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Finite Element Methods in Computational Fluid Dynamics

Exercise 4 – Jan. 2023

Example 4.1

Derive the transport equation for the turbulent kinetic energy $K = \frac{1}{2}\langle \underline{u}' \cdot \underline{u}' \rangle$. The solution should be:

$$\frac{\partial K}{\partial t} + \langle \underline{u} \rangle \cdot \nabla K = -\langle \underline{u}' \otimes \underline{u}' \rangle \nabla \langle \underline{u} \rangle - \nu \langle \nabla \underline{u}' : \nabla \underline{u}' \rangle - \nabla \langle p' \underline{u}' \rangle - \operatorname{div}\left(\frac{1}{2} \langle (\underline{u}' \cdot \underline{u}') \underline{u}' \rangle + \nu \nabla K \right)$$

Hint:

- Begin with inserting the Reynolds decomposition of $\underline{u}$ and p into the Navier-Stokes equations and subtract it from the RANS equations.
- Show that
$$\frac{1}{2}\langle \Delta(\underline{u}' \cdot \underline{u}') \rangle = \langle \nabla \underline{u}' : \nabla \underline{u}' \rangle + \langle \underline{u}' \cdot \Delta \underline{u}' \rangle.$$

Example 4.2

We consider a statistically stationary, fully-developed turbulent channel flow (see figure below). The channel height is $2h$. The end of the plates in z direction is very far away, so that it has no influence on the flow and we consider periodic boundary conditions in x (i.e. an infinite long channel). The density of the fluid is $\rho = const$. The flow is forced by pressure gradient $\frac{\partial \langle p \rangle}{\partial x}$.

1. Simplify the RANS equations for the given case. Consider the continuity equation as well as momentum equations.

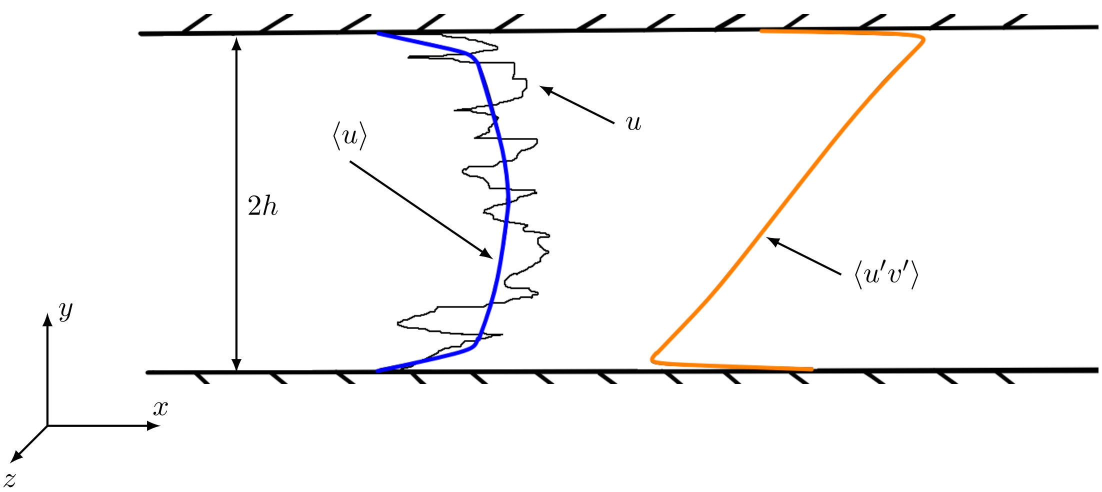

Figure 1: Turbulent channel flow

2. Show that the pressure $\langle p \rangle$ for all (y,z)-planes has reached its maximum at $y = 0/2h$.
3. Show that the $\langle v'w' \rangle$ component of the Reynolds stress tensor is 0 in the whole flow field.
4. Derive the equation for the total shear stress in dependence of y

$$\frac{\tau_\mu + \tau_t}{\tau_w} = f(y), \tag{1}$$

where $\tau_\mu = \mu \frac{\partial \langle u \rangle}{\partial y}$ is the viscous shear stress, $\tau_t = -\rho \langle u'v' \rangle$ the Reynolds shear stress and $\tau_w = \mu \frac{\partial \langle u \rangle}{\partial y}|_{y=0}$ the wall shear stress.

Example 4.3

Implement a RANS solver using two different turbulence models for calculating the eddy viscosity ν_t in NGSolve. Test your implementation for the turbulent channel flow and compare the results of the two models. K-equation model (one transport equation).

- Start with the temporal and spatial discretization of the incompressible RANS equations ($\rho = 1$) using Taylor-Hood elements and first order IMEX scheme.
- Setup the channel flow test case in 2D with $\Omega = (0,5) \times (0,1)$. Use a laminar parabolic flow profile ($v_{bulk} = 1$) for the inlet boundary at $x = 0$, outflow boundary condition at $x = 5$ and no-slip boundary conditions for the lower and upper wall. The Reynolds number should be set to $Re = \frac{1}{\nu} = 10000$.
- Mixing length model (algebraic): The eddy viscosity is calculated via

$$\nu_t = l_m^2 \sqrt{2S(\langle \underline{u} \rangle) : S(\langle \underline{u} \rangle)}, \tag{2}$$

where $S(\langle \underline{u} \rangle)$ is the strain rate tensor. The mixing length l_m is defined as

$$l_m = max(\kappa d, 0.09\delta_{99}). \tag{3}$$

Here, $\kappa = 0.41$, $\delta_{99} = 0.5$ and d is the distance to closest wall.

- K-equation model (one equation model): The transport equation of the turbulent kinetic energy has to be discretized using standard H^1-conforming elements and solved over time. The equation is given as

$$\frac{\partial K}{\partial t} + \langle \underline{u} \rangle \cdot \nabla K = \Pi - \epsilon + \text{div}\big((\nu + \nu_t)\nabla K\big), \tag{4}$$

where the production term Π is modeled using the eddy viscosity assumption

$$\Pi = -\langle \underline{u}' \otimes \underline{u}' \rangle \nabla \langle \underline{u} \rangle = \big(2\nu_t S(\langle \underline{u} \rangle - \frac{2}{3}KI)\big)\nabla \langle \underline{u} \rangle \tag{5}$$

with I the identity matrix and the dissipation term is defined as

$$\epsilon = C_D \frac{K^{3/2}}{l_m} \tag{6}$$

with the model constant $C_D = 0.08$ and the mixing length from the previous model. Use $K_{inlet} = \frac{3}{2} t_{int}^2 \langle \underline{u} \rangle_{inlet}$ with the turbulent intensity $t_{int} = 0.04$ as inlet boundary condition, homogenous Dirichlet boundary condition for the walls and homogenous Neumann boundary condition for the outlet. The eddy viscosity can then be computed via

$$\nu_t = l_m K^{1/2}. \tag{7}$$

Hint: By definition $K \geq 0$. Although, it may happen that the approximation of K has slightly negative values for a short period of time and therefore the computation of $\sqrt{K}$ is not defined. Avoid this problem, by using a bound for K.

- Since ν_t changes over time, the matrices for the IMEX time stepping scheme do not keep constant and have to be reassembled and solved for every step (use the calculated ν_t from the previous step).

After a stationary state has been reached, compare the resulting velocity profiles at $x = 4.5$ of the two different models to the laminar velocity profile at the inlet. What do you observe?