(u,uhat,p),(v,vhat,q) = X.TnT()

n = specialcf.normal(2)

#define a tangential projection with the normal vector
def tang(v):
    return ....
```