Compressed resolvents, Q-functions and h_0-resolvents in almost Pontryagin spaces

HENK DE SNOO * HARALD WORACEK[‡]

To our friend and teacher Heinz Langer, on the occasion of his 80th birthday

Abstract: The interest of this paper lies in the selfadjoint extensions of a symmetric relation in an almost Pontryagin space. More in particular, in their compressed resolvents, Q-functions and h_0-resolvents. We give a systematic approach to each of this three topics, and show an intimate connection bewteen the last two.

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1 Introduction

The notions of compressed resolvents and Q-functions in the title of this paper appear in the theory and applications of symmetric operators in a Hilbert space. As an example, recall the operator theoretic approach to the power moment problems of Hamburger and Stieltjes via symmetric operators. The totality of positive measures possessing prescribed power moments is described in terms of their Cauchy-transforms via the compressed resolvents of selfadjoint extensions of the corresponding symmetric operator. These solutions are parameterised via Krein's formula, which involves the Q-function of the symmetric operator; for details see, e.g., [Akh61].

The starting point of the systematic treatment of compressed resolvents and Q-functions is the paper [Kre44] of M.G.Krein, which has led to generalizations in many directions, involving higher defect numbers, Pontryagin or Krein spaces, and symmetric relations, i.e., multivalued operators (instead of operators). The aim of the present paper is to discuss a generalization of geometric nature: namely that of symmetric relations in an almost Pontryagin space, while for the discussion of the corresponding Q-functions we will assume that the defect index is $(1,1)$.

Roughly speaking, an almost Pontryagin space is a direct and orthogonal sum of a Pontryagin space with a finite dimensional neutral (and hence isotropic) space. Although an almost Pontryagin space differs from a Pontryagin space only by a comparatively 'small' part, namely a finite dimensional degeneracy, there occur several interesting new phenomena. For example, the usual notion of the negative index of a function is not suitable anymore, cf. Definition II.12. As another example we mention that now Q-functions are in duality with 'h_0-resolvents', a notion which is specific for the degenerated case and has no analogue in the Pontryagin space case, cf. Remark III.14. The introduction of almost Pontryagin spaces originates with a generalization of Krein's formula

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in [KW99b]. That work was inspired by degenerated and indefinite versions of interpolation and extrapolation problems, like the power moment problem mentioned above; another instance of the occurence of degeneracy is in the theory of Pontryagin spaces of entire functions. A formal axiomatic treatment of almost Pontryagin spaces can be found in [KWW05]. A continuation concerning sums and couplings of such spaces is given in [SW12]. Finally symmetric and selfadjoint relations in almost Pontryagin spaces were studied in [SW16] with an emphasis on restrictions and factorisations for such relations.

For the benefit of the reader a review of the notions which are treated in this paper is included; see Section 2. This is followed by the three parts of the main text: Part I is about compressed resolvents, Part II is about Q-functions, while Part III is concerned with h_0-resolvents. Many results in Part I and Part II can be seen as generalizations of well known Pontryagin space theorems to the degenerated case. Proofs are generically obtained by tracing the influence of isotropic elements and making appropriate modifications. The matters discussed in Part III are specific for the degenerated case and do not have Pontryagin space analogues. Here is a description of the separate parts.

Part I. Compressed resolvents

With the following four sections:

I.1 Definition and basic properties of compressed resolvents. The definition of compressed resolvents as known from the Pontryagin space situation does not make sense in the almost Pontryagin space setting. One way to overcome this difficulty is to substitute a single operator valued function by a family of scalar valued functions. This fact has been realised earlier, see, e.g., [KW99b]. In this section we give the appropriate definitions and collect some simple facts.

I.2 Intrinsic characterisation. In the Pontryagin space case, it is well known that the fact whether or not an operator valued function is a compressed resolvent, can be characterised intrinsically by means of a certain kernel function, see, e.g., [DLS84, Theorem 2.3]. In this section we provide the almost Pontryagin space analogue of this result.

I.3 Minimality aspects. Let A be selfadjoint relation in a Pontryagin space $\mathcal{P}$ and assume that $\mathcal{P} \supseteq \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space. If A is $\mathcal{H}$-minimal with nonempty resolvent set $\rho(A)$ then its compressed resolvent has no continuous extension beyond the resolvent set. In this section we present an analogue for the almost Pontryagin space situation.

I.4 Generalised resolvents. If the selfadjoint relation A in an almost Pontryagin space $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ extends a closed symmetric relation S in an almost Pontryagin space $\mathcal{A}$, then the compressed resolvent of A to $\mathcal{A}$ is called a generalised resolvent of S. It will be shown that often we can reduce to the case of minimal symmetries.

Part II. Q-functions

With the following five sections:

II.1 Definition of Q-functions. In the degenerated case the definition of a Q-function associated with a symmetric relation is similar, but not as straightforward, as in the Pontryagin space case. In fact, only very specific selfadjoint extensions can be used to produce a Q-function. Q-functions in almost Pontryagin spaces have been introduced in [KW99b]; in this section we recall and supplement this previous work.

II.2 Index of negativity. The usual notion of the negative index of a function as defined, e.g., in [KL77], does not fit the degenerated situation; a fact which already shows up in [KW99b]. In this section we systematically study the adapted notion of negative index. It is interesting to observe that this notion is not anymore defined in the standard way from a reproducing kernel.

II.3 Realization theorem. It is an important fact that each generalised Nevanlinna function can be realised as a Q-function of some symmetry in a Pontryagin space, cf. [KL73]. In this section we prove a degenerate analogue: each generalised Nevanlinna function can be realised as a Q-function of some symmetry acting in an almost Pontryagin space with arbitrarily prescribed degeneracy. For the proof of this fact, we employ some geometric ideas worked out in [SW12] in order to construct a degenerate analogue of the Krein-Langer operator model.

II.4 Minimality aspects. In the Pontryagin space case it is well known that, when dealing with Q-functions, one can restrict attention to minimal symmetries, see, e.g., [KL73]. In this section we show that this statement remains true in the degenerated situation. The proof of this fact relies on a restriction-factorization process for symmetric relations in almost Pontryagin spaces which is elaborated in [SW16].

II.5 Analytic model. In this section we construct a reproducing kernel almost Pontryagin space model for a generalised Nevanlinna function. This is the analogue of the known reproducing kernel Pontryagin space model, see, e.g. [Dij+04, §2].

PART III. h_0-RESOLVENTS

With the following four sections:

III.1 Definition of h_0-resolvents. In this (short) section, we give the definition of h_0-resolvents and provide some simple properties.

III.2 Index of negativity. Again the usual definition of negative index is not suitable. In this section we define and study the proper adapted notion. Already at this stage one can sense that the notion of h_0-resolvents is in some way dual to the notion of Q-functions.

III.3 Duality theorem and h_0-resolvent representations. We show that a function f is a Q-function of a symmetry S if and only if $1/f$ is a h_0-resolvent of S. Let us point out that, in sharp contrast to the Pontryagin space situation, the function $1/f$ is not a Q-function of S. Moreover, we discuss the meaning and relevance of this duality, and give some corollaries. Among them, the realization theorem that every generalised Nevanlinna function is a h_0-resolvent of some symmetry.

III.4 More on minimality. Our aim in this section is twofold. First, we show that, when investigating the totality of all h_0-resolvents of a given symmetry, one can restrict to minimal symmetries. This fact again relies on the restriction-factorization proceedure elaborated in [SW16]. Second, we show that, under a suitable minimality condition and an additional hypothesis on isotropic parts, an h_0-resolvent determines the symmetry uniquely (up to isomorphism).

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2 Almost Pontryagin spaces, symmetric and selfadjoint relations

In this section we first recall some notions and collect basic facts about Pontryagin spaces and linear relations in such spaces (§2.1), and about the spectral theory of symmetric and selfadjoint relations (§2.2). Second, we prove some results on the existence of selfadjoint extensions with nonempty resolvent set of a symmetric relation (§2.3), and discuss minimality (§2.4).

References for the geometry of almost Pontryagin spaces are [KWW05], [SW12], [Wor14a, Appendix A]. For the basic theory of linear relations in Banach space and in particular in Pontryagin spaces we refer to [DS87a].

2.1 Almost Pontryagin spaces

2.1 Definition. An *almost Pontryagin space* is a triple $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ consisting of a linear space $\mathcal{A}$, an inner product $[.,.]$ on $\mathcal{A}$, and a topology $\mathcal{T}$ on $\mathcal{A}$, such that

- **(aPs1)** $\mathcal{T}$ is a Hilbert space topology on $\mathcal{A}$;
- **(aPs2)** $[.,.] : \mathcal{A}\times\mathcal{A} \to \mathbb{C}$ is $\mathcal{T}\times\mathcal{T}$-continuous;
- **(aPs3)** There exists a $\mathcal{T}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension such that $\langle \mathcal{M}, [.,.] \rangle$ is a Hilbert space.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces. A map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is called a *morphism* from $\mathcal{A}_1$ to $\mathcal{A}_2$ if it is linear, isometric, continuous, and maps closed subspaces of $\mathcal{A}_1$ onto closed subspaces of $\mathcal{A}_2$. It is an *isomorphism* if there exists a morphism $\psi : \mathcal{A}_2 \to \mathcal{A}_1$, such that $\psi \circ \phi = \mathrm{id}_{\mathcal{A}_1}$ and $\phi \circ \psi = \mathrm{id}_{\mathcal{A}_2}$. $\Diamond$

Note that $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism if and only if it is linear, isometric, bijective and homeomorphic. Topological notions are always understood w.r.t. the Hilbert space topology $\mathcal{T}$.

We usually suppress explicit notation of the inner product $[.,.]$ and the topology $\mathcal{T}$, and shortly speak of an almost Pontryagin space $\mathcal{A}$. Also, when saying that $\mathcal{A}_1$ *contains* $\mathcal{A}_2$, we mean that $\mathcal{A}_1$ is a closed linear subspace of $\mathcal{A}_2$, that the inner product of $\mathcal{A}_1$ is the restriction of the inner product of $\mathcal{A}_2$, and that the topology of $\mathcal{A}_1$ is the restriction of the topology of $\mathcal{A}_2$.

The *negative index* of an inner product space $\mathcal{L}$ is defined as

$$\mathrm{ind}_- \mathcal{L} := \sup \big\{ \dim \mathcal{N} : \mathcal{N} \text{ negative subspace of } \mathcal{L} \big\} \in \mathbb{N}_0 \cup \{\infty\},$$

where a subspace $\mathcal{N}$ of $\mathcal{L}$ is called *negative*, if $[x,x] < 0$, $x \in \mathcal{N} \setminus \{0\}$. We denote by $\mathcal{L}^\circ$ the isotropic part of $\mathcal{L}$, i.e. $\mathcal{L}^\circ := \mathcal{L} \cap \mathcal{L}^\perp$, and $\mathrm{ind}_0 \mathcal{L} := \dim \mathcal{L}^\circ$ is called the *degree of degeneracy* of $\mathcal{L}$. The inner product space $\mathcal{L}$ is called *nondegenerated* if $\mathrm{ind}_0 \mathcal{L} = 0$; otherwise $\mathcal{L}$ is called *degenerated*.

An almost Pontryagin space is a Pontryagin space if and only if it is nondegenerated, in which case its topology is uniquely determined by the inner product.

2.2 Definition. Let $\mathcal{A}$ be an almost Pontryagin space. A pair $(\iota, \mathcal{P})$ is called a *canonical Pontryagin space extension of* $\mathcal{A}$, if $\mathcal{P}$ is a Pontryagin space, $\iota : \mathcal{A} \to \mathcal{P}$ is an injective morphism, and

$$\dim \mathcal{P}/\iota(\mathcal{A}) = \operatorname{ind}_0 \mathcal{A}.$$

$\Diamond$

Let $\mathcal{P}$ be a canonical Pontryagin space extension of $\mathcal{A}$, then $\operatorname{ind}_- \mathcal{P} = \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$. Canonical Pontryagin space extensions are in some sense minimal among all Pontryagin spaces which contain $\mathcal{A}$ as a closed subspace: If $\mathcal{P}$ is a Pontryagin space which contains $\mathcal{A}$ as a closed subspace, then $\dim \mathcal{P}/\mathcal{A} \geq \operatorname{ind}_0 \mathcal{A}$ and $\operatorname{ind}_- \mathcal{P} \geq \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$, and $\mathcal{P}$ contains a canonical Pontryagin space extension of $\mathcal{A}$.

Canonical Pontryagin space extensions of a given almost Pontryagin space $\mathcal{A}$ always exist and are unique up to isomorphism, cf. [SW12, §5]. We generically write $(\iota_{\text{ext}}, \mathfrak{P}_{\text{ext}}(\mathcal{A}))$ for one element of this isomorphism class.

2.3. Linear relations: Let $\mathcal{A}$ be an almost Pontryagin space.

(*i*) A linear subspace T of $\mathcal{A}^2$ is called a *linear relation* in $\mathcal{A}$. We say that T is a *closed linear relation*, if T is closed in the product topology of $\mathcal{A}^2$.

(*ii*) The *adjoint* T^* of a linear relation T is defined as

$$T^* := \big\{(x, y) \in \mathcal{A}^2 : \; [y, a] - [x, b] = 0, (a, b) \in T\big\}.$$

Clearly, T^* is a linear relation in $\mathcal{A}$. Since the inner product is continuous, T^* is closed.

(*iii*) For a linear relation T we denote

$$\begin{aligned} \operatorname{dom} T &:= \big\{x \in \mathcal{A} : \; \exists\, y \in \mathcal{A} \text{ s.t. } (x, y) \in T\big\}, \\ \operatorname{ran} T &:= \big\{y \in \mathcal{A} : \; \exists\, x \in \mathcal{A} \text{ s.t. } (x, y) \in T\big\}, \\ \ker T &:= \big\{x \in \mathcal{A} : \; (x, 0) \in T\big\}, \\ \operatorname{mul} T &:= \big\{y \in \mathcal{A} : \; (0, y) \in T\big\}. \end{aligned}$$

We call T an *operator* if $\operatorname{mul} T = \{0\}$. We call T a *bounded operator* if it is an operator and continuous w.r.t./ the Hilbert space topology of $\mathcal{A}$.

(*iv*) Let T and S be linear relations in A and $\lambda, \mu \in \mathbb{C}$. Then we denote

$$\begin{aligned} T + S &:= \big\{(x, y + z) : \; (x, y) \in T, (x, z) \in S\big\}, \\ \lambda T &:= \big\{(x, \lambda y) : \; (x, y) \in T\big\}, \\ T^{-1} &:= \big\{(y, x) : \; (x, y) \in T\big\}. \end{aligned}$$

Moreover, we set $I := \{(x, x) : \; x \in \mathcal{A}\}$, and write $T - \lambda$ for $T - \lambda I$.

(*v*) For a linear relation T in $\mathcal{A}$ set

$$\sigma_p(T) := \big\{z \in \mathbb{C} : \; \ker(T - z) \neq \{0\}\big\} \cup \begin{cases} \{\infty\}, & \operatorname{mul} T \neq \{0\} \\ \emptyset \quad, & \text{otherwise} \end{cases}.$$

$\diamondsuit$

Let $\mathcal{A}$ be an almost Pontryagin space and let T be a linear relation in $\mathcal{A}$. Moreover, let $\tilde{\mathcal{A}}$ be another almost Pontryagin space which contains $\mathcal{A}$. Then, of course, we may consider T also as a linear relation in $\tilde{\mathcal{A}}$. Many properties of T are independent of the space in which T is considered, or depend only in an obvious way on it. Using for $\tilde{\mathcal{A}}$ a canonical Pontryagin space extension $\mathfrak{P}_{\rm ext}(\mathcal{A})$ of $\mathcal{A}$, often allows us to employ Pontryagin space theory.

2.2 Spectral theory

Let us recall the definitions and some properties of three sets associated with a closed linear relation.

2.4. Semi-Fredholm set: Let $\mathcal{A}$ be an almost Pontryagin space, and let T be a closed linear relation in $\mathcal{A}$. The *semi-Fredholm set* $\Phi_+(T)$ and the *index* ind_T of T is defined as

$$\begin{aligned} \Phi_+(T) &:= \big\{ z\in\mathbb{C} :\ \dim\ker(T-z)<\infty, \operatorname{ran}(T-z) \text{ closed} \big\}, \\ \operatorname{Ind}_T(z) &:= \dim\ker(T-z) - \dim\big(\mathcal{A}/\operatorname{ran}(T-z)\big) \in \mathbb{Z}\cup\{-\infty\}, \quad z\in\Phi_+(T). \end{aligned}$$

A proof of the following facts can be found in [DS87a, Theorem 2.4].

- *(i)* The set $\Phi_+(T)$ is open.
- *(ii)* The number $\operatorname{Ind}_T(z)$ is constant on each connected component of $\Phi_+(T)$.
- *(iii)* There exists a (unique) subset $\pi(T)$ of $\Phi_+(T)$, with the following properties:
  1. Let $\mathcal{Z}$ be a connected component of $\Phi_+(T)$. Then each of the numbers $\dim(\mathcal{A}/\operatorname{ran}(T-z))$ and $\dim\ker(T-z)$ is constant on $\mathcal{Z}\setminus\pi(T)$.
  2. Let $z\in\pi(T)$ and let $\mathcal{Z}$ be the component of $\Phi_+(T)$ which contains z. Then at the point z both mentioned dimensions are strictly larger than at points of $\mathcal{Z}\setminus\pi(T)$. In particular, for $z\in\pi(T)$ we have $\ker(T-z)\neq\{0\}$ and $\operatorname{ran}(T-z)\neq\mathcal{A}$.
- *(iv)* The set $\pi(T)$ consists of isolated points only.

$\diamondsuit$

Let $\tilde{\mathcal{A}}$ be an almost Pontryagin space which contains $\mathcal{A}$ (remember: as a closed subspace and with the same inner product and topology). Then the semi-Fredholm set does not depend on whether T is viewed as a linear relation in $\mathcal{A}$ or in $\tilde{\mathcal{A}}$.

The value of ind_T depends on the space in which T is considered, but only in the obvious way. If Ind_T denotes the index of T being considered as a linear relation in $\mathcal{A}$, and $\widetilde{\operatorname{Ind}}_T$ denotes the index of T being considered in $\tilde{\mathcal{A}}$, then

$$\widetilde{\operatorname{ind}}_T(z) = \operatorname{ind}_T(z) - \dim\big(\tilde{\mathcal{A}}/\mathcal{A}\big).$$

2.5. Points of regular type: Let $\mathcal{A}$ be an almost Pontryagin space, and let T be a closed linear relation in $\mathcal{A}$. The set $\gamma(T)$ of *points of regular type* of T is defined as

$$\gamma(T) := \big\{ z\in\mathbb{C} :\ (T-z)^{-1} \text{ is a bounded operator} \big\}.$$

(*i*) We have

$$\gamma(T) = \big\{ z \in \mathbb{C} : \ker(T - z) = \{0\}, \operatorname{ran}(T - z) \text{ closed} \big\},$$

and thus $\gamma(T) \subseteq \Phi_+(T)$. Moreover, $\operatorname{Ind}_T(z) = -\dim\big(\mathcal{A}/\operatorname{ran}(T-z)\big)$ for each $z \in \gamma(T)$.

(*ii*) The set $\gamma(T)$ is open.

The statement (*i*) is easy to see using that T is closed, and (*ii*) is contained in [DS87a, Proposition 2.2]. $\Diamond$

2.6. Resolvent set: Let $\mathcal{A}$ be an almost Pontryagin space, and let T be a closed linear relation in $\mathcal{A}$. The *resolvent set* $\rho(T)$ of T is defined as

$$\rho(T) := \big\{ z \in \mathbb{C} : (T - z)^{-1} \text{ is a bounded everywhere defined operator} \big\}.$$

(*i*) We have

$$\begin{aligned} \rho(T) :=& \big\{ z \in \mathbb{C} : \ker(T - z) = \{0\}, \operatorname{ran}(T - z) = \mathcal{A} \big\} \\ =& \big\{ z \in \gamma(T) : \operatorname{ran}(T - z) \text{ is dense in } \mathcal{A} \big\}. \end{aligned}$$

(*ii*) The set $\rho(T)$ is open.

The statement (*i*) is obvious, and (*ii*) is contained in [DS87a, Proposition 2.3]. $\Diamond$

Next we specifically consider symmetric and selfadjoint relations. For such more detailed information is available.

2.7 Definition. Let $\mathcal{A}$ be an almost Pontryagin space. A linear relation S in $\mathcal{A}$ is called *symmetric*, if $S \subseteq S^*$. Explicitly, this means that

$$[y_1, x_2] = [x_1, y_2], \quad (x_1, y_1), (x_2, y_2) \in S.$$

$\Diamond$

2.8. Spectral properties of symmetric relations: Let $\mathcal{A}$ be an almost Pontryagin space and S a closed symmetric relation in $\mathcal{A}$. Moreover, set $\kappa := \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A} = \operatorname{ind}_- \mathfrak{P}_{\text{ext}}(\mathcal{A})$.

(*i*) The set $\Phi_+(S)$ contains $\mathbb{C} \setminus \mathbb{R}$. In particular, it is either connected or splits into two connected components. In the latter case, these components are $\mathbb{C}^+$ and $\mathbb{C}^-$.

(*ii*) Set

$$\begin{aligned} \alpha_+(S) &:= \dim\ker(S - z), \quad z \in \mathbb{C}^+ \setminus \pi(S) \\ \alpha_-(S) &:= \dim\ker(S - z), \quad z \in \mathbb{C}^- \setminus \pi(S). \end{aligned}$$

Then $\alpha_+(S) = \alpha_-(S)$. This number, let us denote it by $\alpha(S)$, does not exceed κ.

(*iii*) We have

$$|\pi(S) \cap \mathbb{C}^+| \leq \kappa - \alpha(S), \quad |\pi(S) \cap \mathbb{C}^-| \leq \kappa - \alpha(S).$$

These facts follow by applying [DS87a, Proposition 4.3, Proposition 4.4] to the symmetry S considered as a relation in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$.

The *defect numbers* $\mathfrak{n}_+(S)$ and $\mathfrak{n}_-(S)$ of S are defined as

$$\mathfrak{n}_+(S) := -\operatorname{Ind}_S(z),\ z \in \mathbb{C}^+, \qquad \mathfrak{n}_-(S) := -\operatorname{Ind}_S(z),\ z \in \mathbb{C}^-.$$

The pair $(\mathfrak{n}_+(S), \mathfrak{n}_-(S))$ is also called the *defect index* of S.

(*iv*) Let $\tilde{\mathcal{A}}$ be an almost Pontryagin space which contains $\mathcal{A}$. The values of $\mathfrak{n}_+(S)$ and $\mathfrak{n}_-(S)$ depend on the space in which the relation S is considered, but only in the obvious way. If $\mathfrak{n}_\pm(S)$ denote the defect numbers of T being considered as a linear relation in $\mathcal{A}$, and $\widetilde{\mathfrak{n}_\pm}(S)$ denote the defect numbers of T being considered in $\tilde{\mathcal{A}}$, then

$$\widetilde{\mathfrak{n}_+}(S) = \mathfrak{n}_+(S) + \dim\big(\tilde{\mathcal{A}}/\mathcal{A}\big), \quad \widetilde{\mathfrak{n}_-}(S) = \mathfrak{n}_-(S) + \dim\big(\tilde{\mathcal{A}}/\mathcal{A}\big).$$

This is obvious.

(*v*) One of the following alternatives holds:

$$\begin{array}{ll} 1.: & \gamma(S) = \emptyset, \quad \sigma_p(S) = \mathbb{C} \cup \{\infty\} \\ 2.: & \gamma(S) \cap \mathbb{C}^+ = \mathbb{C}^+ \setminus \sigma_p(S), \quad |\mathbb{C}^+ \setminus \gamma(S)| \leq \kappa \end{array}$$

The second alternative takes place if and only if $\alpha(S) = 0$.

The same holds when $\mathbb{C}^+$ is replaced by $\mathbb{C}^-$.

(*vi*) The set $\gamma(S)$ is either connected or splits into two connected components. In the latter case, these components are $\gamma(S) \cap \mathbb{C}^+$ and $\gamma(S) \cap \mathbb{C}^-$.

Item (*v*) follows from [DS87a, Proposition 4.5] applied with S as a relation in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$. For (*vi*): if $\alpha(S) > 0$ use (*v*), if $\alpha(S) = 0$ combine [DS87a, Propositions 4.3, 4.4].

(*vii*) One of the following alternatives holds:

$$\begin{array}{ll} 1.: & \rho(S) \cap \mathbb{C}^+ = \emptyset \\ 2.: & \rho(S) \cap \mathbb{C}^+ = \gamma(S) \cap \mathbb{C}^+ = \mathbb{C}^+ \setminus \sigma_p(S), \quad |\mathbb{C}^+ \setminus \rho(S)| \leq \kappa \end{array}$$

The second alternative takes place if and only if $\alpha(S) = 0$ and $\operatorname{Ind}_S(z) = 0$, $z \in \mathbb{C}^+$.

The same holds when $\mathbb{C}^+$ is replaced by $\mathbb{C}^-$.

(*viii*) The set $\rho(S)$ is either connected or splits into two connected components. In the latter case, these components are $\rho(S) \cap \mathbb{C}^+$ and $\rho(S) \cap \mathbb{C}^-$.

These items are clear from the previous ones. ◊

Let A be linear relation in an almost Pontryagin space $\mathcal{A}$. Then the adjoint A^* of A always contains $\mathcal{A}^\circ \times \mathcal{A}^\circ$, hence, the usual definition " $A = A^*$ " of selfadjointness is not meaningful. It turns out that one rather should use defect numbers.

2.9 Definition. Let $\mathcal{A}$ be an almost Pontryagin space and let A be a linear relation in $\mathcal{A}$. We say that A is *selfadjoint*, if A is closed, symmetric, and $\mathfrak{n}_+(A) = \mathfrak{n}_-(A) = 0$. ◊

2.10 Remark. Let A be a selfadjoint relation in $\mathcal{A}$. Then either $\rho(A) = \emptyset$ or $|\rho(A) \cap \mathbb{C}^{\pm}| \leq \operatorname{ind}_{-} \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$. This follows from 2.8, (*vii*).

Even if $\mathcal{A}$ is a Pontryagin space, $\rho(A)$ may be empty. For example consider the space $\mathbb{C}^2$ endowed with the inner product

$$\Big[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\Big] := x_1\overline{y_2} + x_2\overline{y_1}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{C}^2,$$

and the relation

$$A := \operatorname{span}\Big\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big\} \times \operatorname{span}\Big\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big\}.$$

Contrasting the Pontryagin space case, where symmetry of the spectrum is known (cf. [DS87a, Corollary to Theorem 4.6]), $\rho(A)$ is not necessarily symmetric with respect the real axis if $\mathcal{A}$ is degenerated. For example consider the space $\mathbb{C}^1$ endowed with the inner product $[x, y] := 0$, $x, y \in \mathbb{C}$, let $\lambda \in \mathbb{C}$, and set $A := \operatorname{span}\{(1, \lambda)\}$. Then $\rho(A) = \mathbb{C} \setminus \{\lambda\}$.

This simple example, however, shows in essence the worst that can happen: If A is selfadjoint in the almost Pontryagin space $\mathcal{A}$, and $z \in \mathbb{C} \setminus \rho(A)$ whereas $\overline{z} \in \rho(A)$, then $\ker(A - z) \subseteq \mathcal{A}^{\circ}$. This follows since by symmetry

$$\ker(A - z) \perp \operatorname{ran}(A - \overline{z}) = \mathcal{A}.$$

$\Diamond$

In view of this remark we introduce a notation for the symmetrised set of points of regular type and resolvent set: for a closed symmetric relation S in an almost Pontryagin space denote

$$\gamma_s(S) := \big\{z \in \mathbb{C} : z, \overline{z} \in \gamma(S)\big\}, \quad \rho_s(S) := \big\{z \in \mathbb{C} : z, \overline{z} \in \rho(S)\big\}.$$

The following test is often practical to check whether a point belongs to the resolvent set of a selfadjoint relation.

2.11 Lemma. *Let $\mathcal{A}$ be an almost Pontryagin space and let A be a selfadjoint relation in $\mathcal{A}$. Then*

$$\rho(A) = \gamma(A) = \big\{z \in \mathbb{C} : \operatorname{ran}(A - z) = \mathcal{A}\big\}.$$

Proof. Obviously $\rho(A)$ is contained in the other two written sets. Assume that $z \in \gamma(A)$. Then $z \in \Phi_+(A)$ and hence

$$0 = \operatorname{Ind}_A(z) = \underbrace{\dim\ker(A - z)}_{=0} - \dim\big(\mathcal{A}/\operatorname{ran}(A - z)\big).$$

We see that $\operatorname{ran}(A - z) = \mathcal{A}$, and hence that $z \in \rho(A)$.

Asume now that $\operatorname{ran}(A - z) = \mathcal{A}$. If $z \in \mathbb{R}$ this implies that $\ker(A - z) \subseteq \mathcal{A}^{\circ}$. In particular, $\dim\ker(A - z) < \infty$ and we conclude that $z \in \Phi_+(A)$. If $z \in \mathbb{C} \setminus \mathbb{R}$, certainly also $z \in \Phi_+(A)$. Thus

$$0 = \operatorname{Ind}_A(z) = \dim\ker(A - z) - \underbrace{\dim\big(\mathcal{A}/\operatorname{ran}(A - z)\big)}_{=0}.$$

We see that $\ker(A - z) = \{0\}$, and hence that $z \in \rho(A)$. ❑

In the further chapters of this paper, we will repeatedly use another simple fact in order to show that a relation actually is selfadjoint.

2.12 Lemma. *Let $\mathcal{A}$ be an almost Pontryagin space and let S be a linear relation in $\mathcal{A}$. Then the relation $A := \operatorname{clos}_{\mathcal{A}^2} S$ is selfadjoint, if and only if S is symmetric and there exists $z_+ \in \mathbb{C}^+$ and $z_- \in \mathbb{C}^-$ with*

$$\operatorname{clos}_{\mathcal{A}} \operatorname{ran}(S - z_+) = \operatorname{clos}_{\mathcal{A}} \operatorname{ran}(S - z_-) = \mathcal{A}, \tag{2.1}$$

$$\ker(A - \overline{z_+}) \cap \mathcal{A}^\circ = \ker(A - \overline{z_-}) \cap \mathcal{A}^\circ = \{0\}. \tag{2.2}$$

If A is selfadjoint, then $\{z \in \mathbb{C} \setminus \mathbb{R} : \operatorname{clos}_{\mathcal{A}} \operatorname{ran}(S - z) = \mathcal{A}\} \subseteq \rho(A)$.

Proof. Necessity of the stated conditions is obvious. Assume that S is symmetric, then A is a closed symmetric relation in $\mathcal{A}$. Assume moreover that (2.1) and (2.2) hold. By symmetry we have $\ker(A - \overline{z_\pm}) \perp \operatorname{ran}(A - z_\pm) \supseteq \operatorname{ran}(S - z_\pm)$ and (2.1) yields $\ker(A - \overline{z_\pm}) \subseteq \mathcal{A}^\circ$. Now (2.2) implies that $\ker(A - \overline{z_\pm}) = 0$. Since $\operatorname{ran}(A - z_\pm)$ is closed, we obtain from (2.1) that $\operatorname{ran}(A - z_\pm) = \mathcal{A}$. Hence,

$$\begin{aligned} 0 \leq \dim\ker(A - z_+) - \underbrace{\dim\big(\mathcal{A}/\operatorname{ran}(A - z_+)\big)}_{=0} &= \operatorname{Ind}_A(z_+) \\ = \operatorname{Ind}_A(\overline{z_-}) = \underbrace{\dim\ker(A - \overline{z_-})}_{=0} - \dim\big(\mathcal{A}/\operatorname{ran}(A - \overline{z_-})\big) &\leq 0. \end{aligned}$$

Thus $\mathfrak{n}_+(A) = -\operatorname{Ind}_A(z) = 0$, $z \in \mathbb{C}^+$. The analogous argument applies in the lower halfplane, and it follows that also $\mathfrak{n}_-(A) = 0$.

The last statement follows from Lemma 2.11. ❑

Finally, let us point out explicitly one obvious fact, which is important when studying extensions of symmetric relations.

2.13 Remark. Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be almost Pontryagin spaces, and let S and $\tilde{S}$ be closed symmetric relations in $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. Assume that $\tilde{\mathcal{A}}$ contains $\mathcal{A}$ and that $\tilde{S} \supseteq S$. Then we have $\gamma(\tilde{S}) \subseteq \gamma(S)$. In particular, if $\tilde{S}$ is selfadjoint then $\rho(\tilde{S}) \subseteq \gamma(S)$, and if S and $\tilde{S}$ are both selfadjoint then $\rho(\tilde{S}) \subseteq \rho(S)$.

This follows since $(S-z)^{-1} \subseteq (\tilde{S}-z)^{-1}$, and hence $(\tilde{S}-z)^{-1}$ being a bounded operator in $\tilde{\mathcal{A}}$ implies that $(S - z)^{-1}$ is a bounded operator in $\mathcal{A}$. Remember that the topology of $\mathcal{A}$ is nothing but the restriction of the topology of $\tilde{\mathcal{A}}$. ♢

2.3 Selfadjoint extensions with nonempty resolvent set

In this subjection we investigate selfadjoint extensions of a symmetric relation. First we discuss existence of selfadjoint extensions with nonempty resolvent set which are permitted to act in some possibly larger almost Pontryagin space.

2.14 Proposition. *Let $\mathcal{A}$ be an almost Pontryagin space and S a closed symmetric relation in $\mathcal{A}$. Then there exists an almost Pontryagin space $\tilde{\mathcal{A}}$ which contains $\mathcal{A}$ and a selfadjoint relation $A \subseteq \tilde{\mathcal{A}}^2$ with $A \supseteq S$ and $\rho(A) \neq \emptyset$ if and only if $\gamma(S) \neq \emptyset$ (equivalently, if and only if $\ker(S - z) = \{0\}$ for some $z \in \mathbb{C} \setminus \mathbb{R}$).*

If $\gamma(S) \neq \emptyset$ and $z \in \gamma_s(S)$, then we can choose $\tilde{\mathcal{A}}$ and A such that $\tilde{\mathcal{A}}$ is a Pontryagin space, $\operatorname{ind}_- \tilde{\mathcal{A}} = \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$,

$$\dim \tilde{\mathcal{A}}/\mathfrak{P}_{\text{ext}}(\mathcal{A}) = \begin{cases} \max\left\{\mathfrak{n}_+(S), \mathfrak{n}_-(S), \aleph_0\right\}, & \mathfrak{n}_+(S) \neq \mathfrak{n}_-(S), \\ 0, & \mathfrak{n}_+(S) = \mathfrak{n}_-(S), \end{cases}$$

and $z \in \rho(A)$.

Proof. Necessity is clear; if $A \supseteq S$, then $\rho(A) \subseteq \gamma(S)$. For the proof of sufficiency assume that $\gamma(S) \neq \emptyset$ and let $z \in \gamma_s(S)$ be given. We distinguish the cases that $z \in \mathbb{C} \setminus \mathbb{R}$ and $z \in \mathbb{R}$.

Case $z \in \mathbb{C} \setminus \mathbb{R}$: We pass to the Caley-transform. Set

$$\beta := \{(y - zx, y - \overline{z}x) : \ (x, y) \in S\},$$

then β is a linear and isometric homeomorphism of the closed subspace $\mathcal{D} := \operatorname{ran}(S - z)$ onto the closed subspace $\mathcal{D}' := \operatorname{ran}(S - \overline{z})$, considered as subspaces of $\mathfrak{P}_{\text{ext}}(\mathcal{A})$. The aim is to extend β to a linear and isometric homeomorphism $\tilde{\beta}$ of a suitable Pontryagin space onto itself, since then the inverse Caley-transform A of $\tilde{\beta}$ will be a selfadjoint extension of S with $z, \overline{z} \in \rho(A)$.

Consider first the case that $\mathcal{D}$ (and hence also $\mathcal{D}'$) is nondegenerated. Then $\mathcal{D}$ and $\mathcal{D}'$ are orthocomplemented, the spaces $\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{D}$ and $\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{D}'$ are Pontryagin spaces with the same negative index, and their dimensions are $\mathfrak{n}_+(S) + \operatorname{ind}_0 \mathcal{A}$ and $\mathfrak{n}_-(S) + \operatorname{ind}_0 \mathcal{A}$, respectively. If $\mathfrak{n}_+(S) = \mathfrak{n}_-(S)$, we can choose an isometric isomorphism γ of $\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{D}$ onto $\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{D}'$, and obtain a required extension as ($P : \mathfrak{P}_{\text{ext}}(\mathcal{A}) \to \mathcal{D}^\perp$ denotes the orthogonal projection)

$$\tilde{\beta} := \beta(I - P) + \gamma P.$$

Assume now that $\mathfrak{n}_+(S) \neq \mathfrak{n}_-(S)$. Choose a Hilbert space $\mathcal{H}$ with

$$\dim \mathcal{H} = \max\left\{\mathfrak{n}_+(S), \mathfrak{n}_-(S), \aleph_0\right\},$$

then the spaces $(\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{D})[\dot{+}]\mathcal{H}$ and $(\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{D}')[\dot{+}]\mathcal{H}$ are Pontryagin spaces with the same negative index and dimension. Again we can choose an isometric isomorphism γ between these spaces and obtain $\tilde{\beta}$ in the same way as above.

Assume now that $\mathcal{D}$ is degenerated. We extend β to a nondegenerated domain, then the above case will apply. Choose a decomposition of $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ of the form

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}) = \mathcal{D}_r[\dot{+}]\big(\mathcal{D}^\circ \dot{+} \mathcal{D}_1\big)[\dot{+}]\mathcal{R},$$

where $\mathcal{D}_r$ is a closed and nondegenerated subspace of $\mathcal{D}$ with $\mathcal{D}_r[\dot{+}]\mathcal{D}^\circ = \mathcal{D}$, and where $\mathcal{D}_1$ is skewly linked with $\mathcal{D}^\circ$, see, e.g., [IKL82, Theorem 3.4]. Next choose a decomposition of $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ which fits the action of β: Set $\mathcal{D}_r' := \beta(\mathcal{D}_r)$, then $\mathcal{D}_r'$ is a closed subspace of $\mathcal{D}'$ and $\mathcal{D}_r'[\dot{+}](\mathcal{D}')^\circ = \mathcal{D}'$. Choose a space D_1' which is skewly linked with $(\mathcal{D}')^\circ$ and such that (with appropriate $\mathcal{R}'$)

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}) = \mathcal{D}_r'[\dot{+}]\big((\mathcal{D}')^\circ \dot{+} \mathcal{D}_1'\big)[\dot{+}]\mathcal{R}'.$$

Set $n := \dim \mathcal{D}^\circ$ and choose bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ of $\mathcal{D}^\circ$ and $\mathcal{D}_1$ which are skewly linked, i.e. satisfy

$$[e_i, f_j] = \begin{cases} 1\,, & i = j \\ 0\,, & i \neq j \end{cases}$$

Set $e_i' := \beta e_i$, $i = 1, \dots, n$, and let $\{f_1', \dots, f_n'\}$ be the basis of $\mathcal{D}_1'$ with

$$[e_i', f_j'] = \begin{cases} 1\,, & i = j \\ 0\,, & i \neq j \end{cases}$$

Define an extension $\hat{\beta} : \mathcal{D}_r[\dot{+}]\big(\mathcal{D}^\circ \dot{+} \mathcal{D}_1\big) \to \mathcal{D}_r'[\dot{+}]\big((\mathcal{D}')^\circ \dot{+} \mathcal{D}_1'\big)$ of β by linearity and the requirements that

$$\hat{\beta} f_i = f_i',\ i = 1, \dots, n.$$

Then $\hat{\beta}$ is a linear and isometric homeomorphism with a closed and nondegenerated domain.

Case $z \in \mathbb{R}$: In this case certainly $\mathfrak{n}_+(S) = \mathfrak{n}_-(S)$. Set

$$\mathcal{D} := \operatorname{ran}(S - z), \quad \varphi := (S - z)^{-1}.$$

Then $\mathcal{D}$ is a closed subspace of $\mathcal{A}$ and hence of $\mathfrak{P}_{\text{ext}}(\mathcal{A})$, and φ is a bounded operator from $\mathcal{D}$ into $\mathcal{A}$ with

$$[\varphi x, y] = [x, \varphi y], \quad x, y \in \mathcal{D}. \tag{2.3}$$

The aim is to extend φ to a bounded operator $\tilde{\varphi}$ on all of $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ retaining the symmetry property (2.3), since then $A := \tilde{\varphi}^{-1} + z$ is a selfadjoint extension of S with $z \in \rho(A)$.

Again let us first consider the case that $\mathcal{D}$ is nondegenerated, and denote again by $P : \mathfrak{P}_{\text{ext}}(\mathcal{A}) \to \mathcal{D}^\perp$ the orthogonal projection. Then $P\varphi : \mathcal{D} \to \mathcal{D}^\perp$ is a bounded operator between Pontryagin spaces. Let $(P\varphi)^* : \mathcal{D}^\perp \to \mathcal{D}$ be its (Pontryagin space-) adjoint, and set

$$\tilde{\varphi} := \varphi(I - P) + (P\varphi^*)P.$$

Then $\tilde{\varphi}$ is a bounded operator of $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ into itself and extends φ. Let us check (2.3):

$$\begin{aligned}
[\tilde{\varphi} x, y] &= [\varphi(I - P)x + (P\varphi)^* Px, y] \\
&= [\varphi(I - P)x, (I - P)y] + [\varphi(I - P)x, Py] + [(P\varphi)^* Px, (I - P)y] \\
&\quad + \underbrace{[(P\varphi)^* Px, Py]}_{=0} \\
&= [(I - P)x, \varphi(I - P)y] + [(I - P)x, (P\varphi)^* Py] + [Px, \varphi(I - P)y] \\
&\quad + \underbrace{[Px, (P\varphi)^* Py]}_{=0} \\
&= [x, \varphi y], \quad x, y \in \mathfrak{P}_{\text{ext}}(\mathcal{A}).
\end{aligned}$$

Assume now that $\mathcal{D}$ is degenerated. Again we are going to extend φ to a nondegenerated domain retaining (2.3), so that then the above case applies. In order to achieve such an extension, we show that any operator φ_0 with (2.3) can be extended to every space containing the domain of φ_0 with codimension one. Since there exist nondegenerated subspaces containing $\mathcal{D}$ with finite codimension, we can iteratively apply this fact and achieve a required extension of φ to a nondegenerated domain.

Let $\varphi_0 : \mathcal{D}_0 \to \mathfrak{P}_{\text{ext}}(\mathcal{A})$ be a bounded operator and assume that it satisfies (2.3). Moreover, let $x_0 \in \mathfrak{P}_{\text{ext}}(\mathcal{A}) \setminus \mathcal{D}_0$ and set $\mathcal{D}_1 := \mathcal{D}_0 + \operatorname{span}\{x_0\}$. The linear functional

$$\lambda : \begin{cases} \mathcal{D}_1 & \to \quad \mathbb{C} \\ x + \alpha x_0 & \mapsto \quad [\varphi_0 x, x_0] \end{cases}$$

is continuous. Choose an element $y_0 \in \mathfrak{P}_{\text{ext}}(\mathcal{A})$ which represents it as $[.,y_0]$, and define an extension φ_1 of φ_0 to $\mathcal{D}_1$ by linearity and the requirement that $\varphi_1 x_0 = y_0$. Then φ_1 is a bounded operator of $\mathcal{D}_1$ into $\mathfrak{P}_{\text{ext}}(\mathcal{A})$.

To check (2.3) for φ_1, let two elements $x + \alpha x_0$, $x' + \alpha' x_0 \in \mathcal{D}_1$ be given. Here $x, x' \in \mathcal{D}_0$ and $\alpha, \alpha' \in \mathbb{C}$. Using (2.3) for the map φ_0 and the definition of y_0, we compute

$$\begin{aligned} \big[\varphi_1(x+\alpha x_0), x' + \alpha' x_0\big] = & \\ &= [\varphi_0 x, x'] + [\varphi_0 x, \alpha' x_0] + [\alpha y_0, x'] + \underbrace{[\alpha y_0, \alpha' x_0]}_{=0} = \\ &= [x, \varphi_0 x'] + [x, \alpha' y_0] + [\alpha x_0, \varphi_0 x'] + \underbrace{[\alpha x_0, \alpha' y_0]}_{=0} = \\ &= \big[x + \alpha x_0, \varphi_1(x' + \alpha' x_0)\big] \end{aligned}$$

This completes the task. ❑

In the second result of this subsection, we investigate existence of canonical extensions, i.e., extensions A of S which act in the same space as S. The proof is again of geometric nature.

2.15 Proposition. *Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed symmetric relation in $\mathcal{A}$ with $\gamma(S) \neq \emptyset$ and $\mathfrak{n}_+(S) = \mathfrak{n}_-(S) < \infty$. There exists a selfadjoint relation $A \subseteq \mathcal{A}^2$ with $A \supseteq S$ and $\rho(A) \neq \emptyset$ if and only if*

$$\exists\, \mu \in \gamma_s(S) \setminus \mathbb{R} : \ (S-\mu)^{-1}\big(\mathcal{A}^\circ \cap \operatorname{ran}(S-\mu)\big) \subseteq \mathcal{A}^\circ. \tag{2.4}$$

If this condition is satisfied, the choice of A can be made such that $\mu \in \rho_s(A)$.

Proof. To see necessity of (2.4), assume that a selfadjoint extension A of S with $\rho(A) \neq \emptyset$ is given. Choose $\mu \in \rho_s(A) \setminus \mathbb{R}$. Then, clearly, $\mu \in \gamma_s(S)$. For $x \in \mathcal{A}^\circ$ we have

$$[(A-\mu)^{-1}x, y] = [x, (A-\overline{\mu})^{-1}y] = 0, \quad y \in \mathcal{A},$$

and hence $(A-\mu)^{-1}\mathcal{A}^\circ \subseteq \mathcal{A}^\circ$. Thus also the inclusion (2.4) holds.

We turn to the proof of sufficiency. Assume that $\mu \in \mathbb{C} \setminus \mathbb{R}$ and (2.4) holds. We consider the Cayley transform of S with base point μ; set

$$\beta := \{(y - \mu x, y - \overline{\mu} z x) : \ (x,y) \in S\}, \qquad R := \operatorname{ran}(S-\mu), \quad R' := \operatorname{ran}(S-\overline{\mu}).$$

Then β is an isometric, bijective, and homeomorphic map between the closed subspaces R and R'.

Next, we choose decompositions of $\mathcal{A}$ which are compatible with the action of β. Set

$$D_1 := R \cap \mathcal{A}^\circ$$

and choose (we write $E\#D$ to express that E and D are skewly linked subspaces)

$$\begin{array}{rll} D_2 & \text{s.t.} & D_2 \dot{+} D_1 = R^\circ \\ R_1 & \text{s.t.} & R_1 \dot{+} R^\circ = R, R_1 \text{ closed} \\ E_1 & \text{s.t.} & E_1 \dot{+} D_1 = \mathcal{A}^\circ \\ E_2 & \text{s.t.} & E_2 \perp R_1,\ E_2 \# D_2 \\ Q & \text{s.t.} & Q \dot{+} \mathcal{A}^\circ = \big(R_1 [\dot{+}] (D_2 \dot{+} E_2)\big)^\perp, Q \text{ closed} \end{array}$$

Set

$$R_1' := \beta(R_1), \quad D_1' := \beta(D_1), \quad D_2' := \beta(D_2).$$

Then, remember our hypothesis (2.4) and the properties of Caley transform (see [SW16, §2.4])

$$D_1' \subseteq \mathcal{A}^\circ, \quad D_2' \dot{+} D_1' = (R')^\circ = R^\circ, \quad R_1' \dot{+} R^\circ = R, R_1' \text{ closed}.$$

Choose

$$\begin{array}{rll} E_1' & \text{s.t.} & E_1' \dot{+} D_1' = \mathcal{A}^\circ \\ E_2' & \text{s.t.} & E_2' \perp R_1',\ E_2' \# D_2' \\ Q' & \text{s.t.} & Q' \dot{+} \mathcal{A}^\circ = \big(R_1' [\dot{+}] (D_2' \dot{+} E_2')\big)^\perp, Q \text{ closed} \end{array}$$

Since β is bijective and isometric, we have

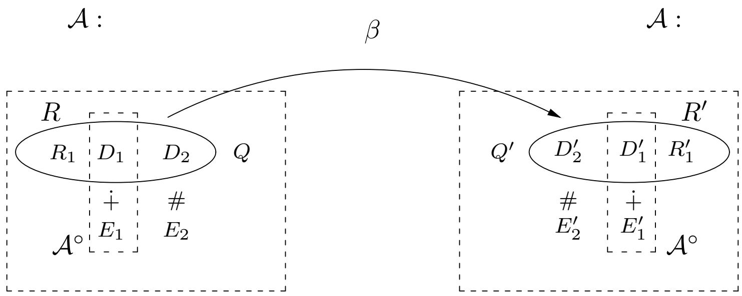
$$\dim D_1 = \dim D_1',\ \dim D_2 = \dim D_2', \quad \operatorname{ind}_- R_1 = \operatorname{ind}_- R_1'.$$

Together with the fact that $\mathfrak{n}_+(S) = \mathfrak{n}_-(S) < \infty$, this implies

$$\dim E_1 = \dim E_1', \quad \dim E_2 = \dim E_2',$$

$$\dim Q = \dim Q' < \infty, \quad \operatorname{ind}_- Q = \operatorname{ind}_- Q'.$$

We can picture these spaces as follows (columns are pairwise orthogonal):



Choose bijective and isometric maps

$$\begin{aligned}
&\beta_1 : E_1 \to E_1' \\
&\beta_2 : D_2 \dot{+} E_2 \to D_2' \dot{+} E_2' \text{ with } \beta_2|_{D_2\dot{+}E_2} = \beta|_{D_2\dot{+}E_2} \\
&\beta_3 : Q \to Q'
\end{aligned}$$

and define a map $\tilde{\beta} : \mathcal{A} \to \mathcal{A}$ by linearity and the requirements that

$$\tilde{\beta}|_R = \beta, \quad \tilde{\beta}|_{E_1} = \beta_1, \quad \tilde{\beta}|_{E_2} = \beta_2|_{E_2}, \quad \tilde{\beta}|_Q = \beta_3.$$

Clearly, $\tilde{\beta}$ is a bijective and isometric map of $\mathcal{A}$ onto itself which extends β. Since R and R' are closed and have finite codimension, $\tilde{\beta}$ is also homeomorphic. Its inverse Cayley transform $\tilde{S}$ is thus a selfadjoint relation in $\mathcal{A}$ with $\mu \in \rho_s(\tilde{S})$ which extends S. ❑

2.4 Minimality aspects

Minimality of a selfadjoint relation with respect to a subset is defined in the usual way.

2.16 Definition. Let $\mathcal{A}$ be an almost Pontryagin space, $A \subseteq \mathcal{A}^2$ a selfadjoint relation, and $M \subseteq \mathcal{A}$. Then we say that A is *M-minimal*, if

$$\mathcal{A} = \operatorname{cls}\Big(M \cup \bigcup_{z\in\rho(A)} (A-z)^{-1}M\Big). \tag{2.5}$$

◊

The set $\rho(A)$ in (2.5) can be substituted by much smaller sets. This is a well known consequence of analyticity: if A is a selfadjoint relation in a Pontryagin space and Ω is an open subset of $\rho(A)$ which intersects each connected component of $\rho(A)$, then

$$\operatorname{cls}\Big(M \cup \bigcup_{z\in\rho(A)} (A-z)^{-1}M\Big) = \operatorname{cls}\Big(M \cup \bigcup_{z\in\Omega} (A-z)^{-1}M\Big). \tag{2.6}$$

The same holds in the degenerated situation (as is seen with the same argument).

2.17 Lemma. *Let $\mathcal{A}$ be an almost Pontryagin space, $A \subseteq \mathcal{A}^2$ a selfadjoint relation, and $M \subseteq \mathcal{A}$. If Ω is an open subset of $\rho(A)$ which intersects each connected component of $\rho(A)$, then the equality* (2.6) *holds.*

Proof. If $\rho(A) = \emptyset$, there is nothing to prove. Hence, assume that $\rho(A) \neq \emptyset$. Let $y \in M$ and $x \in \mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\operatorname{cls}(M \cup \bigcup_{z\in\Omega}(A-z)^{-1}M)$. The function

$$z \mapsto [(A-z)^{-1}y, x], \quad z \in \rho(A),$$

is analytic and vanishes on Ω. Hence, it vanishes on all of $\rho(A)$, i.e.,

$$x \in \mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\operatorname{cls}\Big(M \cup \bigcup_{z\in\rho(A)} (A-z)^{-1}M\Big).$$

Since $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ is nondegenerated, passing once more to orthogonal complements yields the inclusion '$\subseteq$' in (2.6). The reverse inclusion is obvious. ❑

The concept of a minimal – synonymously, completely nonselfadjoint – symmetric relation in an almost Pontryagin space is also defined in the usual way (only taking care of possible unsymmetrically located spectral points).

2.18 Definition. Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed linear relation in $\mathcal{A}$ with $\gamma(S) \neq \emptyset$. We say that S is *minimal*, if

$$\bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z) = \{0\}.$$

$\Diamond$

Again, the usual consequence of analyticity holds true.

2.19 Lemma. *Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed symmetric relation in $\mathcal{A}$ with $\gamma(S) \neq \emptyset$. Moreover, let Ω be an open subset of $\gamma_s(S)$ which intersects each component of $\gamma_s(S)$. Then*

$$\bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z) = \bigcap_{z \in \Omega} \operatorname{ran}(S - z).$$

Proof. Fix $z_0 \in \gamma_s(S)$ such that $z_0 \in \mathbb{R}$ if $\gamma_s(S) \cap \mathbb{R} \neq \emptyset$. Choose a selfadjoint extension A_0 of S acting in a Pontryagin space $\tilde{\mathcal{A}}_0 \supseteq \mathcal{A}$ such that $z_0 \in \rho(A_0)$, cf. Proposition 2.14. For each $z, w \in \rho(A_0)$, the operator

$$I + (\overline{z} - \overline{w})(A_0 - \overline{z})^{-1} : \tilde{\mathcal{A}}_0[-] \operatorname{ran}(S - w) \to \tilde{\mathcal{A}}_0[-] \operatorname{ran}(S - z)$$

is bijective.

Choose $w \in \rho(A_0) \cap \Omega$, and set $M := \tilde{\mathcal{A}}[-] \operatorname{ran}(S - w)$. Then, using Lemma 2.17,

$$\tilde{\mathcal{A}}_0[-] \operatorname{ran}(S - z) \subseteq \operatorname{cls}\Big(M \cup \bigcup_{\zeta \in \rho(A_0)} (A_0 - \zeta)^{-1} M\Big)$$
$$= \operatorname{cls}\Big(M \cup \bigcup_{\zeta \in \rho(A_0) \cap \Omega} (A_0 - \zeta)^{-1} M\Big) \subseteq \operatorname{cls} \bigcup_{\zeta \in \Omega} \big(\tilde{\mathcal{A}}_0[-] \operatorname{ran}(S - \zeta)\big), \quad z \in \rho(A_0).$$

Note here that $\rho(A_0)$ is connected if $\gamma_s(S)$ is. Passing to orthogonal complements yields

$$\bigcap_{\zeta \in \Omega} \operatorname{ran}(S - \zeta) \subseteq \bigcap_{\zeta \in \rho(A_0)} \operatorname{ran}(S - \zeta).$$

Now let $z \in \gamma_s(S)$ be given. Choose a selfadjoint extension A of S acting in some Pontryagin space $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ such that $z \in \rho(A)$. Replacing A_0 by A and Ω by $\rho(A_0) \cap \rho(A)$ yields

$$\tilde{\mathcal{A}}[-] \operatorname{ran}(S - z) \subseteq \operatorname{cls} \bigcup_{\zeta \in \rho(A_0) \cap \rho(A)} \big(\tilde{\mathcal{A}}[-] \operatorname{ran}(S - \zeta)\big).$$

Together with the above, thus,

$$\bigcap_{\zeta \in \Omega} \operatorname{ran}(S - \zeta) \subseteq \bigcap_{\zeta \in \rho(A_0)} \operatorname{ran}(S - \zeta) \subseteq \bigcap_{\zeta \in \rho(A_0) \cap \rho(A)} \operatorname{ran}(S - \zeta) \subseteq \operatorname{ran}(S - z).$$

The inclusion "$\supseteq$" in the present assertion follows. The reverse inclusion is trivial. ❑

It is often useful to pass to minimal symmetries. As a consequence of [SW16] also this can be done in the standard way.

2.20 Lemma. *Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed linear relation in $\mathcal{A}$ with $\gamma(S) \neq \emptyset$. Set*

$$\mathcal{C} := \bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z), \quad \mathcal{D} := \mathcal{C}^{\perp}, \qquad \mathcal{A}_1 := \mathcal{D}/_{\mathcal{C}^{\circ}},$$

let $\pi : \mathcal{D} \to \mathcal{A}_1$ denote the canonical projection, and set

$$S_1 := (\pi \times \pi)\big(S \cap (\mathcal{D} \times \mathcal{D})\big).$$

Then S_1 is a minimal closed symmetric relation in the almost Pontryagin space $\mathcal{A}_1$, $\gamma_s(S) \subseteq \gamma(S_1)$, and $\mathfrak{n}_\pm(S_1) \leq \mathfrak{n}_\pm(S)$.

Proof. Let $z, w \in \gamma_s(S)$. If $x \in \operatorname{ran}(S - w) \cap \operatorname{ran}(S - z)$, write

$$x = b - wa = b' - za' \text{ with some } (a, b), (a', b') \in S.$$

Then

$$a = \frac{1}{w - z}\big((b - b') - z(a - a')\big) \in \operatorname{ran}(S - z).$$

We see that

$$(S - w)^{-1}\Big(\bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z)\Big) \subseteq \bigcap_{z \in \gamma_s(S)\setminus\{w\}} \operatorname{ran}(S - z) = \bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z). \quad (2.7)$$

The set $\gamma_s(S)$ is symmetric w.r.t. the real line, and hence

$$[(S - w)^{-1}x, y] = [x, (S - \overline{w})^{-1}y] = 0, \quad x \in \mathcal{D} \cap \operatorname{ran}(S - w), y \in \bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z).$$

This shows that

$$(S - w)^{-1}(\mathcal{D} \cap \operatorname{ran}(S - w)) \subseteq \mathcal{D},$$

and together with (2.7) that

$$(S - w)^{-1}(\mathcal{C}^{\circ}) \subseteq \mathcal{C}^{\circ}.$$

Now [SW16, Proposition 3.2] applies and yields

$$\gamma_s(S) \subseteq \gamma(S_1) \quad \text{and} \quad \operatorname{ran}(S_1 - z) = \pi(\mathcal{D} \cap \operatorname{ran}(S - z)), \; z \in \gamma_s(S). \qquad (2.8)$$

Using that $\ker \pi = \mathcal{C}^{\circ} \subseteq \mathcal{D} \cap \operatorname{ran}(S - z)$, $z \in \gamma_s(S)$, we obtain

$$\bigcap_{z \in \gamma_s(S_1)} \operatorname{ran}(S_1 - z) \subseteq \bigcap_{z \in \gamma_s(S)} \pi\big(\mathcal{D} \cap \operatorname{ran}(S - z)\big)$$
$$= \pi\Big(\mathcal{D} \cap \bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z)\Big) = \pi(\mathcal{C}^{\circ}) = \{0\}.$$

The inequalities $\mathfrak{n}_\pm(S_1) \leq \mathfrak{n}_\pm(S)$ are clear from (2.8). ❑

PART I
COMPRESSED RESOLVENTS

I.1 Definition and basic properties of compressed resolvents

Let $\mathcal{P}$ and $\tilde{\mathcal{P}}$ be Pontryagin spaces with $\mathcal{P} \subseteq \tilde{\mathcal{P}}$, and let A be a selfadjoint relation in $\tilde{\mathcal{P}}$ with nonempty resolvent set. Denoting by P the orthogonal projection of $\tilde{\mathcal{P}}$ onto $\mathcal{P}$, the operator valued function

$$T(z) := P(A-z)^{-1}|_{\mathcal{P}}, \quad z \in \rho(A), \tag{I.1}$$

is called the *compressed resolvent of* A.

Viewing this notion another way, we may say that a function T, defined on some open subset of the complex plane and taking values in the set of all bounded linear operators on $\mathcal{P}$, is a compressed resolvent if it admits the representation (I.1) with some selfadjoint relation A acting in some Pontryagin space $\tilde{\mathcal{P}} \supseteq \mathcal{P}$.

Seeking for an analogue in the degenerated setting one immediately runs into the problem that an orthogonal projection of $\tilde{\mathcal{P}}$ onto $\mathcal{P}$ does not exist unless $\mathcal{P}^\circ \subseteq \tilde{\mathcal{P}}^\circ$. Hence the right side of (I.1) at once becomes meaningless. This difficulty has been recognised and a possible way out was proposed in [KW99b]. The idea is to substitute the operator valued function (I.1) by the family of scalar valued functions

$$\Big\{ z \mapsto \big[(A-z)^{-1}x, y\big] : x, y \in \mathcal{A} \Big\}.$$

I.1 Definition. Let $\mathcal{E}$ be an inner product space, let Ω be an open and nonempty subset of $\mathbb{C}$, and let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$. Then we say that R is a *compressed resolvent*, if it satisfies the following axioms.

(CR1) For each fixed $x, y \in \mathcal{E}$ the function

$$R(x,y;.) : \begin{cases} \Omega & \to & \mathbb{C} \\ z & \mapsto & R(x,y;z) \end{cases}$$

is continuous.

(CR2) There exists an almost Pontryagin space $\mathcal{A}$, a linear and isometric map $\iota : \mathcal{E} \to \mathcal{A}$, and a selfadjoint relation $A \subseteq \mathcal{A}^2$ with nonempty resolvent set, such that

$$R(x,y;z) = \big[(A-z)^{-1}\iota x, \iota y\big], \quad x, y \in \mathcal{A}, \ z \in \Omega \cap \rho(A). \tag{I.2}$$

If $\mathcal{A}$, ι, and A are as in (CR2), we say that the triple $\langle \mathcal{A}, \iota, A \rangle$ *induces* R. $\Diamond$

The generalisation compared to the Pontryagin space situation is twofold. One, we allow $\mathcal{E}$ to be an arbitrary inner product space, in particular, $\mathcal{E}$ may be a degenerated almost Pontryagin space. Two, the space $\mathcal{A}$ may be degenerated; this is of interest when thinking of minimality issues and increases ease in handling.

I.2 Lemma. *Let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$ be a compressed resolvent.*

(*i*) *There exists a triple $\langle \mathcal{A}_0, \iota_0, A_0 \rangle$ which induces R with $\mathcal{A}_0$ being a Pontryagin space and A_0 being $\iota_0\mathcal{E}$-minimal.*

(*ii*) *Assume that R is induced by $\langle \mathcal{A}, \iota, A \rangle$. Then there exists a triple $\langle \mathcal{A}_1, \iota_1, A_1 \rangle$ which induces R with A_1 being $\iota_1\mathcal{E}$-minimal and* $\ker \iota_1 = \ker \iota$.

(*iii*) *Assume that R is induced by $\langle \mathcal{A}, \iota, A \rangle$. Then there exists a triple $\langle \mathcal{A}_2, \iota_2, A_2 \rangle$ which induces R with $\mathcal{A}_2$ being a Pontryagin space and* $\ker \iota_2 = \ker \iota$.

Proof. Let $\langle \mathcal{A}, \iota, A \rangle$ be a triple which induces R, and set

$$\mathcal{D} := \operatorname{cls}\Big(\iota\mathcal{E} \cup \bigcup_{z \in \rho(A)} (A - z)^{-1}\iota\mathcal{E}\Big), \quad \mathcal{B}_0 := \mathcal{D}^\circ,\ \mathcal{B}_1 := \{0\}.$$

By the resolvent identity and analyticity of the resolvent of A

$$(A - z)^{-1}\mathcal{D} \subseteq \mathcal{D}, \quad z \in \rho(A).$$

Thus also $(A - \overline{z})^{-1}\mathcal{D}^\perp \subseteq \mathcal{D}^\perp$, $z \in \rho(A)$. We conclude that $(A - z)^{-1}\mathcal{B}_0 \subseteq \mathcal{B}_0$, $z \in \rho_s(A)$. The space $\mathcal{B}_1$ is trivially invariant. Set

$$\mathcal{A}_j := \mathcal{D}/_{\mathcal{B}_j}, \quad j = 0, 1,$$

let $\pi_j : \mathcal{D} \to \mathcal{A}_j$ be the canonical projection, and set

$$A_j := (\pi_j \times \pi_j)\big(A \cap (\mathcal{D} \times \mathcal{D})\big).$$

Applying [SW16, Proposition 3.2] shows that A_j is selfadjoint and that

$$\begin{aligned} &\rho_s(A) \subseteq \rho(A_j), \\ &\big[(A_j - z)^{-1}\pi_j x, \pi_j y\big]_{\mathcal{A}_j} = \big[(A - z)^{-1}x, y\big]_{\mathcal{A}}, \quad x, y \in \mathcal{D},\ z \in \rho_s(A). \end{aligned}$$

In particular, therefore,

$$R(x, y; z) = \big[(A - z)^{-1}\iota x, \iota y\big]_{\mathcal{A}} = \big[(A_j - z)^{-1}(\pi_j \circ \iota)x, (\pi_j \circ \iota)y\big]_{\mathcal{A}_j}, \qquad x, y \in \mathcal{E},\ z \in \Omega \cap \rho_s(A).$$

Setting $\iota_j := \pi_j \circ \iota$, we may say that $\langle \mathcal{A}_j, \iota_j, A_j \rangle$ induces R.

Again referring to [SW16, Proposition 3.2], we have

$$(A_j - z)^{-1}(\pi(\iota\mathcal{E})) = \pi\big((A - z)^{-1}\iota\mathcal{E}\big), \quad z \in \rho_s(A).$$

This shows that

$$\operatorname{cls}\Big(\iota_j\mathcal{E} \cup \bigcup_{z \in \rho(A_j)} (A_j - z)^{-1}\iota_j\mathcal{E}\Big) \supseteq \pi\Big(\operatorname{span}\big(\iota\mathcal{E} \cup \bigcup_{z \in \rho_s(A)} (A - z)^{-1}\iota\mathcal{E}\big)\Big).$$

By definition the linear span appearing on the right side is dense in $\mathcal{D}$, and π_j being continuous and surjective implies that it image under π_j is dense in $\mathcal{A}_j$. This means that A_j is $\iota_j\mathcal{E}$-minimal. Obviously, $\mathcal{A}_0$ is nondegenerated and $\ker \iota_1 = \ker \iota$.

Item (*iii*) follows immediately from (*ii*) using Proposition 2.14. ❑

Simply by its nature a compressed resolvent has a couple of algebraic and analytic properties.

I.3 Lemma. *Let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$ be a compressed resolvent. Then R has the following properties.*

(C1) *For each $z \in \Omega$ the function*

$$R(.,.;z) : \left\{ \begin{array}{ccc} \mathcal{E}^2 & \to & \mathbb{C} \\ (x,y) & \mapsto & R(x,y;z) \end{array} \right.$$

is a sesquilinear form.

(C2) *For each $z \in \mathbb{C}$ with $z, \overline{z} \in \Omega$ it holds that*

$$R(x,y;z) = \overline{R(y,x;\overline{z})}, \quad x,y \in \mathcal{E}. \tag{I.3}$$

(C3) *For each $x, y \in \mathcal{E}$ the function*

$$R(x,y;.) : \left\{ \begin{array}{ccc} \Omega & \to & \mathbb{C} \\ z & \mapsto & R(x,y;z) \end{array} \right.$$

is analytic in Ω.

Proof. Choose $\langle \mathcal{A}, \iota, A \rangle$ which induces R. The property asserted in (C1) obviously holds for $z \in \Omega \cap \rho(A)$, and the property in (C2) for $\{z \in \mathbb{C} : z, \overline{z} \in \Omega \cap \rho(A)\}$. Since $\rho(A)$ contains both halfplanes $\mathbb{C}^+$ and $\mathbb{C}^-$ with possible exception of finitely many points, we have

$$\Omega \subseteq \overline{\Omega \cap \rho(A)}, \quad \{z \in \mathbb{C} : z, \overline{z} \in \Omega\} \subseteq \overline{\{z \in \mathbb{C} : z, \overline{z} \in \Omega \cap \rho(A)\}}.$$

Continuity of $R(x,y;.)$ now implies that (C1) and (C2) hold.

We come to the proof of (C3). Analyticity is again clear on $z \in \Omega \cap \rho(A)$. Fix $x, y \in \mathcal{E}$, and $z_0 \in \Omega \setminus \rho(A)$. If $z_0 \notin \mathbb{R}$, then z_0 is an isolated point of $\Omega \setminus \rho(A)$. Continuity of $R(x,y;.)$ implies that z_0 is a removable singularity. If $z_0 \in \mathbb{R}$, we choose a disk $U_r(z_0)$ centered at z_0 such that $U_r(z_0) \subseteq \Omega$ and $U_r(z_0) \setminus \mathbb{R} \subseteq \rho(A)$. Since $R(x,y;.)$ is continuous, we can refer to Goursat's theorem and conclude that $R(x,y;.)$ is analytic throughout $U_r(z_0)$. ❑

Compressed resolvents enjoy a certain definiteness property. To explain this, we introduce an inner product space.

I.4 Definition. Let $\mathcal{E}$ be an inner product space, let Ω be an open and nonempty subset of $\mathbb{C}$, and let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$. Assume that R has the properties (C1)–(C3). Then we denote

$$\mathring{\Omega} := \Omega \,\dot{\cup} \{\infty\},$$

$$\mathcal{L}_R := \big\{ (x_i)_{i\in\mathring{\Omega}} :\ x_i \in \mathcal{E},\ x_i = 0 \text{ for all but finitely many } i \in \mathring{\Omega} \big\},$$

$$\big[(x_i)_{i\in\mathring{\Omega}}, (y_i)_{i\in\mathring{\Omega}} \big]_R := \ [x_\infty, y_\infty]_{\mathcal{E}} + \sum_{z\in\Omega} R(x_z, y_\infty; z) + \sum_{w\in\Omega} \overline{R(y_w, x_\infty; w)} +$$

$$+ \sum_{z,w\in\Omega} \frac{R(x_z, y_w; z) - \overline{R(y_w, x_z; w)}}{z - \overline{w}}, \quad (x_i)_{i\in\mathring{\Omega}}, (y_i)_{i\in\mathring{\Omega}} \in \mathcal{L}_R.$$

Here, the quotient in the last summand is interpreted as a derivative if $z = \overline{w}$, namely, as $\frac{\partial}{\partial z} R(x, y; z)$ with $x = x_z$ and $y = y_{\overline{z}}$. This is possible by the symmetry property (C2) and analyticity (C3). ◊

Linear operations on $\mathcal{L}_R$ are defined in the canonical way, and then $[.,.]_R$ becomes an inner product on $\mathcal{L}_R$. The fact that $[.,.]_R$ is sesquilinear follows from (C1), and the fact that it is hermitian from (C2).

To shorten notation, we write $x\delta_i$ for the element of $\mathcal{L}_R$ whose i-th component is equal to x and all other components are equal to 0.

I.5 *Remark.* The inner product $[.,.]_R$ has a continuity property which follows immediately from analyticity of R and turns out to be important: *Let $\mathcal{E}$ be an inner product space, let Ω be an open and nonempty subset of $\mathbb{C}$, and let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$. Assume that R has the properties* (C1)–(C3). *For each $x, y \in \mathcal{E}$ the maps*

$$(z, w) \mapsto \big[x\delta_z, y\delta_w \big]_R, \quad (z, w) \in \Omega \times \Omega,$$

$$z \mapsto \big[x\delta_z, y\delta_\infty \big]_R, \quad z \in \Omega,$$

are continuous. ◊

Now we can prove the announced definiteness property of compressed resolvents.

I.6 Lemma. *Let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$ be a compressed resolvent. Then*

(C4) $\operatorname{ind}_- \langle \mathcal{L}_R, [.,.]_R \rangle < \infty$.

Proof. Choose $\langle \mathcal{A}, \iota, A \rangle$ which induces R. Consider the subspace

$$\mathcal{M} := \big\{ (x_i)_{i\in\mathring{\Omega}} \in \mathcal{L}_R : x_i = 0, i \in \Omega \setminus \rho(A) \big\}$$

of $\mathcal{L}_R$, and define a map $\varphi : \mathcal{M} \to \mathcal{A}$ by

$$\varphi\big((x_i)_{i\in\mathring{\Omega}} \big) := \iota x_\infty + \sum_{z\in\Omega\cap\rho(A)} (A - z)^{-1} \iota x_z, \quad (x_i)_{i\in\mathring{\Omega}} \in \mathcal{M}.$$

The definition of $[.,.]_R$ ensures that φ is isometric, and hence

$$\operatorname{ind}_- \langle \mathcal{M}, [.,.]_R \rangle \leq \operatorname{ind}_- \mathcal{A} < \infty.$$

The continuity property Remark I.5 implies that $\operatorname{ind}_- \mathcal{L}_R = \operatorname{ind}_- \mathcal{M}$. ❑

I.2 Intrinsic characterisation

In the situation that $\mathcal{E}$ is a Hilbert space, (operator valued) compressed resolvents (I.1) can be characterised by means of a certain kernel function, cf. [DLS84, Theorem 2.3]. Namely, a function R which takes values in the set of all bounded operators on the Hilbert space $\mathcal{E}$ is a compressed resolvent (of a seladjoint extension acting in a Pontryagin space), if and only if the operator valued kernel

$$\frac{R(z)-R(w)^*}{z-\overline{w}} - R(w)^*R(z)$$

has a finite number of negative squares.

In the general case, one has to switch from the above kernel function to the inner product $[.,.]_R$.

I.7 Theorem. *Let $\mathcal{E}$ be an inner product space, let Ω be an open and nonempty subset of $\mathbb{C}$, and let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$. Then R is a compressed resolvent if and only if it satisfies* (C1)–(C4).

Proof. Necessity was seen in Lemma I.3 and Lemma I.6. Hence, assume that a function R with (C1)–(C4) is given.

Let $\langle \lambda, \mathcal{A} \rangle$ be a Pontryagin space completion of $\mathcal{L}_R$, i.e., a Pontryagin space $\mathcal{A}$ together with an isometric map λ of $\mathcal{L}_R$ onto a dense subspace of $\mathcal{A}$. Moreover, let $\kappa : \mathcal{E} \to \mathcal{L}_R$ the canonical embedding

$$\kappa : x \mapsto x\delta_\infty, \quad x \in \mathcal{E},$$

and set $\iota := \lambda \circ \kappa$.

To shorten notation, denote $x\varepsilon_i := \lambda(x\delta_i)$, $x \in \mathcal{E}$, $i \in \mathring{\Omega}$. We define

$$A := \operatorname{cls}_{\mathcal{A}^2}\Big(\big\{(x\varepsilon_z, x\varepsilon_\infty + zx\varepsilon_z) : x \in \mathcal{E}, z \in \Omega\big\} \\ \cup \big\{(x\varepsilon_z - x\varepsilon_w, zx\varepsilon_z - wx\varepsilon_w) : x \in \mathcal{E}, z, w \in \Omega\big\}\Big).$$

The first thing to show is that A is symmetric; this is done by computation plugging in the definitions. Consider two elements $(x\varepsilon_z, x\varepsilon_\infty + zx\varepsilon_z), (y\varepsilon_w, y\varepsilon_\infty + wy\varepsilon_w)$ with $z, w \in \Omega$, $z \neq \overline{w}$. Using isometry of λ and the definition of $[.,.]_R$, we compute

$$\begin{aligned} &\big[x\varepsilon_\infty + zx\varepsilon_z, y\varepsilon_w\big]_{\mathcal{A}} - \big[x\varepsilon_z, y\varepsilon_\infty + wy\varepsilon_w\big]_{\mathcal{A}} \\ &\quad = [x\delta_\infty, y\delta_w]_R - [x\delta_z, y\delta_\infty]_R + (z-\overline{w})[x\delta_z, y\delta_w]_R \\ &\quad = \overline{R(y,x;w)} - R(x,y;z) + (z-\overline{w})\frac{R(x,y;z) - \overline{R(y,x;w)}}{z-\overline{w}} = 0. \end{aligned} \tag{I.4}$$

If $z = \overline{w}$ the terms involving $[x\varepsilon_z, y\varepsilon_w]_{\mathcal{A}}$ cancel, and the expression (I.4) also vanishes.

Consider two elements $(x\varepsilon_z, x\varepsilon_\infty + zx\varepsilon_z), (y\varepsilon_v - y\varepsilon_w, vy\varepsilon_v - wy\varepsilon_w)$ with

$z, v, w \in \Omega$, $z \neq \overline{v}$, $z \neq \overline{w}$. Then

$$\begin{aligned}\big[x\varepsilon_\infty + zx\varepsilon_z, y\varepsilon_v - y\varepsilon_w\big]_{\mathcal{A}} - \big[x\varepsilon_z, vy\varepsilon_v - wy\varepsilon_w\big]_{\mathcal{A}} \qquad (I.5)\\ = \overline{R(x,y;\overline{w})} - \overline{R(y,x;w)} \\ + (z-\overline{v})\frac{R(x,y;z) - \overline{R(y,x;v)}}{z-\overline{v}} - (z-\overline{w})\frac{R(x,y;z) - \overline{R(y,x;w)}}{z-\overline{w}} \\ = 0.\end{aligned}$$

Again, if $z = \overline{v}$ or $z = \overline{w}$, the corresponding summands cancel from the beginning and the expression (I.5) also vanishes.

Consider two elements $(x\varepsilon_z - x\varepsilon_w, zx\varepsilon_z - wx\varepsilon_w), (y\varepsilon_v - y\varepsilon_u, vy\varepsilon_v - uy\varepsilon_u)$ with $z, w, v, u \in \Omega$, $z \neq \overline{v}$, $z \neq \overline{u}$, $w \neq \overline{v}$, $w \neq \overline{u}$. Then

$$\begin{aligned}\big[zx\varepsilon_z + wx\varepsilon_w, y\varepsilon_v - y\varepsilon_u\big]_{\mathcal{A}} - \big[x\varepsilon_z - x\varepsilon_w, vy\varepsilon_v - uy\varepsilon_u\big]_{\mathcal{A}} \qquad (I.6)\\ = (z-\overline{v})\frac{R(x,y;z) - \overline{R(y,x;v)}}{z-\overline{v}} - (w-\overline{v})\frac{R(x,y;w) - \overline{R(y,x;v)}}{w-\overline{v}} \\ - (z-\overline{u})\frac{R(x,y;z) - \overline{R(y,x;u)}}{z-\overline{u}} - (w-\overline{u})\frac{R(x,y;w) - \overline{R(y,x;u)}}{w-\overline{u}} \\ = 0.\end{aligned}$$

Again, if one of the mentioned conditions on z, w, v, u is violated, the corresponding summands cancel from the beginning and the expression (I.6) also vanishes. Alltogether, we see that A is symmetric.

In the second step we show that A has a selfadjoint extension with nonempty resolvent set. Let $z \in \Omega$. Then, for each $x \in \mathcal{E}$,

$$\begin{aligned}x\varepsilon_\infty &= (x\varepsilon_\infty + zx\varepsilon_z) - z(x\varepsilon_z) \in \operatorname{ran}(A - z),\\ x\varepsilon_w &= \frac{1}{z-w}\Big((zx\varepsilon_z - wx\varepsilon_w) - z(x\varepsilon_z - x\varepsilon - w)\Big) \in \operatorname{ran}(A-z), \quad w \in \Omega \setminus \{z\}.\end{aligned}$$

From Remark I.5 we obtain that $\lim_{w\to z} x\varepsilon_w = x\varepsilon_z$, and see that

$$\operatorname{clos}_{\mathcal{A}} \operatorname{ran}(A-z) \supseteq \operatorname{clos}_{\mathcal{A}} \lambda(\mathcal{L}_R) = \mathcal{A}, \quad z \in \Omega.$$

This implies $\ker(A - \overline{z}) \subseteq \operatorname{ran}(A-z)^\perp = \mathcal{A}^\circ = \{0\}$, $z \in \Omega$. Proposition 2.14 yields that there exists a Pontryagin space $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ and a selfadjoint relation $\tilde{A}$ in $\tilde{\mathcal{A}}$ extending A with $\rho(\tilde{A}) \neq \emptyset$.

Finally, we show that $\langle \tilde{\mathcal{A}}, \iota, \tilde{A}\rangle$ induces R. This, however, is built in the definition: we have $(x\varepsilon_z, x\varepsilon_\infty + zx\varepsilon_z) \in A$, and hence

$$(\tilde{A} - z)^{-1} x\varepsilon_\infty = x\varepsilon_z, \quad x \in \mathcal{E}, z \in \rho(\tilde{A}) \cap \Omega. \qquad (I.7)$$

Thus

$$\big[(\tilde{A}-z)^{-1}\iota x, \iota y\big] = [x\varepsilon_z, y\varepsilon_\infty]_{\mathcal{A}} = R(x,y;z), \quad x, y \in \mathcal{E}, z \in \rho(\tilde{A}) \cap \Omega.$$

❑

I.8 Corollary. *Let $\mathcal{E}$ be an inner product space, let Ω be an open and nonempty subset of $\mathbb{C}$, and let $R : \mathcal{E}^2 \times \Omega \to \mathbb{C}$. Assume that R satisfies* (C1)–(C4)*. Then there exists an open set $\tilde{\Omega} \subseteq \mathbb{C}$ and a function $\tilde{R} : \mathcal{E}^2 \times \tilde{\Omega} \to \mathbb{C}$, such that $\tilde{\Omega}$ contains both halfplanes $\mathbb{C}^+$ and $\mathbb{C}^-$ with possible exception of finitely many points and is symmetric w.r.t. the real axis, that $\tilde{R}$ satisfies* (C1)–(C4)*, and*

$$\tilde{R}(x, y; z) = R(x, y; z), \quad x, y \in \mathcal{E},\ z \in \Omega \cap \tilde{\Omega}.$$

Proof. The function R is a compressed resolvent, hence is induced by some selfadjoint relation A acting in a Pontryagin space. Use $\tilde{\Omega} := \rho(A)$. ❑

I.3 Minimality aspects

The following is shown in [DLS84, Lemma 1.1]: Let $\mathcal{H}$ be a Hilbert space, $\mathcal{P}$ be a Pontryagin space with $\mathcal{P} \supseteq \mathcal{H}$, and let A be an $\mathcal{H}$-minimal selfadjoint relation in $\mathcal{P}$ with nonempty resolvent set. Then the (operator valued) compressed resolvent (I.1) has no continuous extension beyond $\rho(A)$.

Our next theorem is the analogue for the presently considered almost Pontryagin space situation. Its proof uses the same argument as [DLS84, Lemma 1.1], however, some additions are necessary due to possible presence of isotropic elements.

I.9 Theorem. *Let $\mathcal{A}$ be an almost Pontryagin space, $\mathcal{E} \subseteq \mathcal{A}$, and let A be an $\mathcal{E}$-minimal selfadjoint relation in $\mathcal{A}$ with nonempty resolvent set. Denote by Ω the largest open subset of $\mathbb{C}$ such that each function $z \mapsto [(A-z)^{-1}x, y]$, $x, y \in \mathcal{E}$, has a continuous extension to Ω. Then*

$$\Omega \setminus \sigma_p\big(A \cap (\mathcal{A}^\circ)^2\big) = \rho(A).$$

Note that the relation $A \cap (\mathcal{A}^\circ)^2$ is selfadjoint in $\mathcal{A}^\circ$, and has nonempty resolvent set. This follows by applying [SW16, Proposition 3.2] with "$\mathcal{D} := \mathcal{A}^\circ, \mathcal{B} := \{0\}$". Since $\dim \mathcal{A}^\circ < \infty$, the spectrum $\sigma(A \cap (\mathcal{A}^\circ)^2)$ consists of at most $\operatorname{ind}_0 \mathcal{A}$ points which are all eigenvalues. Moreover, the inclusion $\Omega \setminus \sigma_p\big(A \cap (\mathcal{A}^\circ)^2\big) \supseteq \rho(A)$ is of course trivial.

Proof of Theorem I.9. Set $\tilde{\mathcal{A}} := \mathfrak{P}_{\text{ext}}(\mathcal{A})$, and choose a selfadjoint extension $\tilde{A}$ of A which acts in $\tilde{\mathcal{A}}$ and has nonempty resolvent set. This is possible by Proposition 2.14. Denote by $\mathfrak{M}$ the algebra generated by the semiring of all intervals whose endpoints are not critical points of $\tilde{A}$, and let $\tilde{E}$ be the projection valued spectral measure of $\tilde{A}$, cf. [Lan82], [DS87b].

Step 1: We show that, for each $\Delta \in \mathfrak{M}$,

$$\tilde{E}(\Delta)\mathcal{A} \subseteq \mathcal{A}, \qquad (A-z)^{-1}\tilde{E}(\Delta)\mathcal{A} \subseteq \tilde{E}(\Delta)\mathcal{A},\ z \in \rho(A).$$

Since $\tilde{A} \supseteq A$, we have $\rho(\tilde{A}) \subseteq \rho(A)$ (remember Remark 2.13) and $(A-z)^{-1} = (\tilde{A}-z)^{-1}|_{\mathcal{A}}$, $z \in \rho(\tilde{A})$. In particular, $(\tilde{A}-z)^{-1}\mathcal{A} \subseteq \mathcal{A}$, $z \in \rho(\tilde{A})$.

For each finite open interval $\Delta = (a, b) \in \mathfrak{M}$, the spectral projection $\tilde{E}(\Delta)$ can be obtained as the limit of integrals

$$\tilde{E}(\Delta) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int\limits_{\gamma_{\delta,\varepsilon}} (\tilde{A}-\zeta)^{-1}\, d\zeta$$

where $\gamma_{\delta,\varepsilon}$ is the – piecewise continuous – path consisting of two line segments

$$\gamma_{\delta,\varepsilon}(t) := \begin{cases} (a+\delta+i\varepsilon)t + (b-\delta+i\varepsilon)(1-t) & , \quad t \in (0,1), \\ (b-\delta-i\varepsilon)(t-1) + (a+\delta-i\varepsilon)(2-t), & t \in (1,2). \end{cases}$$

This representation readily implies that $\tilde{E}(\Delta)\mathcal{A} \subseteq \mathcal{A}$. Moreover, for $z \in \rho(\tilde{A})$,

$$(A-z)^{-1}\tilde{E}(\Delta)\mathcal{A} = (\tilde{A}-z)^{-1}\tilde{E}(\Delta)\mathcal{A} = \tilde{E}(\Delta)(\tilde{A}-z)^{-1}\mathcal{A} \subseteq \tilde{E}(\Delta)\mathcal{A}.$$

By continuity the required inclusion $(A-z)^{-1}\tilde{E}(\Delta)\mathcal{A} \subseteq \tilde{E}(\Delta)\mathcal{A}$ holds for all $z \in \rho(A)$.

Step 2: We show that

$$\tilde{E}(\Delta)\mathcal{A} \subseteq \mathcal{A}^\circ, \quad \Delta = (a,b) \in \mathfrak{M},\ \Delta \subseteq \Omega. \tag{I.8}$$

Let $x, y \in \mathcal{E}$. Since $[(A-\zeta)^{-1}x, y]$ is analytic across Δ, we have

$$[\tilde{E}(\Delta)x, y] = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_{\delta,\varepsilon}} [(A-\zeta)^{-1}x, y]\, d\zeta = 0.$$

If $x = (A-w)^{-1}x'$ with $x' \in \mathcal{E}$, $y \in \mathcal{E}$, and $w \in \rho(A) \setminus \Delta$, then

$$\big[(A-\zeta)^{-1}(A-w)^{-1}x', y\big] = \frac{1}{\zeta - w}\Big([(A-\zeta)^{-1}x', y] - [(A-w)^{-1}x', y]\Big)$$

whenever $\zeta \in \gamma_{\delta,\varepsilon}$ with ε, δ sufficiently small. Again, this expression is analytic across Δ, and it follows that $[\tilde{E}(\Delta)x, y] = 0$. The same argument applies if $x \in \mathcal{E}$, $y = (A-w)^{-1}y'$, or if $x = (A-w)^{-1}x'$, $y = (A-v)^{-1}y'$. Alltogether we obtain that $[\tilde{E}(\Delta)x, y] = 0$ for all x, y in $\mathcal{E} \cup \bigcup_{w \in \rho(A)\setminus\Delta} (A-w)^{-1}\mathcal{E}$. However, the linear span of this set is dense in $\mathcal{A}$, and we conclude that (I.8) holds.

Step 3: Let $z_0 \in \Omega \setminus \sigma(A \cap (\mathcal{A}^\circ)^2)$, $z_0 \in \mathbb{R}$, be given. Choose a finite open interval $\Delta \in \mathfrak{M}$ such that

$$z_0 \in \Delta \subseteq \overline{\Delta} \subseteq \Omega \setminus \sigma\big(A \cap (\mathcal{A}^\circ)^2\big).$$

Applying [SW16, Proposition 3.2] with "$\mathcal{D} := \tilde{E}(\Delta)\mathcal{A}, \mathcal{B} := \{0\}$" shows that the relation

$$A_0 := A \cap (\tilde{E}(\Delta)\mathcal{A})^2$$

is selfadjoint in $\tilde{E}(\Delta)\mathcal{A}$ and has nonempty resolvent set. Clearly we have $\sigma(A_0) = \sigma_p(A_0) \neq \emptyset$ unless $\tilde{E}(\Delta)\mathcal{A} = \{0\}$. Here the spectrum is understood in the extended plane $\mathbb{C} \cup \{\infty\}$. Since $A_0 \subseteq A \cap (\mathcal{A}^\circ)^2$,

$$\sigma(A_0) \subseteq \sigma\big(A \cap (\mathcal{A}^\circ)^2\big).$$

The relation

$$\tilde{A}_0 := \tilde{A} \cap (\tilde{E}(\Delta)\tilde{\mathcal{A}})^2$$

is a bounded selfadjoint operator whose spectrum is contained in $\overline{\Delta}$, cf. [Lan82], [DS87b]. Since $A_0 \subseteq \tilde{A}_0$, the relation A_0 is an operator. Moreover, we have

$$\sigma(A_0) \subseteq \sigma\big(\tilde{A}_0\big).$$

The fact that $\sigma(A\cap(\mathcal{A}^\circ)^2)\cap\overline{\Delta}=\emptyset$ now implies that $\sigma(A_0)=\emptyset$ and hence that $\tilde{E}(\Delta)\mathcal{A}=\{0\}$.

We have shown that $\mathcal{A}\subseteq\ker\tilde{E}(\Delta)$. Since $\ker\tilde{E}(\Delta)$ is nondegenerated, this implies that $\ker\tilde{E}(\Delta)=\tilde{\mathcal{A}}$, i.e. $\tilde{E}(\Delta)=0$. Thus $\Delta\subseteq\rho(\tilde{A})\subseteq\rho(A)$. In particular, $z_0\in\rho(A)$.

Step 4: Let $z_0\in\Omega\setminus\sigma(A\cap(\mathcal{A}^\circ)^2)$, $z_0\notin\mathbb{R}$, be given. Assume on the contrary that $z_0\in\sigma(A)$. Then $z_0\in\sigma_p(A)$ and $\{z_0\}$ is an isolated spectral set of A. For a sufficiently small circle γ centered at z_0, the Riesz projection $P_{\{z_0\}}$ is given as

$$P_{\{z_0\}}=\frac{1}{2\pi i}\int_\gamma(A-\zeta)^{-1}\,d\zeta.$$

By Cauchy's theorem, we have $[P_{\{z_0\}}x,y]=0$, $x,y\in\mathcal{E}$. The same argument which led to (I.8) in Step 2 above, now gives

$$P_{\{z_0\}}\mathcal{A}\subseteq\mathcal{A}^\circ.$$

This shows that all eigenvectors of A with eigenvalue z_0 belong to $\mathcal{A}^\circ$. In turn, $z_0\in\sigma(A\cap(\mathcal{A}^\circ)^2)$. We have reached a contradiction and conclude that $z_0\in\rho(A)$. ❑

I.*10 Remark.* One might expect the uniqueness statement: If A is $\mathcal{E}$-minimal, then A is determined up to isomorphism by its compressed resolvent on $\mathcal{E}$. However, this is not the case. Just think of operators acting on a finite dimensional neutral space. ♢

I.4 Generalised resolvents

I.11 Definition. Let $\mathcal{A}$ be an almost Pontryagin space and $S\subseteq\mathcal{A}^2$ a closed symmetric relation with $\gamma(S)\neq\emptyset$. Moreover, let $\tilde{\mathcal{A}}$ be an almost Pontryagin space with $\tilde{\mathcal{A}}\supseteq\mathcal{A}$ and $A\subseteq\tilde{\mathcal{A}}^2$ a selfadjoint relation with $\rho(A)\neq\emptyset$ and $A\supseteq S$. Then we call the compressed resolvent induced by $\langle\tilde{\mathcal{A}},\subseteq,A\rangle$ a *generalised resolvent of* S. ♢

Equivalently, we could use compressed resolvents induced by $\langle\tilde{\mathcal{A}},\iota,A\rangle$ with $\ker\iota=\{0\}$. In order to simplify notation, we think from the start of $\mathcal{A}$ as a subspace of $\tilde{\mathcal{A}}$.

The following is an immediate consequence of Lemma I.2.

I.*12 Remark.* Let $\mathcal{A}$ be an almost Pontryagin space and $S\subseteq\mathcal{A}^2$ a closed symmetric relation with $\gamma(S)\neq\emptyset$.

(*i*) Every generalised resolvent is induced by a triple $\langle\tilde{\mathcal{A}},\subseteq,A\rangle$ where A is $\mathcal{A}$-minimal.

(*ii*) Every generalised resolvent is induced by a triple $\langle\tilde{\mathcal{A}},\subseteq,A\rangle$ where $\tilde{\mathcal{A}}$ is a Pontryagin space.

Unlike in Lemma I.2 we cannot ensure that in the same time A is $\mathcal{A}$-minimal and $\tilde{\mathcal{A}}$ is a Pontryagin space. However, using the usual Pontryagin space uniqueness result, one can show that uniqueness prevails after factorising the isotropic part.

We do not now whether every compressed resolvent induced by some triple $\langle\tilde{\mathcal{A}},\iota,A\rangle$ is a generalised resolvent of the relation $(\iota\times\iota)(S)$. ♢

When studying generalised resolvents, one can often reduce to the case of minimal symmetries. This is a consequence of [SW16, Theorem 7.1], hence is a deeper fact.

I.13 Proposition. *Let $\mathcal{A}$ be an almost Pontryagin space, $S \subseteq \mathcal{A}^2$ a closed symmetric relation with $\gamma(S) \neq \emptyset$, and let $\mathcal{A}_1$ and S_1 be the almost Pontryagin space and symmetry defined in Lemma* 2.20*. Then the families of generalised resolvents of S and S_1 coincide up to an identification via the canonical projection π.*

Proof. Recall the definitions from Lemma 2.20:

$$\mathcal{C} := \bigcap_{z \in \gamma_s(S)} \operatorname{ran}(S - z), \quad \mathcal{D} := \mathcal{A}[-]\mathcal{C}^\perp, \qquad \mathcal{A}_1 := \mathcal{D}/_{\mathcal{C}^\circ},$$

let $\pi : \mathcal{D} \to \mathcal{A}_1$ denote the canonical projection, and

$$S_1 := (\pi \times \pi)\big(S \cap (\mathcal{D} \times \mathcal{D})\big).$$

In (2.7) we saw that $(S - w)^{-1}(\mathcal{C}) \subseteq \mathcal{C}$, $w \in \gamma_s(S)$.

Let $\tilde{\mathcal{A}}$ be an almost Pontryagin space which contains $\mathcal{A}$ and $A \subseteq \tilde{\mathcal{A}}^2$ a selfadjoint relation with $A \supseteq S$ and $\rho(A) \neq \emptyset$. Then $(A - w)^{-1}(\mathcal{C}) \subseteq \mathcal{C}$, $w \in \rho_s(A) \cap \gamma_s(S)$. This implies that also

$$(A - w)^{-1}(\tilde{\mathcal{A}}[-]\mathcal{C}) \subseteq \tilde{\mathcal{A}} \cap \mathcal{C},\ (A - w)^{-1}(\mathcal{C}^\circ) \subseteq \mathcal{C}^\circ, \quad w \in \rho_s(A) \cap \gamma_s(S).$$

Set

$$\tilde{\mathcal{D}} := \tilde{\mathcal{A}}[-]\mathcal{C}, \quad \tilde{\mathcal{A}}_1 := \tilde{\mathcal{D}}/_{\mathcal{C}^\circ},$$

let $\tilde{\pi} : \tilde{\mathcal{D}} \to \tilde{\mathcal{A}}_1$ denote the canonical projection, and

$$A_1 := (\tilde{\pi} \times \tilde{\pi})\big(A \cap (\tilde{\mathcal{D}} \times \tilde{\mathcal{D}})\big).$$

An application of [SW16, Proposition 3.2] shows that A_1 is selfadjoint with $\rho_s(A) \cap \gamma_s(S) \subseteq \rho(A_1)$, and that

$$\big[(A_1 - w)^{-1}\tilde{\pi}x, \tilde{\pi}y\big] = \big[(A - w)^{-1}x, y\big], \quad x, y \in \tilde{\mathcal{D}}, w \in \rho_s(A) \cap \gamma_s(S).$$

We have $\ker \tilde{\pi} = \ker \pi$, and hence $\tilde{\mathcal{A}}_1 \supseteq \mathcal{A}_1$ and $\tilde{\pi}x = \pi x$, $x \in \mathcal{D}$, and $A_1 \supseteq S_1$. Thus the generalised resolvent R of S induced by A and the generalised resolvent R_1 of S_1 induced by A_1 are related as

$$R_1(\pi x, \pi y; z) = R(x, y; z), \quad x, y \in \mathcal{D}, z \in \rho_s(A) \cap \gamma_s(S).$$

In order to show that every generalised resolvent of S_1 occurs in this way, we employ the deeper result [SW16, Theorem 7.1]. The necessary hypothesis [SW16, (7.1),(7.2)] for an application of this theorem are fulfilled. Remember here that we showed in the proof of Lemma 2.20 that $(S - w)^{-1}(\mathcal{C}^\circ) \subseteq \mathcal{C}^\circ$, $w \in \gamma_s(S)$.

Let a selfadjoint extension A_1 of S_1 in an almost Pontryagin space $\tilde{\mathcal{A}}_1 \supseteq \mathcal{A}_1$ be given. Since we factorise by the whole space $\mathcal{C}^\circ$, the condition [SW16, (4.7)] for existence of an almost Pontryagin space $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ with $(\tilde{\mathcal{A}}[-]\mathcal{C})/\mathcal{C}^\circ = \mathcal{A}_1$ is satisfied. Let $\tilde{\mathcal{A}}$ be one such (exists by [SW16, Theorem 4.2]). Again since we factorise by all of $\mathcal{C}^\circ$, item (*ii*) of [SW16, Theorem 7.1] applies, and we find a selfadjoint relation A in $\tilde{\mathcal{A}}$ such that (notation as above)

$$A_1 = (\tilde{\pi} \times \tilde{\pi})\big(A \cap (\tilde{\mathcal{D}} \times \tilde{\mathcal{D}})\big).$$

❑

PART II
Q-FUNCTIONS

II.1 Definition of Q-functions

To start with, let us recall how one proceeds in the nondegenerated case, see, e.g., [KL73].

II.*1. Q-functions in Pontryagin spaces:* Let $\mathcal{P}$ be a Pontryagin space, and let $S \subseteq \mathcal{P}^2$ be a closed symmetric relation with $\gamma(S) \neq \emptyset$ which has defect index $(1,1)$. Choose a selfadjoint extension $\mathring{A}$ of S in $\mathcal{P}$ with $\rho(\mathring{A}) \neq \emptyset$, choose $z_0 \in \rho(\mathring{A})$, and choose a defect element $\chi(z_0)$ of S, i.e. an element $\chi(z_0) \in \mathcal{P}$ with $\operatorname{span}\{\chi(z_0)\} = \operatorname{ran}(S - \overline{z_0})^\perp$. Let $\chi(z)$ be the family of defect elements of S generated from $\chi(z_0)$ by means of the formula

$$\chi(z) = \big(I + (z - z_0)(\mathring{A} - z)^{-1}\big)\chi(z_0), \quad z \in \rho(\mathring{A}).$$

Then there exists a function q which satisfies

$$\frac{q(z) - q(\overline{w})}{z - \overline{w}} = \big[\chi(z), \chi(w)\big]. \tag{II.1}$$

Each function which is constructed in this way from some choices of $\mathring{A}$, z_0, and $\chi(z_0)$ is called a Q-function of S. It depends essentially on $\mathring{A}$, z_0, and $\chi(z_0)$. However, once such choices are made, it is by the relation (II.1) uniquely determined up to a real additive constant. $\Diamond$

If $\mathcal{A}$ is an almost Pontryagin space with $\operatorname{ind}_0 \mathcal{A} > 0$ and $S \subseteq \mathcal{A}^2$ is a closed symmetric relation with $\gamma(S) \neq \emptyset$ which has defect index $(1,1)$, a similar construction can be carried out, cf. [KW99b, §2]. Contrasting the nondegenerated case, not every choice of an extension $\mathring{A}$ is suitable. Our aim in this section is to review this construction and provide some supplementary details.

II.*2. Setup for the definition of Q-functions in a degenerated almost Pontryagin space:* Let $\mathcal{A}$ be an almost Pontryagin space with $\Delta := \operatorname{ind}_0 \mathcal{A} > 0$ and let $S \subseteq \mathcal{A}^2$ be a closed symmetric relation in $\mathcal{A}$ with defect index $(1,1)$. Assume that S satisfies the regularity conditions

$$\exists\, z_+ \in \mathbb{C}^+, z_- \in \mathbb{C}^- : \ \operatorname{ran}(S - z_\pm) + \mathcal{A}^\circ = \mathcal{A} \tag{II.2}$$

$$\forall\, h \in \mathcal{A}^\circ : \ S \cap \big(\operatorname{span}\{h\} \times \operatorname{span}\{h\}\big) = \{0\} \tag{II.3}$$

$\Diamond$

Before we proceed to the actual definition of Q-functions, let us discuss these conditions.

The significance of (II.2) becomes apparent when considering the relation ($\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ denotes the canonical projection)

$$S_{\mathrm{fac}} := (\pi \times \pi)(S),$$

as seen from the next lemma which provides a somewhat more complete version of [KW99b, Remark 1].

II.**3 Lemma.** *Let $\mathcal{A}$ be an almost Pontryagin space with $\Delta := \operatorname{ind}_0 \mathcal{A} > 0$ and let $S \subseteq \mathcal{A}^2$ be a closed symmetric relation in $\mathcal{A}$.*

(*i*) *The relation S_{fac} is closed and symmetric. If $\gamma(S) \cap \mathbb{C}^+ \neq \emptyset$, then $\mathfrak{n}_+(S_{\mathrm{fac}}) \leq \mathfrak{n}_+(S)$. The analogous statement holds for $\mathbb{C}^-$.*

(*ii*) *The relation S satisfies* (II.2) *if and only if S_{fac} is selfadjoint and has nonempty resolvent set.*

(*iii*) *If S satisfies* (II.2), *then*

$$\rho(S_{\mathrm{fac}}) = \big\{ z \in \mathbb{C} : \operatorname{ran}(S - z) + \mathcal{A}^\circ = \mathcal{A} \big\}. \tag{II.4}$$

In particular, $\operatorname{ran}(S - z) + \mathcal{A}^\circ = \mathcal{A}$ *holds for all $z \in \mathbb{C} \setminus \mathbb{R}$ with possible exception of at most* $2 \operatorname{ind}_- \mathcal{A}$ *points located symmetrically with respect to the real line.*

Proof. Since $\ker \pi = \mathcal{A}^\circ$ is finite dimensional, π maps closed subspaces onto closed subspaces. Clearly, $\pi \times \pi : \mathcal{A} \times \mathcal{A} \to (\mathcal{A}/_{\mathcal{A}^\circ}) \times (\mathcal{A}/_{\mathcal{A}^\circ})$ has the same property. Moreover, π is isometric.

From the above said we see that S_{fac} is a closed symmetric relation in $\mathcal{A}/_{\mathcal{A}^\circ}$. It holds that

$$\operatorname{ran}(S_{\mathrm{fac}} - z) = \big\{ \pi y - z \cdot \pi x : (x, y) \in S \big\} = \pi\big(\operatorname{ran}(S - z)\big), \quad z \in \mathbb{C}. \tag{II.5}$$

In particular, therefore

$$\dim \Big[(\mathcal{A}/_{\mathcal{A}^\circ}) \Big/ \operatorname{ran}(S_{\mathrm{fac}} - z) \Big] \leq \dim \Big[\mathcal{A} \Big/ \operatorname{ran}(S - z) \Big].$$

Assume that $\gamma(S) \cap \mathbb{C}^+ \neq \emptyset$, and choose z in this set. Then

$$\mathfrak{n}_+(S) = \dim \Big[\mathcal{A} \Big/ \operatorname{ran}(S - z) \Big] \geq \\ \geq \dim \Big[(\mathcal{A}/_{\mathcal{A}^\circ}) \Big/ \operatorname{ran}(S_{\mathrm{fac}} - z) \Big] - \dim \ker(S_{\mathrm{fac}} - z) = \mathfrak{n}_+(S_{\mathrm{fac}}).$$

The case of $\mathbb{C}^-$ instead of $\mathbb{C}^+$ follows in the same way.

Assume that S_{fac} is selfadjoint. Then, for each $z \in \rho(S_{\mathrm{fac}})$, we see from (II.5) that $\operatorname{ran}(S - z) + \mathcal{A}^\circ = \mathcal{A}$. In particular, (II.2) holds if $\rho(S_{\mathrm{fac}}) \neq \emptyset$. Conversely, assume that (II.2) holds; our aim is to apply Lemma 2.12 (with "$S = A := S_{\mathrm{fac}}$"). However, again referring to (II.5), we have $\operatorname{ran}(S_{\mathrm{fac}} - z_\pm) = \mathcal{A}/_{\mathcal{A}^\circ}$, and this is (2.1). Since $\mathcal{A}/_{\mathcal{A}^\circ}$ is nondegenerated, (2.2) trivially holds. It follows that S_{fac} is selfadjoint. The relation (II.4) follows from (II.5) and Lemma 2.11. ❑

The regularity condition (II.3) also has a very clear meaning. It ensures that $S \cap (\mathcal{A}^\circ)^2$ is a shift operator, cf. [KW99b, Proposition 1] (we recall in Remark II.6, (*i*)). Moreover, in conjunction with (II.2), it gives rise to points of regular type of S.

II.4 **Lemma.** *Let $\mathcal{A}$ be an almost Pontryagin space with $\Delta := \operatorname{ind}_0 \mathcal{A} > 0$ and let $S \subseteq \mathcal{A}^2$ be a closed symmetric relation in $\mathcal{A}$. If S satisfies* (II.3) *then*

$$\big\{z \in \mathbb{C} \setminus \mathbb{R} : \operatorname{ran}(S - \overline{z}) + \mathcal{A}^\circ = \mathcal{A}\big\} \subseteq \gamma(S). \tag{II.6}$$

Proof. Assume that z belongs to the set on the left side of (II.6). Let $x \in \ker(S - z)$ be given. Then $x \perp \operatorname{ran}(S - \overline{z})$, and hence $x \in \mathcal{A}^\circ$. Condition (II.3) implies that $x = 0$. Since $\gamma(S) \setminus \mathbb{R} = (\mathbb{C} \setminus \mathbb{R}) \setminus \sigma_p(S)$, we conclude that $z \in \gamma(S)$. ❑

The regularity conditions (II.2) and (II.3) guarantee existence of the necessary ingredients for building a Q-function of S.

II.*5. Choices to be made:* Assume that $\mathcal{A}$ and S are given according to II.2.

(Bas) There exist elements h_l, $l = 0, \dots, \Delta - 1$, such that $\{h_0, \dots, h_{\Delta-1}\}$ is a basis of $\mathcal{A}^\circ$, and that

$$(h_l, h_{l+1}) \in S, \quad l = 0, \dots, \Delta - 2. \tag{II.7}$$

The element h_0 is by these requirements uniquely determined up to scalar multiples. Once a choice of h_0 is made, the elements $h_1, \dots, h_{\Delta-1}$ are unique.

(Ext) There exist selfadjoint relations $\mathring{A} \subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A})^2$ with nonempty resolvent set which extend the relation $S' := \operatorname{span}(S \cup \{(0, h_0)\})$. For each such relation there exist families $(\chi(z))_{z \in \rho(\mathring{A})}$ of elements $\chi(z) \in \mathfrak{P}_{\text{ext}}(\mathcal{A})$, $z \in \rho(\mathring{A})$, such that $\chi(z) \perp \operatorname{ran}(S - \overline{z})$,

$$\chi(z) = \big(I + (z - w)(\mathring{A} - z)^{-1}\big)\chi(w), \quad z, w \in \rho(\mathring{A}), \tag{II.8}$$

and

$$\big[\chi(z), h_l\big] = z^l, \quad z \in \rho(\mathring{A}),\ l = 0, \dots, \Delta - 1. \tag{II.9}$$

A proof of these statements can be found in [KW99b] (putting together Proposition 1, Corollary 1, and Proposition 2). Let us remark that the relation $\mathring{A}$ and the family $(\chi(z))_{z \in \rho(\mathring{A})}$ in (Ext) is not unique. ♢

In the next remark, we collect some simple but noteworthy facts.

II.*6 Remark.* Let data be given according to II.2 and II.5.

(*i*) We have

$$S \cap (\mathcal{A}^\circ)^2 = \operatorname{span}\big\{(h_l, h_{l+1}) : l = 0, \dots, \Delta - 2\big\},$$

cf. [KW99b, Proposition 1].

(*ii*) Due to (Bas) we have

$$\operatorname{ran}(S - z) + \mathcal{A}^\circ = \operatorname{ran}(S - z) + \operatorname{span}\{h_0\}, \quad z \in \mathbb{C}. \tag{II.10}$$

In particular, $\operatorname{ran}(S - z) + \operatorname{span}\{h_0\} = \mathcal{A}$, $z \in \rho(S_{\text{fac}})$.

(*iii*) The relation (II.8) can be written equivalently as

$$\big(\chi(z) - \chi(w), z\chi(z) - w\chi(w)\big) \in \mathring{A}, \quad z, w \in \rho(\mathring{A}).$$

(*iv*) The property (II.9) is equivalent to

$$\exists\, z_0 \in \rho(\mathring{A}) : \; [\chi(z_0), h_0] = 1.$$

To see this, compute

$$[\chi(z), h_0] = \big[\chi(z_0) + (z - z_0)(\mathring{A} - z)^{-1}\chi(z_0), h_0\big] = \\ = [\chi(z_0), h_0] + (z - z_0)\big[\chi(z_0), \underbrace{(\mathring{A} - \overline{z})^{-1} h_0}_{=0}\big] = 1,$$

and note that

$$[\chi(z), h_{l+1}] = [z\chi(z), h_l]$$

since $h_{l+1} - \overline{z} h_l \in \operatorname{ran}(S - \overline{z})$.

(*v*) The relation S can be described as

$$S = \big\{(x; y) \in \mathcal{A}^2 : \quad y - \overline{z}x \perp \chi(z),\; z \in \rho(\mathring{A})\big\}.$$

To see this, observe that S has defect index $(\Delta + 1, \Delta + 1)$ in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$, and $\chi(z) \notin \mathcal{A}^\circ$. Therefore,

$$\mathfrak{P}_{\text{ext}}(\mathcal{A})[-] \operatorname{ran}(S - \overline{z}) = \mathcal{A}^\circ \dot{+} \operatorname{span}\{\chi(z)\}, \quad z \in \rho(\mathring{A}). \tag{II.11}$$

(*vi*) Since $\rho(\mathring{A})$ is dense in $\gamma_s(S)$ and (II.11) holds, the relation S is minimal if and only if

$$\operatorname{cls}\big(\mathcal{A}^\circ \cup \{\chi(z) : z \in \rho(\mathring{A})\}\big) = \mathfrak{P}_{\text{ext}}(\mathcal{A}).$$

♢

The next statement is a refinement of what was shown in the first part of the proof of [KW99b, Proposition 2].

II.**7 Lemma.** *Let $\mathcal{A}$ and S be given according to* II.2*, choose h_l and $\mathring{A}$ according to* II.5*, and denote again $S' := S + \operatorname{span}\{(0; h_0)\}$.*

(*i*) *The relation S' is selfadjoint in $\mathcal{A}$ and $\rho(\mathring{A}) \subseteq \rho(S_{\text{fac}}) = \rho(S') \subseteq \gamma(S)$.*

(*ii*) *We have*

$$\big[(\mathring{A} - z)^{-1}x, y\big] = \big[(S_{\text{fac}} - z)^{-1}\pi x, \pi y\big], \quad x, y \in \mathcal{A},\; z \in \rho(\mathring{A}).$$

Proof. The inclusion $\rho(S_{\text{fac}}) \subseteq \rho(S')$ is exactly what is shown in the first part of the proof of [KW99b, Proposition 2]. In particular, we see that S' is selfadjoint in $\mathcal{A}$. Due to (II.10), we have

$$\operatorname{ran}(S' - z) = \operatorname{ran}(S - z) + \mathcal{A}^\circ, \quad z \in \mathbb{C}.$$

Hence, $z \in \rho(S')$ implies that $z \in \rho(S_{\text{fac}})$. The inclusions $\rho(\mathring{A}) \subseteq \rho(S') \subseteq \gamma(S)$ hold since $S \subseteq S' \subseteq \mathring{A}$.

For the proof of (*ii*), let $x, y \in \mathcal{A}$ and $z \in \rho(\mathring{A})$ be given. Set $u := (\mathring{A} - z)^{-1}x$. Since $\mathring{A}$ extends S' and $z \in \rho(S')$, we have $u = (S' - z)^{-1}x$. Since

$$S_{\text{fac}} = (\pi \times \pi)(S) = (\pi \times \pi)(S'),$$

and $z \in \rho(S_{\text{fac}})$, it follows that $\pi u = (S_{\text{fac}} - z)^{-1}\pi x$. Isometry of π now yields

$$[(\mathring{A} - z)^{-1}x, y] = [u, y] = \big[\pi u, \pi y\big] = \big[(S_{\text{fac}} - z)^{-1}\pi x, \pi y\big].$$

❑

Let us now state the definition of a Q-function, cf. [KW99b, (2.9)].

II.**8 Definition.** Let $\mathcal{A}$ and S be given according to II.2. For each choice of $h_0, \mathring{A}, \chi$ according to II.5, there exists a function q which satisfies

$$\frac{q(z) - q(\overline{w})}{z - \overline{w}} = \big[\chi(z), \chi(w)\big], \quad z, w \in \rho(\mathring{A}). \tag{II.12}$$

Each function which is constructed in this way is called a Q-function of S. $\Diamond$

II.*9 Remark.*

(i) Functions q as in Definition II.8 depend on the choice of $h_0, \mathring{A}, \chi$. Once $h_0, \mathring{A}, \chi$ is fixed, they are uniquely determined by the relation (II.12) up to a real additive constant.

(ii) Keeping $w \in \rho(\mathring{A})$ fixed in the defining relation (II.12) of a Q-function, it follows that q is defined and analytic (at least) on $\rho(\mathring{A})$. In fact, we have the representation

$$q(z) = q(\overline{w}) + (z - \overline{w})\big[\chi(z), \chi(w)\big], \quad z \in \rho(\mathring{A}).$$

(iii) Let us mention one instance of freedom in choice of $h_0, \mathring{A}, \chi$: If $h_0, \mathring{A}, \chi$ satisfy the condition in II.5, (Ext), then for each $\lambda \in \mathbb{C}$, also the choice $h_0, \mathring{A}, \chi + \lambda h_0$ is admissible.

(iv) As a consequence of the previous item, the family of all Q-functions of a given symmetry S in an almost Pontryagin space $\mathcal{A}$ with $\operatorname{ind}_0 \mathcal{A} > 0$ always contains functions which are not real constants. If we are given a choice of $h_0, \mathring{A}, \chi$ such the Q-function built with this data is a real constant, then we can consider the choice $h_0, \mathring{A}, \tilde{\chi} := \chi + h_0$ and will have

$$[\tilde{\chi}(z), \tilde{\chi}(w)] = \underbrace{[\chi(z), \chi(w)]}_{=0} + 2 + \underbrace{[h_0, h_0]}_{=0} = 2.$$

Hence the Q-function $\tilde{q}$ built with $h_0, \mathring{A}, \tilde{\chi}$ is linear.

We are not going further into details concerning the variety of all possible choices for $h_0, \mathring{A}, \chi$. $\Diamond$

II.2 Index of negativity

Let us recall the common notion of the negative index of a function, cf. [KL77].

II.**10 Definition.** Let f be a function which is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies $f(\overline{z}) = \overline{f(z)}$. Let $\rho(f)$ be its domain of holomorphy, and set

$$N_f(z, w) := \begin{cases} \frac{f(z) - f(\overline{w})}{z - \overline{w}}, & z \neq \overline{w} \\ f'(z), & z = \overline{w} \end{cases} \quad \text{for} \quad z, w \in \rho(f). \tag{II.13}$$

Then we denote by $\operatorname{ind}_- f \in \mathbb{N}_0 \cup \{\infty\}$ the supremum of the numbers of negative squares of quadratic forms

$$Q_f(\xi_1, \ldots, \xi_n) := \sum_{i,j=1}^{n} N_f(z_i, z_j)\xi_i\overline{\xi_j},$$

where $n \in \mathbb{N}_0$ and $z_1, \ldots, z_n \in \rho(f)$.

The function f is called a generalised Nevanlinna function if it is meromorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfies $f(\overline{z}) = \overline{f(z)}$, and has the property that $\operatorname{ind}_- f < \infty$. The set of all generalised Nevanlinna functions is denoted by $\mathcal{N}_{<\infty}$. ♢

Let us recall one fact from nondegenerated theory, see, e.g., [KL73].

II.*11. Negative index of Q-functions; nondegenerated case:* Let $\mathcal{P}$ be a Pontryagin space, S a closed symmetric relation in $\mathcal{P}$ with defect index $(1, 1)$, and q a Q-function of S. Then

$$\operatorname{ind}_- q \leq \operatorname{ind}_- \mathcal{P}. \tag{II.14}$$

If S is minimal, then equality holds. ♢

We are seeking for an analogue in the degenerated situation. Consider $\mathcal{A}$ and S given according to II.2. It is obvious from the defining relation (II.12) that

$$\operatorname{ind}_- q \leq \operatorname{ind}_- \mathfrak{P}_{\text{ext}}(\mathcal{A}) = \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}.$$

However, even if S is minimal, equality may fail. Hence, in order to precisely capture negative indices, another notion than $\operatorname{ind}_- q$ is needed. The appropriate number was introduced in cf. [KW99b, Definition 1].

II.**12 Definition.** Let f be a function which is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies $f(\overline{z}) = \overline{f(z)}$, and denote by $\rho(f)$ its domain of holomorphy. Moreover, let $\Delta \in \mathbb{N}$. Then we denote by $\operatorname{ind}_-^\Delta f$ the supremum of the numbers of negative squares of quadratic forms

$$Q_f^\Delta(\xi_1, \ldots, \xi_n; \eta_0, \ldots, \eta_{\Delta-1}) := \sum_{i,j=1}^{n} N_f(z_i, z_j)\xi_i\overline{\xi_j} + \sum_{k=0}^{\Delta-1} \sum_{i=1}^{n} \operatorname{Re}\left(z_i^k \xi_i \overline{\eta_k}\right),$$

where $n \in \mathbb{N}_0$ and $z_1, \ldots, z_n \in \rho(f)$. ♢

Let us point out that, unlike Q_f, the quadratic form Q_f^Δ is not generated in the standard way from some reproducing kernel.

II.*13 Remark.* Due to the form of the second summand in the definition of Q_f^Δ, we always have $\operatorname{ind}_-^\Delta f \geq \Delta$. Moreover, a dimension argument shows

$$\operatorname{ind}_-^\Delta f - \Delta \leq \operatorname{ind}_- f \leq \operatorname{ind}_-^\Delta f. \tag{II.15}$$

In particular, $\operatorname{ind}_-^\Delta f < \infty$ if and only if $f \in \mathcal{N}_{<\infty}$.

It can be shown by examples that $\operatorname{ind}_- f$ may assume every value permitted by (II.15). ♢

Now we can prove the analogue of II.11.

II.**14 Proposition.** *Let $\mathcal{A}$ and S be given according to* II.2, *let q be a Q-function of S, and set $\Delta := \operatorname{ind}_0 \mathcal{A}$. Then*

$$\operatorname{ind}_-^\Delta q \leq \operatorname{ind}_- \mathcal{A} + \Delta.$$

If S is minimal, equality holds.

Proof. Let $h_0, \mathring{A}, \chi$ be those choices according to II.5 which give rise to the Q-function q. Consider the linear span

$$\operatorname{span}\big(\{h_0,\dots,h_{\Delta-1}\}\cup\{\chi(z):\, z\in\rho(\mathring{A})\}\big)\subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A}). \tag{II.16}$$

The inner product on this span is determined by the quadratic form Q_q^Δ:

$$\Big[\sum_{i=1}^n \xi_i\chi(z_i)+\sum_{l=0}^{\Delta-1}\eta_l h_l, \sum_{j=1}^n \xi_j\chi(z_j)+\sum_{k=0}^{\Delta-1}\eta_k h_k\Big]=$$

$$=\sum_{i,j=1}^n \big[\chi(z_i),\chi(z_j)\big]\xi_i\overline{\xi_j}+\sum_{k=0}^{\Delta-1}\sum_{i=1}^n \big[\chi(z_i),h_k\big]\xi_i\overline{\eta_k}+\sum_{l=0}^{\Delta-1}\sum_{j=1}^n \big[h_l,\chi(z_j)\big]\eta_l\overline{\xi_j}=$$

$$=Q_q^\Delta\big(\xi_1,\dots,\xi_n;2\eta_0,\dots,2\eta_{\Delta-1}\big).$$

If follows that

$$\operatorname{ind}_-^\Delta q\le \operatorname{ind}_-\mathfrak{P}_{\text{ext}}(\mathcal{A})=\operatorname{ind}_-\mathcal{A}+\Delta.$$

Assume that S is minimal. Then the linear span (II.16) is dense in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$, remember Remark II.6, (vi). Hence, negative indices coincide. ❑

II.3 Realization theorem

Our aim in this section is to construct a degenerate analogue of the Krein-Langer operator model for a generalised Nevanlinna function.

II.15 Theorem. *Let $q\in\mathcal{N}_{<\infty}$ and $\Delta\in\mathbb{N}$ be given. Then there exists an almost Pontryagin space $\mathcal{A}_\Delta(q)$ with $\operatorname{ind}_0\mathcal{A}_\Delta(q)=\Delta$ and a minimal closed symmetric relation $S_\Delta(q)\subseteq\mathcal{A}_\Delta(q)^2$ with defect index $(1,1)$ which satisfies the regularity conditions* (II.2) *and* (II.3)*, such that q is a Q-function of $S_\Delta(q)$.*

For later reference we state one portion of the proof as a separate lemma.

II.16 Lemma. *Assume that the following data are given:*

1. *A Pontryagin space $\mathcal{P}$.*
2. *A selfadjoint relation $A\subseteq\mathcal{P}^2$ with nonempty resolvent set.*
3. *A subset Ω of $\rho(A)$ which has an accumulation point in each connected component of $\rho(A)$.*
4. *A family $\chi_0(z)$, $z\in\Omega$, of elements of $\mathcal{P}$ with*

$$\big(\chi_0(z)-\chi_0(w), z\chi_0(z)-w\chi_0(w)\big)\in A,\quad z,w\in\Omega. \tag{II.17}$$

5. *A number $\Delta\in\mathbb{N}$, and elements $h_0,\dots,h_{\Delta-1}\in\mathcal{P}$, with*

$$(h_l,h_{l+1})\in A,\ l=0,\dots,\Delta-2,\quad (0,h_0)\in A, \tag{II.18}$$

$$\operatorname{span}\{h_0,\dots,h_{\Delta-1}\}\ \textit{is neutral}, \tag{II.19}$$

$$[\chi_0(z),h_l]=z^l,\quad z\in\Omega,\ l=0,\dots,\Delta-1. \tag{II.20}$$

Using these data, define

$$\mathcal{A} := \mathcal{P}[-]\operatorname{span}\{h_0,\dots,h_{\Delta-1}\},$$
$$S := \big\{(x,y)\in A\cap\mathcal{A}^2 : \quad \forall\, z\in\Omega :\ y-\overline{z}x \perp \chi_0(z)\big\}.$$

Then $\mathcal{A}$ and S have all properties required in II.2, *and $\rho(S_{\mathrm{fac}})\supseteq\rho(A)$. The elements h_l have the property* II.5, (Bas). *There exists a family $\chi(z)$, $z\in\rho(A)$, of elements of $\mathcal{P}$ such that*

$$\chi(z)=\chi_0(z),\quad z\in\Omega,$$

and such that the data h_0, A, χ has the property II.5, (Ext).

Proof. It is clear that $\mathcal{A}$ is an almost Pontryagin space, and that $\mathcal{A}^\circ = \operatorname{span}\{h_0,\dots,h_{\Delta-1}\}$. Assume that $h=\sum_{l=0}^{\Delta-1}\lambda_l h_l=0$. Then

$$0=[\chi_0(z),h]=\sum_{l=0}^{\Delta-1}\overline{\lambda_l}z^l,\quad z\in\Omega.$$

Since Ω certainly contains infinitely many points, we conclude that $\lambda_0 = \ldots = \lambda_{\Delta-1}=0$. Thus $\{h_0,\dots,h_{\Delta-1}\}$ is linearly independent, and we obtain $\operatorname{ind}_0\mathcal{A}=\Delta$. Clearly $\dim\mathcal{P}/\mathcal{A}=\Delta$, and hence $\mathcal{P}$ is a canonical Pontryagin space extension of $\mathcal{A}$.

The fact that S is a closed symmetric relation is again clear. We have

$$[\chi_0(z),h_{l+1}-\overline{z}h_l]=z^{l+1}-z\cdot z^l=0,\quad z\in\Omega,\ l=0,\dots,\Delta-2,$$

and hence $(h_l,h_{l+1})\in S$. This already shows that the elements h_l satisfy II.5, (Bas).

The fact that A is a selfadjoint relation with nonempty resolvent set which extends $\operatorname{span}(S\cup\{(0,h_0)\})$ holds by assumption. Choose $w\in\Omega$, and define

$$\chi(z):=\big(I+(z-w)(A-z)^{-1}\big)\chi_0(w),\quad z\in\rho(A).$$

Due to (II.17), we have $\chi(z)=\chi_0(z)$, $z\in\Omega$. Let $(x,y)\in S$, then

$$[\chi(z),y-\overline{z}x]=0,\quad z\in\Omega.$$

By analyticity this equality holds for all $z\in\rho(A)$, i.e. $\chi(z)\perp\operatorname{ran}(S-\overline{z})$, $z\in\rho(A)$. Similarly, the equality $[\chi(z),h_l]=z^l$, $z\in\Omega$, transfers by analyticity to all of $\rho(A)$. We see that h_0,A,χ satisfy the conditions listed in II.5, (Ext).

It remains to show that S has defect index $(1,1)$ and satisfies the regularity conditions (II.2) and (II.3). Let $h\in\mathcal{A}^\circ$ be given, and write $h=\sum_{l=0}^{\Delta-1}\lambda_l h_l$. Assume that $(\mu_1 h,\mu_2 h)\in S$, then

$$0=[\chi(z),(\mu_2-\overline{z}\mu_1)h]=(\mu_2-z\mu_1)\cdot\sum_{l=0}^{\Delta-1}\lambda_l z^l,\quad z\in\rho(A).$$

This yields that either $\mu_1=\mu_2=0$ or $\lambda_l=0$, $l=0,\dots,\Delta-1$. Hence, (II.3) is satisfied.

Next, we have $(A-z)^{-1}\operatorname{span}\{h_0,\dots,h_{\Delta-1}\} \subseteq \operatorname{span}\{h_0,\dots,h_{\Delta-1}\}$, $z \in \rho(A)$, and hence also $(A-z)^{-1}\mathcal{A} \subseteq \mathcal{A}$, $z \in \rho(A)$. The elements $\chi(z)$ are related among each other by (II.8), and hence, for $(x,y) \in A$ and $z,w \in \rho(A)$,

$$[y-\overline{z}x,\chi(z)]-[y-\overline{w}x,\chi(w)] = [y,\chi(z)-\chi(w)]-[x,z\chi(z)-w\chi(w)] = 0, \quad \text{(II.21)}$$

i.e. the value of $[\chi(z), y-\overline{z}x]$ does not depend on z. It follows that

$$S = \big\{(x,y) \in A \cap \mathcal{A}^2 : \quad \exists\, z \in \rho(A) : \, y-\overline{z}x \perp \chi(z)\big\}.$$

Let $y \in \mathcal{A}$ and $z \in \rho(A)$ be given. Set $x := (A-z)^{-1}y$, then $x \in \mathcal{A}$, and

$$\big(x, y+zx-[y,\chi(\overline{z})]h_0\big) \in A \cap \mathcal{A}^2.$$

Moreover,

$$\big[(y+zx-[y,\chi(\overline{z})]h_0)-zx, \chi(\overline{z})\big] = 0,$$

and hence $(x, y+zx-[y,\chi(\overline{z})]h_0) \in S$. Thus $y \in \operatorname{ran}(S-z)+\operatorname{span}\{h_0\}$. We conclude that

$$\operatorname{ran}(S-z)+\operatorname{span}\{h_0\} = \mathcal{A}, \quad z \in \rho(A). \qquad \text{(II.22)}$$

In particular, (II.2) holds. Moreover, Lemma II.3, (iii), implies $\rho(S_{\text{fac}}) \supseteq \rho(A)$.

Since $[\chi(z), h_0] = 1$, $z \in \rho(A)$, we have $h_0 \notin \operatorname{ran}(S-\overline{z})$ for all such values of z. It follows from (II.22) that S has defect index $(1,1)$. ❑

Proof of Theorem II.15. Let $q \in \mathcal{N}_{<\infty}$ and $\Delta \in \mathbb{N}$ be given. We start from the Krein-Langer model associated with q. Denote by $\rho(q)$ the maximal domain of holomorphy of q in $\mathbb{C}$, and let $\mathcal{L}(q)$ be the linear space generated by formal elements $\{e_z : z \in \rho(q)\}$ as a basis. On $\mathcal{L}(q)$ define an inner product by the requirement that

$$[e_z,e_w] := \begin{cases} \frac{q(z)-\overline{q(w)}}{z-\overline{w}}, & z,w \in \rho(q),\, z \neq \overline{w} \\ q'(z), & z = \overline{w} \in \rho(q) \end{cases}$$

Moreover, let $A(q) \subseteq \mathcal{L}(q)^2$ be defined as

$$A(q) := \operatorname{span}\big\{(e_z-e_w, ze_z-we_w) : \, z,w \in \rho(q)\big\}. \qquad \text{(II.23)}$$

Step 1; Definition of data as in Lemma II.16*:* Let D be a linear space with dimension Δ, and choose a basis $\{\delta_0,\dots,\delta_{\Delta-1}\}$ of D. Moreover, let D be endowed with the trivial inner product $[x,y] := 0$, $x,y \in D$. Define a map $c : D \to \mathcal{L}(q)^*$ by conjugate linearity and

$$c(\delta_l) : \begin{cases} \mathcal{L}(q) & \to \quad \mathbb{C} \\ \sum \lambda_i e_{z_i} & \mapsto \quad \sum \lambda_i z_i^l \end{cases}$$

Explicitly, thus, the map c acts as

$$\Big[c\big(\sum_{l=0}^{\Delta-1}\alpha_l\delta_l\big)\Big]\Big(\sum_{i=1}^{n}\lambda_i e_{z_i}\Big) = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{\Delta-1} \end{pmatrix}^* \begin{pmatrix} z_1^0 & \cdots & z_n^0 \\ \vdots & & \vdots \\ z_1^{\Delta-1} & \cdots & z_n^{\Delta-1} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_n \end{pmatrix}$$

In particular, we see that c is injective. By means of [SW12, Definition 3.1] and [SW12, (3.3)] we have an inner product space $D \ltimes_c \mathcal{L}(q)$ together with isometric embeddings $\iota_{c,1} : D \to D \ltimes_c \mathcal{L}(q)$ and $\iota_{c,2} : \mathcal{L}(q) \to D \ltimes_c \mathcal{L}(q)$.

Let $(\iota, \mathcal{P})$ be a Pontryagin space completion of $D \ltimes_c \mathcal{L}(q)$, i.e. a Pontryagin space $\mathcal{P}$ together with an isometric map of $D \ltimes_c \mathcal{L}(q)$ onto a dense subspace of $\mathcal{P}$, cf. [SW12, §6]. Then we have the isometries

$$\eta_1 := \iota \circ \iota_{c,1} : D \to \mathcal{P}, \quad \eta_2 := \iota \circ \iota_{c,2} : \mathcal{L}(q) \to \mathcal{P}.$$

Since $\ker \iota = (D \ltimes_c \mathcal{L}(q))^\circ$, it follows that

$$\ker \eta_1 = \iota_{c,1}^{-1}\big((D \ltimes_c \mathcal{L}(q))^\circ\big), \quad \ker \eta_2 = \iota_{c,2}^{-1}\big((D \ltimes_c \mathcal{L}(q))^\circ\big).$$

Since c is injective, we obtain from [SW12, Proposition 3.5] that $\iota_{c,1}(D) \cap (D \ltimes_c \mathcal{L}(q))^\circ = \{0\}$. Hence η_1 is injective.

Next, set $\Omega := \rho(q)$,

$$h_l := \eta_1(\delta_l),\ l = 0, \dots, \Delta - 1, \quad \chi_0(z) := \eta_2(e_z),\ z \in \rho(q),$$

and let a linear relation $\mathring{A}_\Delta(q) \subseteq \mathcal{P}^2$ be defined as

$$\mathring{A}_\Delta(q) := \operatorname{cls}\Big((\eta_2 \times \eta_2)A(q) \cup \big\{(h_l, h_{l+1}) : l = 0, \dots, \Delta - 2\big\} \cup \big\{(0, h_0)\big\}\Big)$$

Step 2; Checking the hypothesis of Lemma II.16*:* Our first aim is to show that $\mathring{A}_\Delta(q)$ is selfadjoint. The relation $(\eta_2 \times \eta_2)(A(q))$ is symmetric. The elements h_l all belong to the neutral subspace $\eta_1(D)$, and hence the linear span of (h_l, h_{l+1}), $l = 0, \dots, \Delta - 2$, and $(0, h_0)$ is trivially symmetric. Next,

$$[\chi_0(z), h_{l+1}] = [\iota_{c,2}e_z, \iota_{c,1}\delta_{l+1}] = z^{l+1} = [z\iota_{c,2}e_z, \iota_{c,1}\delta_l] = [z\chi_0(z), h_l], \\ l = 0, \dots, \Delta - 2,\ z \in \rho(q),$$

and hence in particular

$$[z\chi_0(z) - w\chi_0(w), h_l] = [\chi_0(z) - \chi_0(w), h_{l+1}], \quad z, w \in \rho(q),\ l = 0, \dots, \Delta - 2.$$

Finally, for $z, w \in \rho(q)$,

$$[\chi_0(z) - \chi_0(w), h_0] = [\iota_{c,2}e_z - \iota_{c,2}e_w, \iota_{c,1}\delta_0]_c = 0 = [z\chi_0(z) - w\chi_0(w), 0].$$

It follows that $\mathring{A}_\Delta(q)$ is symmetric. Moreover, by its definition, $\mathring{A}_\Delta(q)$ is closed.

In order to establish selfadjointness we apply Lemma 2.12. Let $w \in \rho(q)$. It is clear that $\operatorname{ran}(\mathring{A}_\Delta(q) - w)$ contains $\eta_1(D)$. We have

$$e_z \in \operatorname{ran}\big(A(q) - w\big), \quad z \in \rho(q), z \neq w,$$

and it follows that $\{\chi_0(z) : z \in \rho(q), z \neq w\} \subseteq \operatorname{ran}(\mathring{A}_\Delta(q) - w)$. Next,

$$\lim_{z \to w}[\chi_0(z), \chi_0(v)] = \lim_{z \to w} N_q(v, z) = N_q(v, w) = [\chi_0(w), \chi_0(v)], \quad v \in \rho(q),$$
$$\lim_{z \to w}[\chi_0(z), h_l] = \lim_{z \to w} z^l = w^l = [\chi_0(w), h_l], \quad l = 0, \dots, \Delta - 1,$$
$$\lim_{z \to w}[\chi_0(z), \chi_0(z)] = \lim_{z \to w} N_q(z, z) = N_q(w, w) = [\chi_0(w), \chi_0(w)],$$

and hence $\lim_{z\to w}\chi_0(z)=\chi_0(w)$ in the Pontryagin space $\mathcal{P}$, cf. [IKL82, §2]. It follows that $\operatorname{ran}(\mathring{A}_\Delta(q)-w)\supseteq\eta_1(D)+\eta_2(\mathcal{L}(q))$, and is thus dense in $\mathcal{P}$. We conclude that $\mathring{A}_\Delta(q)$ is selfadjoint and that $\rho(\mathring{A}_\Delta(q))\supseteq\rho(q)$.

The relation (II.17) is immediate from (II.23) and the definition of $\chi_0(z)$. Finally, the relations (II.18)–(II.20) are immediate from the definitions of $\mathring{A}_\Delta(q)$ and c.

Step 3; The Q-function representation: We apply Lemma II.16 with the data constructed above. This provides us with an almost Pontryagin space, call it $\mathcal{A}_\Delta(q)$, a symmetric relation $S_\Delta(q)$ therein, and a family $\chi(z)$ extending our family $\chi_0(z)$ to $\rho(\mathring{A}_\Delta(q))$. Lemma II.16 says moreover that we may speak of Q-functions of $S_\Delta(q)$ constructed with $h_l,\mathring{A}_\Delta(q),\chi$. However, by the definition of inner products,

$$[\chi(z),\chi(w)]=\frac{q(z)-\overline{q(w)}}{z-\overline{w}},\quad z,w\in\rho(q),$$

and hence q is a Q-function of $S_\Delta(q)$. By definition of the space $\mathcal{P}$ as completion of $D\ltimes_c\mathcal{L}(q)$, the linear span of $\mathcal{A}_\Delta(q)^\circ\cup\{\chi(z):\,z\in\rho(q)\}$ is dense in $\mathcal{P}$. Hence, $S_\Delta(q)$ is minimal, cf. Remark II.6, (vi). ❑

II.*17 Remark.* The above constructed representation of q has an additional noteworthy property. Since the function q has an analytic continuation to $\rho(\mathring{A}_\Delta(q))$, maximality of $\rho(q)$ implies that $\rho(q)=\rho(\mathring{A}_\Delta(q))$. ◊

II.18 Lemma. *Let $q\in\mathcal{N}_{<\infty}$ and $\Delta\in\mathbb{N}$ be given, and let $\mathcal{A}_\Delta(q)$, $\mathring{A}_\Delta(q)$, $S_\Delta(q)$, $\chi(z)$, and h_l, be as in the proof of Theorem* II.15. *Moreover, set*

$$S_0:=\operatorname{span}\big(\{(h_l;h_{l+1}):\,l=0,\dots,\Delta-2\}\cup\{(0;h_0)\}\big).$$

Then, as relations in $\mathfrak{P}_{\rm ext}(\mathcal{A}_\Delta(q))$, we can describe $\mathring{A}_\Delta(q)$ and $S_\Delta(q)$ as (S_0^* denotes the Pontryagin space adjoint in $\mathcal{P}_{\rm ext}(\mathcal{A}_\Delta(q))$)

$$\begin{aligned}S_\Delta(q)&=\big\{(x,y)\in\mathfrak{P}_{\rm ext}(\mathcal{A}_\Delta(q))^2:\\&\qquad\qquad\forall\,z\in\rho(q):\,y-\overline{z}x\perp\operatorname{span}\{\chi(z),h_0,\dots,h_{\Delta-1}\}\big\}\\ \mathring{A}_\Delta(q)&=\big\{(x,y)\in S_0^*:\ \exists\,\gamma\in\mathbb{C}\,\forall\,z\in\rho(q):\,[y-\overline{z}x,\chi(z)]=\gamma\big\}\end{aligned}\tag{II.24}$$

Proof. The formula for $S_\Delta(q)$ is immediate from its definition, since $\mathcal{A}_\Delta(q)=\mathfrak{P}_{\rm ext}(\mathcal{A}_\Delta(q))[-]\operatorname{span}\{h_0,\dots,h_{\Delta-1}\}$.

Next, we show the inclusion '$\subseteq$' in (II.24). Since $S_0\subseteq\mathring{A}_\Delta(q)$, clearly, $\mathring{A}_\Delta(q)\subseteq S_0^*$. Moreover, as we have already noticed in (II.21), the value of $[y-\overline{z}x,\chi(z)]$ does not depend on $z\in\rho(\mathring{A}_\Delta(q))$ whenever $(x,y)\in\mathring{A}_\Delta(q)$. For the converse inclusion, assume that (x,y) belongs to the set on the right side of (II.24). Then, again by the computation (II.21), we have $(x,y)\in(\eta_2\times\eta_2)(A(q))^*$. Since $(\eta_2\times\eta_2)(A(q))+S_0$ is dense in $\mathring{A}_\Delta(q)$, we obtain $(x,y)\in\mathring{A}_\Delta(q)^*=\mathring{A}_\Delta(q)$. ❑

II.4 Minimality aspects

We have seen in Proposition I.13 that, when investigating the totality of all generalized resolvents of a given symmetry S, it is enough to consider the case that S is minimal. Let us show that also for investigating the totality of all Q-functions of a symmetry S this is the case. We again use the notation introduced in Lemma 2.20: let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed linear relation in $\mathcal{A}$ with $\gamma(S) \neq \emptyset$, set

$$\mathcal{C} := \bigcap_{z\in\gamma_s(S)} \operatorname{ran}(S-z), \quad \mathcal{D} := \mathcal{A}[-]\mathcal{C}, \qquad \mathcal{A}_1 := \mathcal{D}/\mathcal{C}^\circ,$$

let $\pi : \mathcal{D} \to \mathcal{A}_1$ denote the canonical projection, and set

$$S_1 := (\pi \times \pi)\big(S \cap (\mathcal{D} \times \mathcal{D})\big).$$

II.**19 Proposition.** *Let $\mathcal{A}$ and S be given according to* II.2*, and let $\mathcal{A}_1$ and S_1 be defined as above. Then $\mathcal{A}_1$ and S_1 possess all properties required in* II.2*. The set of all Q-functions of S_1 is equal to the set of all Q-functions of S.*

Proof.
Step 1: Choose $h_0, \mathring{A}, \chi$ according to II.5 for S. Then we have $\chi(z) \in \mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{C}$, $z \in \rho(\mathring{A})$. However, no nonzero element of $\mathcal{A}^\circ$ can be orthogonal to all elements $\chi(z)$, and hence $\mathcal{A}^\circ \cap \mathcal{C} = \{0\}$. We conclude that

$$\mathcal{A}_1^\circ = \mathcal{A}^\circ/\mathcal{C}^\circ \cong \mathcal{A}^\circ. \tag{II.25}$$

in particular, $\operatorname{ind}_0 \mathcal{A}_1 = \operatorname{ind}_0 \mathcal{A} > 0$. Moreover,

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_1) = (\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{C})/\mathcal{C}^\circ. \tag{II.26}$$

By Lemma 2.20 the relation S_1 is closed, symmetric, minimal, and $\mathfrak{n}_\pm(S_1) \leq 1$. Let $z \in \gamma_s(S)$ and assume that $\pi(h_0) \in \operatorname{ran}(S_1 - z)$. Remembering (2.8) we find $x \in \mathcal{D} \cap \operatorname{ran}(S-z)$ with $\pi(x) = \pi(h_0)$. However, $\ker\pi = \mathcal{C}^\circ$ and hence $h_0 \in x + \mathcal{C}^\circ \subseteq \operatorname{ran}(S-z)$. We arrived at a contradiction, conclude that $\pi(h_0) \notin \operatorname{ran}(S_1 - z)$, and hence that $\mathfrak{n}_\pm(S_1) = 1$. Since $\pi(h_0) \in \mathcal{A}_1^\circ$, we also see that (II.2) holds. Since S_1 is minimal, S_1 has no eigenvalues, in particular, (II.3) holds.

Step 2: Let $h_0, \mathring{A}, \chi$ be given according to II.5 for S, and let q be a Q-function of S built from this data. We are going to construct data $h_{0,1}, \mathring{A}_1, \chi_1$ according to II.5 for S_1, such that q is a Q-function of S_1 built with $h_{0,1}, \mathring{A}_1, \chi_1$. Define

$$h_{l,1} := \pi(h_l),\ l = 0, \ldots, \Delta-1, \quad \mathring{A}_1 := (\pi \times \pi)\big(\mathring{A} \cap \mathcal{D}^2\big).$$

By (II.25), the elements $h_{0,1}, \ldots h_{\Delta-1,1}$ form a basis of $\mathcal{A}_1^\circ$. Clearly, $(\pi(h_l), \pi(h_{l+1})) \in S_1$, $l = 0, \ldots, \Delta-2$, and hence $h_{0,1}, \ldots h_{\Delta-1,1}$ are an admissible choice according to II.5, (Bas). Since $\mathring{A}$ extends S, we obtain from (2.7) that

$$(\mathring{A} - z)^{-1}(\mathcal{C}) \subseteq \mathcal{C}, \quad z \in \rho(\mathring{A}).$$

From $\mathring{A}$ being selfadjoint, we conclude that also $(\mathring{A} - z)^{-1}(\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{C}) \subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{C})$ and in turn $(\mathring{A} - z)^{-1}(\mathcal{C}^\circ) \subseteq \mathcal{C}^\circ$. Now [SW16, Proposition 3.2] applies and yields that $\mathring{A}_1$ is selfadjoint and satisfies $\rho(\mathring{A}_1) \supseteq \rho(\mathring{A})$. Since $\mathring{A}$ extends $\operatorname{span}(S \cup \{(0, h_0)\})$, $\mathring{A}_1$ extends $\operatorname{span}(S_1 \cup \{(0, h_{0,1})\})$, and hence is an admissible choice according to II.5, (Ext).

In order to define defect elements, choose $z_0 \in \rho(\mathring{A})$, and set

$$\begin{aligned}\chi_1(z_0) &:= \pi(\chi(z_0)),\\ \chi_1(z) &:= \big(I + (z - z_0)(\mathring{A}_1 - z)^{-1}\big)\chi_1(z_0), \quad z \in \rho(\mathring{A}_1), z \neq z_0.\end{aligned}$$

Since $\chi(z_0) \perp \operatorname{ran}(S - \overline{z_0})$ also $\chi_1(z_0) \perp \operatorname{ran}(S_1 - \overline{z}_0)$, and hence actually $\chi_1(z) \perp \operatorname{ran}(S_1 - \overline{z})$ for all $z \in \rho(\mathring{A}_1)$. Moreover, by isometry of π, we have $[\chi_1(z_0), h_{0,1}] = 1$. By Remark II.6, (*iv*), the relation (II.12) holds. Thus the family $(\chi_1(z))_{z\in\rho(\mathring{A}_1)}$ is an admissible choice according to II.5, (Ext).

Since $(\chi(z)-\chi(w), z\chi(z)-w\chi(w)) \in \mathring{A}$ and $\chi(z) \perp \mathcal{C}$, we also have $(\pi(\chi(z))-\pi(\chi(w)), z\pi(\chi(z)) - w\pi(\chi(w))) \in \mathring{A}_1$. Therefore $\chi_1(z) = \pi(\chi(z))$, $z \in \rho(\mathring{A})$. We obtain

$$[\chi_1(z), \chi_1(w)] = [\chi(z), \chi(w)], \quad z, w \in \rho(\mathring{A}), \tag{II.27}$$

whence q is a Q-function of S_1.

Step 3: Let $h_{0,1}, \mathring{A}_1, \chi_1$ be given according to II.5 for S_1, and let q_1 be a Q-function of S_1 built with this data. Let $h'_l \in \mathcal{A}^\circ$ be as in II.5, (Bas), for S. Then $\pi(h'_0), \dots, \pi(h'_{\Delta-1})$ satisfy the requirements II.5, (Bas), for S_1. Thus there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\pi(h'_l) = \lambda h_{l,1}$. Set $h_l := \lambda^{-1} h'_l$, then also $h_0, \dots, h_{\Delta-1}$ satisfy II.5, (Bas), for S, and $\pi(h_l) = h_{l,1}$.

Consider the relation $S' := S + \operatorname{span}\{(0, h_0)\} \subseteq \mathcal{A}^2$ and choose $\mu \in (\rho(S_{\text{fac}}) \cap \rho(\mathring{A}_1)) \setminus \mathbb{R}$. Then the hypothesis [SW16, (7.1),(7.2)] are fulfilled. Moreover, since we factorise the whole space $\mathcal{C}^\circ$, also the hypothesis of item (*ii*) of [SW16, Theorem 7.1] is fulfilled.

Now observe that

$$S'_1 := \operatorname{span}(S_1 + \{(0, h_{0,1})\}) = (\pi \times \pi)(S' \cap \mathcal{D}^2),$$

and apply [SW16, Theorem 7.1] with the extending spaces (remember (II.26))

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}) \supseteq \mathcal{A}, \quad \mathfrak{P}_{\text{ext}}(\mathcal{A}_1) \supseteq \mathcal{A}_1,$$

and the extension $\mathring{A}_1$ of S'_1. This provides a selfadjoint extension $\mathring{A}$ of S' in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ with

$$(\pi \times \pi)\big(\mathring{A} \cap (\mathfrak{P}_{\text{ext}}(\mathcal{A})[-]\mathcal{C})^2\big) = \mathring{A}_1.$$

Choose an element $\chi_0 \in \mathfrak{P}_{\text{ext}}(\mathcal{A})$ such that $\chi_0 \perp \operatorname{ran}(S - \overline{\mu})$ and $[\chi_0, h_0] = 1$. Then $\pi(\chi_0) \perp \operatorname{ran}(S_1 - \overline{\mu})$ and $[\pi(\chi_0), h_{0,1}] = 1$. Hence, $\chi_0 - \chi_1(\mu) \in \mathcal{A}_1^\circ$. By (II.25) and the fact that $\mathcal{A}^\circ \perp \operatorname{ran}(S - \overline{\mu})$, we can choose $\chi(\mu) \in \mathfrak{P}_{\text{ext}}(\mathcal{A})$ such that $\chi(\mu) \perp \operatorname{ran}(S - \overline{\mu})$ and $\pi(\chi(\mu)) = \chi_1(\mu)$. Let $\chi(z)$ be defined by the relation (II.8) using $\mathring{A}$. Again referring to Remark II.6, (*iv*), we obtain that $h_0, \mathring{A}, \chi$ qualify for the definition of a Q-function q of S. Applying Step 2 with this data leads to the equality (II.27), and it follows that q_1 is a Q-function of S. ❑

Next let us show that a minimal symmetry S is, up to isomorphisms, uniquely determined by each of its Q-function. Note that this contrasts the situation for compressed resolvents, cf. Remark I.12.

II.20 Proposition. *Let $\mathcal{A}_1, S_1$ and $\mathcal{A}_2, S_2$ be given according to* II.2*, and assume that S_1 and S_2 are minimal. For $j \in \{1, 2\}$ choose $h_0^j, \mathring{A}_j, \chi_j$ according to* II.5 *for $\mathcal{A}_j, S_j$, and let q_j be a Q-function of S_j built with this data.*

If $q_1 = q_2$ *and* $\operatorname{ind}_0 \mathcal{A}_1 = \operatorname{ind}_0 \mathcal{A}_2$*, then there exists an isometric isomorphism* Φ *of* $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$ *onto* $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$ *with*

$$\begin{aligned}
&\Phi(\mathcal{A}_1) = \mathcal{A}_2,\\
&(\Phi \times \Phi)(S_1) = S_2, \quad (\Phi \times \Phi)(\mathring{A}_1) = \mathring{A}_2,\\
&\Phi(\chi_1(z)) = \chi_2(z), \quad z \in \rho(\mathring{A}_1) = \rho(\mathring{A}_2),\\
&\Phi(h_l^1) = h_l^2, \quad l = 0, \dots, \Delta - 1.
\end{aligned}$$

Proof. Set $\Delta := \dim \mathcal{A}_1^\circ = \dim \mathcal{A}_2^\circ$. We claim that a linear map

$$\Phi : \mathcal{A}_1^\circ \dot{+} \operatorname{span}\big\{\chi_1(z) \colon z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2)\big\} \to \mathcal{A}_2^\circ \dot{+} \operatorname{span}\big\{\chi_2(z) \colon z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2)\big\}$$

is well-defined by the requirements that

$$\begin{aligned}
&\Phi(h_l^1) = h_l^2, \quad l = 0, \dots, \Delta - 1,\\
&\Phi(\chi_1(z)) = \chi_2(z), \quad z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2),
\end{aligned}$$

To see this assume that λ_j and ν_k are scalars such that

$$a := \sum_{k=0}^{\Delta-1} \nu_k h_k^1 + \sum_j \lambda_j \chi_1(w_j) = 0.$$

Evaluating scalar products with h_l^1 and $\chi_1(z)$, respectively, yields

$$\begin{aligned}
&0 = [a, h_l^1] = \sum_j \lambda_j w_j^l, \quad l = 0, \dots, \Delta - 1,\\
&0 = [\chi(z), a] = \sum_{k=0}^{\Delta-1} \overline{\nu_k} z^k + \sum_j \overline{\lambda_j} \frac{q_1(z) - \overline{q_1(w_j)}}{z - \overline{w}_j}, \quad z \in \rho(\mathring{A}_1).
\end{aligned}$$

Since $[\chi_2(z), h_l^2] = z^l$ and $q_1 = q_2$, it follows that

$$\begin{aligned}
&\Big[\sum_{k=0}^{\Delta-1} \nu_k h_k^2 + \sum_j \lambda_j \chi_2(w_j), h_l^2\Big] = \sum_j \lambda_j w_j^l = 0, \quad l = 0, \dots, \Delta - 1,\\
&\Big[\sum_{k=0}^{\Delta-1} \nu_k h_k^2 + \sum_j \lambda_j \chi_2(w_j), \chi_2(z)\Big] = \sum_{k=0}^{\Delta-1} \overline{\nu_k} z^k + \sum_j \overline{\lambda_j} \frac{q_2(z) - \overline{q_2(w_j)}}{z - \overline{w}_j} = 0,\\
&\hspace{20em} z \in \rho(\mathring{A}_1) \cap \rho(\mathring{A}_2).
\end{aligned}$$

Minimiality of S_2 implies that

$$\sum_{k=0}^{\Delta-1} \nu_k h_k^2 + \sum_j \lambda_j \chi_2(w_j) = 0,$$

cf. Remark II.6, (vi).

The fact that Φ is isometric is clear from our assumption that $q_1 = q_2$ and from the fact that $[\chi_1(z), h_l^1] = z^l = [\chi_2(z), h_l^2]$. Moreover, by minimality of S_1 and S_2,

$$\overline{\operatorname{dom} \Phi} = \mathfrak{P}_{\text{ext}}(\mathcal{A}_1), \quad \overline{\operatorname{ran} \Phi} = \mathfrak{P}_{\text{ext}}(\mathcal{A}_2).$$

Thus Φ can be extended to an isometric isomorphism between $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$ and $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$, which we will denote again by Φ. By definition it holds that $\Phi(\mathcal{A}_1^\circ) = \mathcal{A}_2^\circ$, and passing to orthogonal complements yields $\Phi(\mathcal{A}_1) = \mathcal{A}_2$.

In order to show that $(\Phi\times\Phi)(\mathring{A}_1) = \mathring{A}_2$, fix $w \in \rho(\mathring{A}_1)\cap\rho(\mathring{A}_2)$, and compute

$$(\mathring{A}_2 - w)^{-1}\Phi\chi_1(z) = (\mathring{A}_2 - w)^{-1}\chi_2(z) = \frac{\chi_2(w) - \chi_2(z)}{w - z} =$$
$$= \Phi\Big(\frac{\chi_1(w) - \chi_1(z)}{w - z}\Big) = \Phi\Big((\mathring{A}_1 - w)^{-1}\chi_1(z)\Big), \quad z \in \rho(\mathring{A}_1)\cap\rho(\mathring{A}_2), z \neq w,$$

$$(\mathring{A}_2 - w)^{-1}\Phi(h_l^1 - wh_{l-1}^1) = (\mathring{A}_2 - w)^{-1}(h_l^2 - wh_{l-1}^2) = h_{l-1}^2 =$$
$$= \Phi(h_{l-1}^1) = \Phi\Big((\mathring{A}_1 - w)^{-1}(h_l^1 - wh_{l-1}^1)\Big), \quad l = 1,\dots,\Delta-1,$$

$$(\mathring{A}_2 - w)^{-1}\Phi h_0^1 = (\mathring{A}_2 - w)^{-1}h_0^2 = 0 = \Phi\Big((\mathring{A}_1 - w)^{-1}h_0^1\Big).$$

By continuity it follows that $\Phi\circ(\mathring{A}_1 - w)^{-1} = (\mathring{A}_2 - w)^{-1}\circ\Phi$, and hence $(\Phi\times\Phi)(\mathring{A}_1) = \mathring{A}_2$. This also implies that $\rho(\mathring{A}_1) = \rho(\mathring{A}_2)$.

The fact that also $(\Phi\times\Phi)(S_1) = S_2$ is immediate from

$$S_1 = \big\{(a,b)\in\mathring{A}_1 : b - \overline{z}a \perp \mathcal{A}_1^\circ + \operatorname{span}\{\chi(z)\}, z\in\rho(\mathring{A}_1)\big\}$$

and the corresponding representation of S_2. ❑

Finally, we determine the maximal domain of analyticity of a Q-function.

II.**21 Corollary.** *Let $\mathcal{A}$ and S be given according to* II.2*, let $h_0, \mathring{A}, \chi$ be chosen according to* II.5*, and let q be a Q-function of S built with this data. If S is minimal, then $\rho(q) = \rho(\mathring{A})$.*

Proof. Set $\Delta := \operatorname{ind}_0\mathcal{A}$ and consider the space $\mathcal{A}_\Delta(q)$ and the relations $S_\Delta(q), \mathring{A}_\Delta(q)$ constructed in Theorem II.15. Then $S_\Delta(q)$ is minimal, q is a Q-function of $S_\Delta(q)$, and $\operatorname{ind}_0\mathcal{A}_\Delta(q) = \Delta = \operatorname{ind}_0\mathcal{A}$. Hence, by the above Proposition II.20 we have, in particular, $\rho(\mathring{A}) = \rho(\mathring{A}_\Delta(q))$. However, as we observed in Remark II.17, $\rho(\mathring{A}_\Delta(q)) = \rho(q)$. ❑

II.5 Analytic model

In this section we show existence of a degenerate version of the reproducing kernel Pontryagin space model for a minimal symmetry as discussed, e.g., in [Dij+04]

If Ω is a set, we denote by δ_w, $w\in\Omega$, the point evaluation functionals

$$\delta_w : \begin{cases} \mathbb{C}^\Omega & \to \quad \mathbb{C} \\ \ f & \mapsto \quad f(w) \end{cases}$$

Let us recall the following notion, cf. [KWW05, §5].

II.**22 Definition.** Let Ω be a set, and let $\mathcal{A}$ be an almost Pontryagin space. We say that $\mathcal{A}$ is a *reproducing kernel almost Pontryagin space* of functions on Ω, if

(rk aPs1) the elements of $\mathcal{A}$ are functions of Ω into $\mathbb{C}$;

(rk aPs2) for each $w \in \Omega$ the restriction of the point evaluation functional δ_w to $\mathcal{A}$ is continuous (w.r.t. the topology of $\mathcal{A}$).

We speak of a *reproducing kernel almost Pontryagin space of analytic functions* on Ω, if $\Omega \subseteq \mathbb{C}$ is an open subset of $\mathbb{C}$ and all elements of $\mathcal{A}$ are analytic on Ω. ♢

II.**23 Proposition.** *Let $\mathcal{A}$ and S be given according to* II.2*, set $\Delta := \operatorname{ind}_0 \mathcal{A}$, and assume that S is minimal. Then there exists an open and dense subset Ω of $\mathbb{C}$, a reproducing kernel almost Pontryagin space $\mathcal{K}$ of analytic functions on Ω, and an isometric isomorphism $\Lambda : \mathcal{A} \to \mathcal{K}$, such that $\mathcal{K}^\circ = \operatorname{span}\{1, \ldots, z^{\Delta-1}\}$ and*

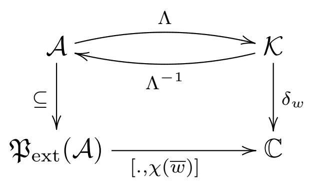
$$(\Lambda \times \Lambda)(S) = \big\{(f, g) \in \mathcal{K}^2 : \ g(z) = zf(z), z \in \Omega\big\}.$$

Proof. Choose $h_0, \mathring{A}, \chi$ according to II.5, set $\Omega := \rho(\mathring{A})$, and define a map $\Lambda : \mathcal{A} \to \mathbb{C}^\Omega$ by

$$\Lambda(x)(z) := [x, \chi(\overline{z})], \quad x \in \mathcal{A}, z \in \Omega.$$

Clearly, Λ is linear. Let us show that Λ is injective. Assume that $x \in \mathcal{A}$ with $\Lambda(x) = 0$. This means that $x \in \mathfrak{P}_{\text{ext}}(\mathcal{A})$ with $x \perp \mathcal{A}^\circ$ and $x \perp \chi(z)$, $z \in \rho(\mathring{A})$. Minimality of S implies that $x = 0$, cf. Remark II.6, (vi). Clearly, each function $\Lambda(x)$ is analytic on Ω.

Define an inner product and a topology on $\mathcal{K} := \Lambda(\mathcal{A}) \subseteq \mathbb{C}^\Omega$ by requiring Λ to be an isometric homeomorphism. Then $\mathcal{K}$ becomes an almost Pontryagin space whose elements are analytic functions on Ω. Since $\mathcal{A}$ is continuously contained in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$, the diagram



yields that point evaluation functionals are continuous. This shows that $\mathcal{K}$ is a reproducing kernel almost Pontryagin space of analytic functions of Ω. Moreover, by (II.9),

$$\mathcal{K}^\circ = \Lambda(\mathcal{A}^\circ) = \Lambda\big(\operatorname{span}\{h_0, \ldots, h_{\Delta-1}\}\big) = \operatorname{span}\{1, \ldots, z^{\Delta-1}\}.$$

The required description of $(\Lambda \times \Lambda)(S)$ follows from Remark II.6, (v), and the definition of Λ. ❑

II.*24 Remark.* The isomorphism Λ can in general not be extended to an isomorphism of $\mathcal{P}_{\text{ext}}(\mathcal{A})$ onto a reproducing kernel Pontryagin space of analytic functions, and hence the known Pontryagin space theory is in general not applicable. The reason for this is that, despite the fact that S is minimal, the linear span of $\{\chi(z) : \ z \in \rho(\mathring{A})\}$, need not be dense in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$. ♢

In the next statement, the below Proposition II.26, we show that the space $\mathcal{K}$ can be constructed intrinsically from a Q-function of S. Namely, as a completion. Recall (cf. [Wor14b, §4]):

II.25 Definition. Let Ω be a set and let $\mathcal{L}$ be an inner product space whose elements are functions of Ω into $\mathbb{C}$. We say that $\mathcal{K}$ is a *reproducing kernel almost Pontryagin space completion* of $\mathcal{L}$, if

(i) $\mathcal{K}$ is a reproducing kernel almost Pontryagin space of functions on Ω;

(ii) $\mathcal{K}$ contains $\mathcal{L}$ isometrically as a dense subspace.

$\Diamond$

We saw in [Wor14b, Theorem 4.1] that a reproducing kernel almost Pontryagin space completion of an inner product space is unique if it exists.

II.26 Proposition. *Let $\mathcal{A}$ and S be given according to* II.2*, set $\Delta := \operatorname{ind}_0 \mathcal{A}$, and assume that S is minimal. Let q be a Q-function of S, set $\Omega := \rho(q)$, and denote by $N_q(z,w)$ the Nevanlinna kernel of q, cf.* (II.13)*. Choose pairwise different points $w_1, \dots, w_\Delta \in \Omega$, and let $\lambda_l : \Omega \to \mathbb{C}$, $l = 1, \dots, \Delta$, be the functions defined by*

$$\begin{pmatrix} \lambda_1(w) \\ \vdots \\ \lambda_\Delta(w) \end{pmatrix} := \begin{pmatrix} w_1^0 & \cdots & w_\Delta^0 \\ \vdots & & \vdots \\ w_1^{\Delta-1} & \cdots & w_\Delta^{\Delta-1} \end{pmatrix}^{-1} \begin{pmatrix} w^0 \\ \vdots \\ w^{\Delta-1} \end{pmatrix}, \quad w \in \Omega.$$

For $w \in \Omega$ let F_w denote the function

$$F_w(z) := N_q(z, \overline{w}) - \sum_{j=1}^{\Delta} \lambda_j(w) N_q(z, \overline{w_j}), \quad z \in \Omega,$$

and consider the linear space

$$\mathcal{L} := \operatorname{span}\Big(\big\{ F_w(z) : w \in \Omega \big\} \cup \big\{ 1, z, \dots, z^{\Delta-1} \big\} \Big) \subseteq \mathbb{C}^\Omega.$$

Then the following statements hold.

(i) *There exists a unique inner product $[.,.]_\mathcal{L}$ on $\mathcal{L}$ with*

$$[F_w(z), F_v(z)]_\mathcal{L} = F_w(\overline{v}) - \sum_{i=1}^{\Delta} \overline{\lambda_i(w)} F_w(\overline{w_i}), \quad w, v \in \Omega,$$
$$\langle \mathcal{L}, [.,.]_\mathcal{L} \rangle^\circ = \operatorname{span}\{1, \dots, z^{\Delta-1}\}.$$

(ii) $\operatorname{ind}_- \langle \mathcal{L}, [.,.]_\mathcal{L} \rangle = \operatorname{ind}_-^\Delta q - \Delta$.

(iii) *The reproducing kernel almost Pontryagin space $\mathcal{K}$ constructed in Proposition* II.23 *is the reproducing kernel almost Pontryagin space completion of $\langle \mathcal{L}, [.,.]_\mathcal{L} \rangle$.*

In the proof of this result we use the following simple notice, which we state explicitly for later reference. Its – obvious – proof is dual to the one of [KW99b, Corollary 3].

II.**27 Lemma.** *Let $\mathcal{A}$ be an almost Pontryagin space, set $\Delta := \operatorname{ind}_0 \mathcal{A}$, and let $\{b_0, \dots, b_{\Delta-1}\}$ be a basis of $\mathcal{A}^\circ$. Let $\tau_i \in \mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$, $i = 1, \dots, \Delta$, let $z_i \in \mathbb{C}$, $i = 1, \dots, \Delta$, be pairwise different, and assume that*

$$[\tau_i, b_l] = z_i^l, \quad i = 1, \dots, \Delta,\ l = 0, \dots, \Delta - 1.$$

Then

$$\mathcal{A} \dot{+} \operatorname{span}\{\tau_i : i = 1, \dots, \Delta\} = \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}).$$

Proof. Assume that $\sum_{i=1}^{\Delta} \lambda_i \tau_i \in \mathcal{A}$, then

$$0 = \Big[\sum_{i=1}^{\Delta} \lambda_i \tau_i, b_l \Big] = \sum_{i=1}^{\Delta} \lambda_i [\tau_i, b_l] = \sum_{i=1}^{\Delta} \lambda_i z_i^l, \quad l = 0, \dots, \Delta - 1.$$

Thus $\lambda_1 = \ldots = \lambda_\Delta = 0$. Since $\dim \mathfrak{P}_{\mathrm{ext}}(\mathcal{A})/\mathcal{A} = \Delta$, the assertion follows. ❑

Proof of Proposition II.26. By the previous lemma we have

$$\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}) = \mathcal{A} \dot{+} \operatorname{span}\{\chi(w_1), \dots, \chi(w_\Delta)\}.$$

Let P be the projection of $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$ onto $\mathcal{A}$ with kernel $\operatorname{span}\{\chi(w_1), \dots, \chi(w_\Delta)\}$. Since $\mathcal{A}$ is closed, P is continuous. The action of P can be computed by elementary linear algebra. Define a map $\lambda : \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}) \to \mathbb{C}^\Delta$ as

$$\lambda(x) := \begin{pmatrix} w_1^0 & \cdots & w_\Delta^0 \\ \vdots & & \vdots \\ w_1^{\Delta-1} & \cdots & w_\Delta^{\Delta-1} \end{pmatrix}^{-1} \begin{pmatrix} [x, h_0] \\ \vdots \\ [x, h_{\Delta-1}] \end{pmatrix}, \quad x \in \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}).$$

We claim that

$$Px = x - (\chi(w_1), \dots, \chi(w_\Delta)) \cdot \lambda(x). \tag{II.28}$$

First, since $\ker \lambda = \operatorname{span}\{h_0, \dots, h_{\Delta-1}\}^\perp = \mathcal{A}$, the right side of (II.28) equals x whenever $x \in \mathcal{A}$. Second, $\lambda(\chi(w)) = (\lambda_1(w), \dots, \lambda_\Delta(w))^T$, in particular

$$\lambda(\chi(w_l)) = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th place}}}{1}, 0, \dots, 0)^T, \quad l = 1, \dots, \Delta,$$

and hence the right side of (II.28) vanishes for $x = \chi(w_l)$, $l = 1, \dots, \Delta$. Together indeed (II.28) holds.

Consider the linear space

$$\mathcal{M} := \operatorname{span}\big(\{\chi(w) : w \in \rho(q)\} \cup \{h_0, \dots, h_{\Delta-1}\}\big) \subseteq \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}).$$

By minimality of S, this space is dense in $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$. Since P maps $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$ surjectively and continuously onto $\mathcal{A}$, its image $P(\mathcal{M})$ is dense in $\mathcal{A}$. Density implies that $\operatorname{ind}_- P(\mathcal{M}) = \operatorname{ind}_- \mathcal{A}$, and referring to Proposition II.14 thus

$$\operatorname{ind}_- P(\mathcal{M}) = \operatorname{ind}_-^\Delta q - \Delta. \tag{II.29}$$

Morover, $P(\mathcal{M})^\circ \subseteq \mathcal{A}^\circ$ by density; the reverse inclusion holds since $\mathcal{A}^\circ \subseteq \mathcal{M}$. Let us compute scalar products: for $w, v \in \Omega$ we have

$$\begin{aligned}
[P\chi(w), P\chi(v)] &= \Big[\chi(w) - \sum_{j=1}^{\Delta} \chi(w_j)\lambda_j(w), \chi(v)\Big] \\
&\quad - \Big[\chi(w) - \sum_{j=1}^{\Delta} \chi(w_j)\lambda_j(w), \sum_{i=1}^{\Delta} \chi(w_i)\lambda_i(v)\Big] \\
&= \Big(N_q(w,v) - \sum_{j=1}^{\Delta} \lambda_j(w) N_q(w_j, v)\Big) \\
&\quad - \sum_{i=1}^{\Delta} \overline{\lambda_i(v)} \Big(N_q(w, w_i) - \sum_{j=1}^{\Delta} \lambda_j(w) N_q(w_j, w_i)\Big) \\
&= F_w(\overline{v}) - \sum_{i=1}^{\Delta} \overline{\lambda_i(v)} F_w(\overline{w_i}).
\end{aligned}$$

Now we apply the isomorphism $\Lambda : \mathcal{A} \to \mathcal{K}$. The subspace $\Lambda(P(\mathcal{M}))$ is a dense linear subspace of $\mathcal{K}$, viewing this the other way, $\mathcal{K}$ is the reproducing kernel almost Pontryagin space completion of $\Lambda(P(\mathcal{M}))$ (endowed with the inner product inherited via Λ).

By the definition of Λ, we have $\Lambda(\chi(w))(z) = N_q(w, \overline{z})$, whence

$$\Lambda(P\chi(w))(z) = F_w(z), \quad w \in \Omega.$$

Clearly, $\Lambda(h_l)(z) = z^l$, $l = 0, \ldots, \Delta - 1$, and we conclude that $\Lambda(P(\mathcal{M})) = \mathcal{L}$ and that the inner product inherited via Λ indeed acts as asserted in (i). The equality in (ii) has been seen in (II.29). ❑

PART III

h_0-resolvents

III.1 Definition of h_0-resolvents

The notion of h_0-resolvents is specific for the degenerated situation. It turns out that it is in many ways dual to the notion of Q-functions.

III.**1 Definition.** Let $\mathcal{A}$, S, and h_0, be given according to II.2 and II.5, (Bas). Let Ω be an open subset of $\mathbb{C}$, and r an analytic function defined on Ω. Then r is called an *h_0-resolvent of S*, if there exists a Pontryagin space $\mathcal{P} \supseteq \mathcal{A}$ and selfadjoint relation A in $\mathcal{P}$ with $\rho(A) \neq \emptyset$ and $A \supseteq S$, such that

$$\Omega \supseteq \rho(A) \quad \text{and} \quad r(z) = \big[(A-z)^{-1}h_0, h_0\big], \quad z \in \rho(A). \tag{III.1}$$

We speak of a *minimal* h_0-resolvent, if the selfadjoint relation in this representation can be chosen $\mathcal{A}^\circ$-minimal, and of a *canonical* h_0-resolvent if it can be chosen to act in $\mathfrak{P}_{\text{ext}}(\mathcal{A})$. ◊

For a function f which is analytic on an open set Ω which is dense in $\mathbb{C}$ there exists a largest open subset Ω' of $\mathbb{C}$ such that f has a continuous (equivalently, analytic) extension to Ω'. We shall generically denote this largest subset by $\rho(f)$.

For the definition of h_0-resolvents a similar notice applies as made in Lemma I.2 for the definition of compressed resolvents.

III.*2 Remark.* Let $\mathcal{A}$, S, and h_0, be given according to II.2 and II.5, (Bas). Every selfadjoint relation with nonempty resolvent set in an almost Pontryagin space can be extended to a selfadjoint relation with nonempty resolvent set acting in its canonical Pontryagin space extension. Hence, we could equally well allow the representing selfadjoint relation A in Definition III.1 to act in an almost Pontryagin space.

Things change when discussing minimality aspects. We always can find a representing relation acting in some almost Pontryagin space which is $\mathcal{A}^\circ$-minimal, cf. Lemma I.2, (ii). But it might be possible that we cannot find one which in the same time is $\mathcal{A}^\circ$-minimal and acts in a Pontryagin space. ◊

Next, let us explicitly state two properties of h_0-resolvents which are immediate consequences of their definition.

III.*3 Remark.* Let r be an h_0-resolvent of S. Then

(i) $r(\overline{z}) = \overline{r(z)}$, $z \in \rho(r)$.

(ii) If r vanishes on some open subset of $\rho(r)$, then r vanishes identically.

Item (i) follows since A is selfadjoint. Second, observe that $\rho(r)$ either is connected or has two components, namely, $\rho(r) \cap \mathbb{C}^+$ and $\rho(r) \cap \mathbb{C}^-$. The assertion (ii) now follows using the symmetry property (i) and analyticity of r. ◊

III.2 Index of negativity

In the context of h_0-resolvents a notion of negative index appears which is dual to ind_-^Δ introduced in Definition II.12.

III.**4 Definition.** Let f be a function which is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies $f(\overline{z}) = \overline{f(z)}$, and denote by $\rho(f)$ its domain of holomorphy. Moreover, let $\Delta \in \mathbb{N}$. Then we denote by ${}^\Delta\mathrm{ind}_- f$ the supremum of the numbers of negative squares of quadratic forms (N_f is the Nevanlinna kernel introduced in (II.13))

$$ {}^\Delta Q_f(\xi_1, \dots, \xi_n; \eta_0, \dots, \eta_{\Delta-1}) := \sum_{i,j=1}^n N_f(z_i, z_j)\xi_i\overline{\xi_j} + \sum_{k=0}^{\Delta-1}\sum_{i=1}^n \mathrm{Re}\,\big(z_i^k f(z_i)\xi_i\overline{\eta_k}\big), $$

where $n \in \mathbb{N}_0$ and $z_1, \dots, z_n \in \rho(f)$. ◊

Note the – small but essential – difference between ${}^\Delta\mathrm{ind}_- f$ and ind_-^Δ: the additional factor $f(z_i)$ in the last term.

Again it is clear that ${}^\Delta\mathrm{ind}_- f \geq \Delta$ provided f does not vanish identically. Moreover,

$$ {}^\Delta\mathrm{ind}_- f - \Delta \leq \mathrm{ind}_- f \leq {}^\Delta\mathrm{ind}_- f, $$

in particular, ${}^\Delta\mathrm{ind}_- f < \infty$ if and only if $f \in \mathcal{N}_{<\infty}$.

Duality of ${}^\Delta\mathrm{ind}_-$ and ind_-^Δ becomes manifest in the following relation.

III.**5 Lemma.** *Let f be meromorphic in $\mathbb{C} \setminus \mathbb{R}$, $f(\overline{z}) = \overline{f(z)}$, and assume that f does not vanish identically. Moreover, let $\Delta \in \mathbb{N}$. Then we have*

$$ {}^\Delta\mathrm{ind}_- f = \mathrm{ind}_-^\Delta\Big(-\frac{1}{f}\Big). $$

Proof. Set $\Omega := \rho(f) \cap \rho(f^{-1})$. Using the kernel relation $N_f(z,w) = f(z)N_{-f^{-1}}(z,w)f(\overline{w})$, $z, w \in \Omega$, we obtain

$$ {}^\Delta Q_f\big(\xi_1, \dots, \xi_n; \eta_0, \dots, \eta_{\Delta-1}\big) = Q^\Delta_{-f^{-1}}\big(f(z_1)\xi_1, \dots, f(z_n)\xi_n; \eta_0, \dots, \eta_{\Delta-1}\big). $$

Hence, the numbers of negative squares of the quadratic forms ${}^\Delta Q_f$ and $Q^\Delta_{-f^{-1}}$ where $n \in \mathbb{N}$ and $z_1, \dots, z_n \in \Omega$ are equal.

Since f does not vanish identically, it does not vanish on any open set of its domain of holomorphy. Hence, Ω is dense in both sets $\rho(f)$ and $\rho(-f^{-1})$. Both quadratic forms ${}^\Delta Q_f$ and $Q^\Delta_{-f^{-1}}$ depend continuously on $z_1, \dots, z_n$, and it follows that ${}^\Delta\mathrm{ind}_- f$ and $\mathrm{ind}_-^\Delta(-\frac{1}{f})$ coincide. ❑

The following fact is the dual to Proposition II.14.

III.**6 Proposition.** *Let $\mathcal{A}$, S, and h_0, be given according to* II.2 *and* II.5, (Bas). *Let r be an h_0-resolvent of S, and let A be a selfadjoint relation acting in a Pontryagin space $\mathcal{P}$ which induces r. Then*

$$ {}^\Delta\mathrm{ind}_- r \leq \mathrm{ind}_- \mathcal{P}. $$

If A is $\mathcal{A}^\circ$-minimal, then equality holds.

In the proof we use the following observation.

III.**7 Lemma.** *Let $\mathcal{A}$, S, and h_0, be given according to* II.2 *and* II.5, (Bas). *Moreover, let $\mathcal{P} \supseteq \mathcal{A}$ be a Pontryagin space, and $A \subseteq \mathcal{P}^2$ a selfadjoint extension of S with $\rho(A) \neq \emptyset$. Then A is $\mathcal{A}^\circ$-minimal or $\mathcal{A}$-minimal, respectively, if and only if there exists an open subset Ω of $\rho(A)$ which intersects all connected components of $\rho(A)$, such that*

$$\mathcal{P} = \operatorname{cls}\Big(\mathcal{A}^\circ \cup \big\{(A-z)^{-1}h_0 : z \in \Omega\big\}\Big),$$

or

$$\mathcal{P} = \operatorname{cls}\Big(\mathcal{A} \cup \big\{(A-z)^{-1}h_0 : z \in \Omega\big\}\Big),$$

respectively.

If A is $\mathcal{A}$-minimal, then the h_0-resolvent induced by A does not vanish on any open subset of its domain of analyticity.

Proof. Since A extends S, and S contains the elements (h_l, h_{l+1}), $l = 0, \dots, \Delta - 1$, we have

$$\operatorname{span}\big(\mathcal{A}^\circ \cup (A-z)^{-1}\mathcal{A}^\circ\big) = \operatorname{span}\big(\mathcal{A}^\circ \cup \{(A-z)^{-1}h_0\}\big), \quad z \in \rho(A).$$

Remembering Lemma 2.17, the assertion concerning $\mathcal{A}^\circ$-minimality follows. By Remark II.6, (ii),

$$\operatorname{span}\big(\mathcal{A} \cup (A-z)^{-1}\mathcal{A}\big) = \operatorname{span}\big(\mathcal{A} \cup \{(A-z)^{-1}h_0\}\big), \quad z \in \rho(A) \cap \rho(S_{\text{fac}}),$$

and the assertion on $\mathcal{A}$-minimality follows.

Finally, assume that A is $\mathcal{A}$-minimal, and that r vanishes on some open subset of its domain of analyticity $\rho(r)$. As we have noted in Remark III.3, it then vanishes identically. We obtain $(A-z)^{-1}h_0 \perp h_0$, $z \in \rho(A)$. Since in any case $\mathcal{A} \perp h_0$, this implies that

$$\mathcal{A} \cup \big\{(A-z)^{-1}h_0 : z \in \rho(A)\big\} \subseteq \mathcal{P}[-]\operatorname{span}\{h_0\},$$

which contradicts minimality. ❑

Proof of Proposition III.6. Since A induces r,

$$\begin{aligned}&\big[(A-z)^{-1}h_0, (A-w)^{-1}h_0\big] \\ &\quad = \Big[\frac{(A-z)^{-1} - (A-\overline{w})^{-1}}{z-\overline{w}}h_0, h_0\Big] = \frac{r(z) - \overline{r(w)}}{z - \overline{w}}, \quad z, w \in \rho(A). \end{aligned} \tag{III.2}$$

We claim that

$$\big[(A-z)^{-1}h_0, h_l\big] = r(z)z^l, \quad z \in \rho(A),\ l = 0, \dots, \Delta - 1. \tag{III.3}$$

To see this use induction: The case $l = 0$ is clear. Let $0 < l \leq \Delta - 1$, then

$$\begin{aligned}\big[(A-z)^{-1}h_0, h_l\big] &= \big[(A-z)^{-1}h_0, h_l - \overline{z}h_{l-1}\big] + \big[(A-z)^{-1}h_0, \overline{z}h_{l-1}\big] \\ &= \big[h_0, (A-\overline{z})^{-1}(h_l - \overline{z}h_{l-1})\big] + z\big[(A-z)^{-1}h_0, h_{l-1}\big].\end{aligned}$$

Since $A \supseteq S$, we have $(A-\overline{z})^{-1}(h_l - \overline{z}h_{l-1}) = h_{l-1}$ and thus the first summand vanishes. The relation (III.3) follows.

Using (III.2) and (III.3), we compute

$$\Big[\sum_{i=1}^{n}\xi_i(A-z_i)^{-1}h_0+\sum_{l=0}^{\Delta-1}\eta_l h_l,\sum_{j=1}^{n}\xi_j(A-z_j)^{-1}h_0+\sum_{k=0}^{\Delta-1}\eta_k h_k\Big]=$$

$$=\sum_{i,j=1}^{n}\big[(A-z_i)^{-1}h_0,(A-z_j)^{-1}h_0\big]\xi_i\overline{\xi_j}+\sum_{k=0}^{\Delta-1}\sum_{i=1}^{n}\big[(A-z_i)^{-1}h_0,h_k\big]\xi_i\overline{\eta_k}+$$

$$+\sum_{l=0}^{\Delta-1}\sum_{j=1}^{n}\big[h_l,(A-z_j)^{-1}h_0\big]\eta_l\overline{\xi_j}={}^{\Delta}Q_r\big(\xi_1,\dots,\xi_n;2\eta_1,\dots,2\eta_{\Delta-1}\big).$$

The required inequality of negative indices follows.

If A is $\mathcal{A}^\circ$-minimal, then the linear span of all elements $(A-z)^{-1}h_0$, $z\in\rho(A)$, together with $h_0,\dots,h_{\Delta-1}$ is dense in $\mathcal{P}$. Thus, in this case, negative indices coincide. ❑

III.3 Duality theorem and h_0-resolvent representations

The theorem below shows a striking instance of the duality between Q-functions and h_0-resolvents. The idea for its proof is partially extracted from [KW99a]. This result is highly specific for the degenerated situation, as we will explain in detail in Remark III.14 below, and it is one of the three major ingredients for our proof of the Krein formula given in the forthcoming paper [SW].

III.8 Theorem. *Let $\mathcal{A}$, S, and h_0, be given according to* II.2 *and* II.5, (Bas). *Then the assignment $f\mapsto -f^{-1}$ is a bijection between the sets*

$$\mathcal{Q}:=\Big\{q:\ \substack{q \text{ is } Q\text{-function of } S\\ \text{constructed with } h_0},\ q\not\equiv 0\Big\}$$

and

$$\mathcal{R}:=\Big\{r:\ r \text{ is canonical } h_0\text{-resolvent of } S,\ r\not\equiv 0\Big\}.$$

Since the assignment $q\mapsto -q^{-1}$ is involutory, it is enough to show that "$-\mathcal{Q}^{-1}\subseteq\mathcal{R}$" and "$-\mathcal{R}^{-1}\subseteq\mathcal{Q}$".

Proof of Theorem III.8*; "$-\mathcal{Q}^{-1}\subseteq\mathcal{R}$".* Let q be a Q-function of S constructed with h_0 which does not vanish identically. Denote by $\mathring{A},\chi$ the data according to II.5, (Ext), from which q is defined, and set $r:=-q^{-1}$. We have to construct a selfadjoint extension A of S in $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$ such that $r(z)=[(A-z)^{-1}h_0,h_0]$.

Define elements $\eta(z)$ and a relation $A\subseteq\mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$ by

$$\eta(z):=\frac{-1}{q(z)}\chi(z),\quad z\in\rho(\mathring{A})\cap\rho(-q^{-1}),$$
$$A:=\operatorname{cls}\Big(S\cup\big\{\big(\eta(z),h_0+z\eta(z)\big):\ z\in\rho(\mathring{A})\cap\rho(-q^{-1})\big\}\Big).$$

We show that A is symmetric. First, the relation S is symmetric. Second,

$$[\eta(z),\eta(w)]=\frac{-1}{q(z)}\frac{q(z)-\overline{q(w)}}{z-\overline{w}}\overline{\frac{-1}{q(w)}}=\frac{r(z)-\overline{r(w)}}{z-\overline{w}},$$
$$[\eta(z),h_0]=r(z)[\chi(z),h_0]=r(z),$$

and hence

$$[h_0 + z\eta(z), \eta(w)] - [\eta(z), h_0 + w\eta(w)]$$
$$= \overline{r(w)} + z\frac{r(z) - \overline{r(w)}}{z - \overline{w}} - r(z) - \overline{w}\frac{r(z) - \overline{r(w)}}{z - \overline{w}} = 0,$$

i.e. the relation $\{(\eta(z), h_0 + z\eta(z)) : z \in \rho(\mathring{A}) \cap \rho(-q^{-1})\}$ is symmetric. Finally, for $(a,b) \in S$ and $z \in \rho(\mathring{A}) \cap \rho(-q^{-1})$, we have

$$[b, \eta(z)] - [a, h_0 + z\eta(z)] = \overline{r(z)}\big([b, \chi(z)] - [a, z\chi(z)]\big) = 0.$$

To show that A is actually selfadjoint we employ Lemma 2.12. Let $z \in \rho(\mathring{A}) \cap \rho(-q^{-1})$. By the definition of A we have

$$S \subseteq A, \quad (\eta(z), h_0) \in A - z, \quad (\eta(z) - \eta(w), (z-w)\eta(w)) \in A - z,\ z \neq w.$$

Hence we obtain, for the last equality remember Lemma II.27,

$$\operatorname{ran}(A - z) \supseteq \operatorname{ran}(S - z) + \operatorname{span}\{h_0\} + \operatorname{span}\big\{\eta(w) : w \in \rho(\mathring{A}) \cap \rho(-q^{-1}), w \neq z\big\}$$
$$= \mathcal{A} + \operatorname{span}\big\{\eta(w) : w \in \rho(\mathring{A}) \cap \rho(-q^{-1}), w \neq z\big\} = \mathfrak{P}_{\text{ext}}(\mathcal{A}).$$

We conclude that indeed A is selfadjoint. Moreover, Lemma 2.11 yields $\rho(A) \supseteq \rho(\mathring{A}) \cap \rho(-q^{-1})$.

The fact that A induces r as h_0-resolvent is built in the definition. Since $(\eta(z), h_0) \in A - z$, we have $(A - z)^{-1}h_0 = \eta(z)$, and hence

$$\big[(A - z)^{-1}h_0, h_0\big] = [\eta(z), h_0] = r(z)[\chi(z), h_0] = r(z).$$

❑

Instead of proving just the inclusion "$-\mathcal{R}^{-1} \subseteq \mathcal{Q}$" we show the following more detailed assertion. This refinement is needed in [SW].

III.**9 Proposition.** *Let $\mathcal{A}$, S, and h_0, be given according to* II.2 *and* II.5, (Bas). *Moreover, let $\mathcal{P}$ be a Pontryagin space with $\mathcal{P} \supseteq \mathcal{A}$, and let $A \subseteq \mathcal{P}^2$ be an $\mathcal{A}$-minimal selfadjoint relation in $\mathcal{P}$ with $\rho(A) \neq \emptyset$ and $A \supseteq S$. Finally, set*

$$\tilde{\mathcal{A}} := \mathcal{P}[-]\mathcal{A}^\circ, \quad r(z) := [(A - z)^{-1}h_0, h_0].$$

Then there exists a closed symmetric relation $\tilde{S}$ in $\tilde{\mathcal{A}}$ with defect index $(1,1)$ *which extends S and satisfies the regularity conditions* (II.2)*, such that $-r(z)^{-1}$ is a Q-function of $\tilde{S}$ built with h_0 and some $\tilde{A} \subseteq \mathcal{P}^2$, $\tilde{\chi}(z) \in \mathcal{P}$.*

Proof. Note that, by Lemma III.7, r does not vanish on any open subset of its domain of analyticity $\rho(r)$. We proceed by reversing the argument which led to the inclusion "$-\mathcal{Q}^{-1} \subseteq \mathcal{R}$". Define elements $\tilde{\chi}(z)$ and a relation $\tilde{A}$ in $\mathcal{P}$ by

$$\tilde{\chi}(z) := \frac{1}{r(z)}(A - z)^{-1}h_0, \quad z \in \rho(A) \cap \rho(-r^{-1}), \tag{III.4}$$

$$\tilde{A} := \operatorname{cls}\Big(S \cup \big\{(0, h_0)\big\} \cup \Big\{\big(\tilde{\chi}(z) - \tilde{\chi}(w), z\tilde{\chi}(z) - w\tilde{\chi}(w)\big) : z, w \in \rho(A) \cap \rho(-\frac{1}{r})\Big\}\Big).$$

First, let us compute $[\tilde\chi(z),h_l]$ for $z\in\rho(A)\cap\rho(-r^{-1})$ and $l=0,\dots,\Delta-1$. For $l=0$ we have

$$[\tilde\chi(z),h_0]=\frac{1}{r(z)}[(A-z)^{-1}h_0,h_0]=1.$$

If $l>0$, then $(h_{l-1},h_l)\in S$ and thus

$$[\tilde\chi(z),h_l-\overline{z}h_{l-1}]=\frac{1}{r(z)}\big[(A-z)^{-1}h_0,h_l-\overline{z}h_{l-1}\big]=\frac{1}{r(z)}[h_0,h_{l-1}]=0.$$

Therefore $[\tilde\chi(z),h_l]=z[\tilde\chi(z),h_{l-1}]$, and we obtain inductively that

$$[\tilde\chi(z),h_l]=z^l,\quad z\in\rho(A)\cap\rho(-r^{-1}),\ l=0,\dots,\Delta-1. \tag{III.5}$$

Our second aim is to show that $\tilde A$ is symmetric. First, the relation S is symmetric. Next, we have $(q:=-r^{-1})$

$$\begin{aligned}[\tilde\chi(z),\tilde\chi(w)]&=\frac{1}{r(z)}\big[(A-z)^{-1}h_0,(A-w)^{-1}h_0\big]\frac{1}{\overline{r(w)}}=\\&=q(z)\Big[\frac{(A-z)^{-1}-(A-\overline{w})^{-1}}{z-\overline{w}}h_0,h_0\Big]\overline{q(w)}=\\&=q(z)\frac{r(z)-\overline{r(w)}}{z-\overline{w}}\overline{q(w)}=\frac{q(z)-q(\overline{w})}{z-\overline{w}},\end{aligned} \tag{III.6}$$

and hence the usual computation will show that

$$\operatorname{span}\big\{(\tilde\chi(z)-\tilde\chi(w),z\tilde\chi(z)-w\tilde\chi(w)):\ z,w\in\rho(A)\cap\rho(-r^{-1})\big\}$$

is symmetric. Next, let $(a,b)\in S$ and $z\in\rho(A)\cap\rho(-r^{-1})$. Then

$$[\tilde\chi(z),b]-[z\tilde\chi(z),a]=[\tilde\chi(z),b-\overline{z}a]=\frac{1}{r(z)}\big[(A-z)^{-1}h_0,b-\overline{z}a\big]=\frac{1}{r(z)}[h_0,a]=0.$$

Finally, since $[\tilde\chi(z),h_0]=1$, in particular, $h_0\perp\operatorname{dom}\tilde A$. Altogether, we see that $\tilde A$ is symmetric.

Third, we need to establish that $\tilde A$ is in fact selfadjoint. By the definition of $\tilde A$ we have

$$\operatorname{ran}(\tilde A-z)\supseteq\operatorname{ran}(S-z)\cup\{h_0\}\cup\big\{\tilde\chi(w):\ w\in\rho(A)\cap\rho(-r^{-1}),w\neq z\big\}$$

whenever $z\in\rho(A)\cap\rho(-r^{-1})$. If $w\in\rho(S_{\text{fac}})\cap\rho(A)\cap\rho(-r^{-1})$, we have

$$\operatorname{ran}(S-w)+\operatorname{span}\{h_0\}=\mathcal{A},$$

$$(A-w)^{-1}\mathcal{A}=(A-w)^{-1}\operatorname{ran}(S-w)+\operatorname{span}\big\{(A-w)^{-1}h_0\big\}\subseteq\mathcal{A}+\operatorname{span}\{\tilde\chi(w)\}.$$

Hence,

$$\operatorname{ran}(\tilde A-z)\supseteq\mathcal{A}\quad\cup\bigcup_{\substack{w\in\rho(S_{\text{fac}})\cap\rho(A)\cap\\ \cap\rho(-r^{-1}),w\neq z}}(A-w)^{-1}\mathcal{A},\quad z\in\rho(S_{\text{fac}})\cap\rho(A)\cap\rho(-r^{-1}).$$

Since A is $\mathcal{A}$-minimal, the linear span of the right hand side is dense in $\mathcal{P}$. Lemma 2.12 yields that $\tilde A$ is selfadjoint and that

$$\rho(\tilde A)\supseteq\big(\rho(S_{\text{fac}})\cap\rho(A)\cap\rho(-r^{-1})\big)\setminus\mathbb{R}.$$

Now we are ready to invoke Lemma II.16, namely with the data

$$\mathcal{P}, \quad \tilde{\mathcal{A}}, \quad \rho(A)\cap\rho(-r^{-1}), \quad \tilde{\chi}(z), \quad \Delta, \quad h_0,\dots,h_{\Delta-1}.$$

This provides us with a symmetry, call it $\tilde{S}$, which acts in the almost Pontryagin space $\tilde{\mathcal{A}}$. Moreover, all necessary properties are satisfied in order that Q-functions of $\tilde{S}$ can be defined using $h_0, \tilde{A}, \tilde{\chi}$. The computation (III.6) shows that the function $q = -r^{-1}$ is one such.

In order to finish the proof, it only remains to show that $\tilde{S}$ extends S. However, if $(x,y)\in S$, then $(x,y)\in \tilde{A}\cap\mathcal{A}^2$ and

$$[\tilde{\chi}(z), y-\overline{z}x] = \frac{1}{r(z)}\big[(A-z)^{-1}h_0, y-\overline{z}x\big] = \frac{1}{r(z)}[h_0,x] = 0, \quad z\in\rho(A)\cap\rho(-r^{-1}).$$

❑

Theorem III.8*; "$-\mathcal{R}^{-1}\subseteq\mathcal{Q}$".* Let r be a canonical h_0-resolvent, $r\not\equiv 0$, and let A be a selfadjoint extension of S in $\mathfrak{P}_{\rm ext}(\mathcal{A})$ which induces r. The elements $\tilde{\chi}(z)$, $z\in\rho(A)\cap\rho(-r^{-1})$, defined by (III.4) satisfy (III.5). Lemma II.27 thus gives the last equality in

$$\mathfrak{P}_{\rm ext}(\mathcal{A}) \supseteq \operatorname{cls}\Big(\mathcal{A}\cup\big\{(A-z)^{-1}h_0 : z\in\rho(A)\cap\rho(-r^{-1})\big\}\Big) \supseteq$$
$$\supseteq \mathcal{A} + \operatorname{span}\big\{\tilde{\chi}(z) : z\in\rho(A)\cap\rho(-r^{-1})\big\} = \mathfrak{P}_{\rm ext}(\mathcal{A}),$$

and we see that A is $\mathcal{A}$-minimal. Hence, an application of Proposition III.9 with $\mathfrak{P}_{\rm ext}(\mathcal{A})$ and A is justified. Since we work in the canonical Pontryagin space extension, $\tilde{\mathcal{A}} = \mathcal{A}$. Since S and $\tilde{S}$ both have defect index $(1,1)$, we have $\tilde{S} = S$. ❑

Let us immediately exploit Theorem III.8.

III.**10 Corollary.** *Let $\mathcal{A}$ and S be given according to* II.2*. Then the following are equivalent.*

(*i*) *S is minimal.*

(*ii*) *Every canonical h_0-resolvent which does not vanish identically is minimal.*

(*iii*) *There exists a minimal canonical h_0-resolvent of S.*

Proof. We start with one general observation. Let A be a selfadjoint extension of S in $\mathfrak{P}_{\rm ext}(\mathcal{A})$, $\rho(A)\neq\emptyset$, and denote $r(z) := [(A-z)^{-1}h_0, h_0]$. Then, certainly, $(A-z)^{-1}h_0\notin\mathcal{A}$ whenever $z\in\rho(A)\cap\rho(-r^{-1})$. Since $(A-z)^{-1}h_0 \perp \operatorname{ran}(S-\overline{z})$, we obtain

$$\operatorname{ran}(S-\overline{z})^{\perp} = \mathcal{A}^{\circ} + \operatorname{span}\big\{(A-z)^{-1}h_0\big\}, \quad z\in\rho(A)\cap\rho(-r^{-1}).$$

If r does not vanish identically, the set $\Omega := \rho(A)\cap\rho(-r^{-1})$ is an open subset of $\rho(A)$ which intersects all components of $\rho(A)$ and possesses the same property for $\gamma_s(S)$. Thus, assuming $r\not\equiv 0$,

$$\Big(\bigcap_{z\in\gamma(S)}\operatorname{ran}(S-z)\Big)^{\perp} = \Big(\bigcap_{z\in\Omega}\operatorname{ran}(S-z)\Big)^{\perp} =$$
$$= \operatorname{cls}\Big(\mathcal{A}^{\circ}\cup\big\{(A-z)^{-1}h_0 : z\in\Omega\big\}\Big) = \operatorname{cls}\Big(\mathcal{A}^{\circ}\cup\big\{(A-z)^{-1}h_0 : z\in\rho(A)\big\}\Big).$$

Thus, S is minimal if and only if A is $\mathcal{A}^\circ$-minimal

The implication "$(i) \Rightarrow (ii)$" is now clear. Remembering that minimal h_0-resolvent cannot vanish identically, also "$(iii) \Rightarrow (i)$" follows. For "$(ii) \Rightarrow (iii)$", remember that there always exist Q-functions of S which do not vanish identically, cf. Remark II.9, (iv). By the already proved inclusion "$-\mathcal{Q}^{-1} \subseteq \mathcal{R}$" of Theorem III.8 thus also nonvanishing h_0-resolvents always exist. ❑

As another consequence, we obtain a representation theorem dual to Theorem II.15.

III.**11 Corollary.** *Let $r \in \mathcal{N}_{<\infty} \setminus \{0\}$ and $\Delta \in \mathbb{N}$ be given. Then r is a minimal canonical h_0-resolvent of the symmetric relation $S_\Delta(-r^{-1})$ in the almost Pontryagin space $\mathcal{A}_\Delta(-r^{-1})$.*

Proof. The function $-r^{-1}$ belongs to the class $\mathcal{N}_{<\infty}$. By Theorem II.15 it is a Q-function of the symmetry $S_\Delta(-r^{-1})$ in the almost Pontryagin space $\mathcal{A}_\Delta(-r^{-1})$. By Theorem III.8, therefore, r is a canonical h_0-resolvent of $S_\Delta(-r^{-1})$. Since $S_\Delta(-r^{-1})$ is minimal, r is a minimal h_0-resolvent. ❑

III.*12 Remark.* Existence of h_0-resolvent representations of a given function $r \in \mathcal{N}_{<\infty}$ was in essence established already in [KW99a, Theorem 4.2]. With little additional effort, one could deduce from this earlier result the following less specific version of Corollary III.11: *Let $r \in \mathcal{N}_{<\infty} \setminus \{0\}$ and $\Delta \in \mathbb{N}$ be given. Then there exists some almost Pontryagin space $\mathcal{A}$ with* $\operatorname{ind}_0 \mathcal{A} = \Delta$ *and some closed symmetric relation S in $\mathcal{A}$ with defect index* $(1,1)$ *which satisfes the regularity conditions* (II.2)*, such that r is a minimal canoncial h_0-resolvent of S.* However, it turns out important to explictly know the space and relation which gives such representation. ◊

The following observation shows that, though in a certain duality, Q-functions and h_0-resolvents are of intrinsically different nature.

III.*13 Remark.* Assume that $\mathcal{A}$, S, and h_0 are given according to II.2 and II.5, (Bas). Those selfadjoint extensions $\mathring{A}$ which give rise to Q-functions of S are transversal to those selfadjoint extensions A which give rise to nontrivial canoncial h_0-resolvents. For if $\mathring{A}$ satisfies II.5, (Ext), then certainly $[(\mathring{A} - z)^{-1}h_0, h_0] = 0$. Moreover,

$$\operatorname{cls}\Big(\mathcal{A}^\circ \cup \bigcup_{z\in\rho(\mathring{A})} (\mathring{A}-z)^{-1}\mathcal{A}^\circ\Big) = \mathcal{A}^\circ, \quad \operatorname{cls}\Big(\mathcal{A} \cup \bigcup_{z\in\rho(\mathring{A})} (\mathring{A}-z)^{-1}\mathcal{A}\Big) = \mathcal{A},$$

i.e., these closed linear spans are as small as they can possibly be. On the other hand, if A induces a canonical h_0-resolvent which does not vanish identically, then certainly

$$\operatorname{cls}\Big(\mathcal{A}^\circ \cup \bigcup_{z\in\rho(\mathring{A})} (A-z)^{-1}\mathcal{A}^\circ\Big) \nsubseteq \mathcal{A}.$$

Assuming that S is minimal, actually

$$\operatorname{cls}\Big(\mathcal{A}^\circ \cup \bigcup_{z\in\rho(\mathring{A})} (A-z)^{-1}\mathcal{A}^\circ\Big) = \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}),$$

i.e., this closed linear span is as large as it possily can be. ◊

Let us now explain that Theorem III.8 indeed expresses a phenomenon which is highly specific for the degenerated situation.

III.*14 Remark.*

№1. A result from 'nondegenerated theory':
Let $q \in \mathcal{N}_{<\infty}$ be given. Then also the function $-q^{-1}$ belongs to $\mathcal{N}_{<\infty}$, actually $\operatorname{ind}_-(-q^{-1}) = \operatorname{ind}_- q$. There exist Pontryagin spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ and minimal symmetries $S_1 \subseteq \mathcal{P}_1^2$ and $S_2 \subseteq \mathcal{P}_2^2$, such that q is a Q-function of S_1 and $-q^{-1}$ is a Q-function of S_2. Actually, and this is the point we wish to make, one can choose $\mathcal{P}_1 = \mathcal{P}_2$ and $S_1 = S_2$.

№2. An observation for 'degenerated theory': Let besides $q \in \mathcal{N}_{<\infty}$ a number $\Delta \in \mathbb{N}$ be given. There exist almost Pontryagin spaces $\mathcal{A}_1$ and $\mathcal{A}_2$, $\operatorname{ind}_0 \mathcal{A}_1 = \operatorname{ind}_0 \mathcal{A}_2 = \Delta$, and minimal symmetries $S_1 \subseteq \mathcal{A}_1^2$, $S_2 \subseteq \mathcal{A}_2^2$, such that q is a Q-function of S_1 and $-q^{-1}$ is a Q-function of S_2. However, we will in general not have the slightest chance to choose $\mathcal{A}_1 = \mathcal{A}_2$ and $S_1 = S_2$, since already equality of negative indices fails: In general

$$\operatorname{ind}_- \mathcal{A}_2 = \operatorname{ind}_-^{\Delta}(-q^{-1}) \quad \text{will not be equal to} \quad \operatorname{ind}_-^{\Delta} q = \operatorname{ind}_- \mathcal{A}_1 .$$

This shows that an 'exact degenerate analogue' of *№1* cannot hold true.

№3. The significance of Theorem III.8*:* We end up with the question what a proper degenerate analogue of *№1* could be. The relation

$${}^{\Delta}\operatorname{ind}_-(-q^{-1}) = \operatorname{ind}_-^{\Delta} q$$

suggests that using h_0-resolvents of S_2 instead of Q-functions, there might be a chance to achieve $\mathcal{A}_1 = \mathcal{A}_2$ and $S_1 = S_2$. And, as we just saw in Corollary III.11, this indeed works out.

№4. Observation for 'nondegenerated theory': The question appears whether there is an 'exact nondegenerate analogue' of Theorem III.8. The answer is no. For example, consider a function $q \in \mathcal{N}_{<\infty}$ with

$$\lim_{y\to+\infty} \frac{1}{y} q(iy) = 0, \quad \limsup_{y\to+\infty} y|\operatorname{Im} q(iy)| = +\infty. \tag{III.7}$$

Let $\mathcal{P}$ and $S \subseteq \mathcal{P}^2$ be a Pontryagin space and a minimal symmetry such that q is a Q-function of S. Then S is densely defined, and hence every selfadjoint extension $A \subseteq \mathcal{P}^2$ of S is an operator. Thus also the function $-q^{-1}$ will possess the corresponding asymptotics (III.7). However, if $A \subseteq \mathcal{P}^2$ is a selfadjoint extension of S and $u \in \mathcal{P}$, then the function $r(z) := [(A-z)^{-1}u, u]$ will satisfy $\lim_{y\to+\infty} yr(iy) = i[u,u]$. We see that $-q^{-1}$ certainly cannot be represented as a u-resolvent of S. $\Diamond$

III.4 More on minimality

Putting together Theorem III.8 with Proposition II.19, we obtain that one can often assume that S is minimal.

III.**15 Corollary.** *Let $\mathcal{A}$ and S be given according to* II.2*, and let $\mathcal{A}_1$ and S_1 be as in Proposition* II.19*. Then the set of all canonical h_0-resolvents of S_1 is equal to the set of all canonical h_0-resolvents of S.*

Proof. By Theorem III.8 and Proposition II.19 the sets of all those h_0-resolvents of S and S_1, respectively, which do not vanish identically, coincide. However, the zero function is always a h_0-resolvent since $[(A-z)^{-1}h_0, h_0] = 0$ if A is chosen according to II.5, (Ext). ❑

Let us next show that a symmetry S is, up to isomorphisms, uniquely determined by each of its minimal h_0-resolvents. This is the statement dual to Proposition II.20.

III.**16 Proposition.** *Let $\mathcal{A}_1, S_1$ and $\mathcal{A}_2, S_2$ be given according to* II.2. *For $j \in \{1,2\}$ let h_0^j be an element as is* II.5, (Bas), *for $\mathcal{A}_j, S_j$, and let $A_j \subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A}_j)^2$ be an $\mathcal{A}^\circ$-minimal selfadjoint extension of S_j. Moreover, denote by r_j the h_0^j-resolvent of S_j induced by A_j.*

If $r_1 = r_2$ and $\operatorname{ind}_0 \mathcal{A}_1 = \operatorname{ind}_0 \mathcal{A}_2$, then there exists an isometric isomorphism Φ of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$ onto $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$ with

$$\begin{aligned}
&\Phi(\mathcal{A}_1) = \mathcal{A}_2,\\
&(\Phi\times\Phi)(S_1) = S_2, \quad (\Phi\times\Phi)(A_1) = A_2,\\
&\Phi(h_l^1) = h_l^2, \quad l = 0,\dots,\Delta-1.
\end{aligned}$$

In particular $\rho(A_1) = \rho(A_2)$.

Proof. We show that a linear map

$$\Phi : \mathcal{A}_1^\circ \dot{+} \operatorname{span}\big\{(A_1-z)^{-1}h_0^1 : z\in\rho(A_1)\cap\rho(A_2)\big\} \longrightarrow \mathcal{A}_2^\circ \dot{+} \operatorname{span}\big\{(A_2-z)^{-1}h_0^2 : z\in\rho(A_1)\cap\rho(A_2)\big\}$$

is well-defined by the requirements that

$$\Phi(h_l^1) = h_l^2, \quad l = 0,\dots,\Delta-1,$$

$$\Phi((A_1-z)^{-1}h_0^1) = (A_2-z)^{-1}h_0^2, \quad z\in\rho(A_1)\cap\rho(A_2),$$

where $\Delta := \dim \mathcal{A}_1^\circ = \dim \mathcal{A}_2^\circ$. Assume that $\lambda_j, \nu_k \in \mathbb{C}$ and

$$\sum_{k=0}^{\Delta-1} \nu_k h_k^1 + \sum_j \lambda_j (A_1 - w_j)^{-1} h_0^1 = 0.$$

Using (III.2) and (III.3), we obtain

$$0 = \Big[\sum_{k=0}^{\Delta-1} \nu_k h_k^1 + \sum_j \lambda_j (A_1-w_j)^{-1}h_0^1, h_l\Big] = \sum_j \lambda_j r_1(w_j) w_j^l, \quad l = 0,\dots,\Delta,$$

$$\Big[\sum_{k=0}^{\Delta-1} \nu_k h_k^1 + \sum_j \lambda_j (A_1-w_j)^{-1}h_0^1, (A_1-z)^{-1}\Big] = \\
= \sum_{k=0}^{\Delta-1} \nu_k \overline{r_1(z)}\overline{z}^l + \sum_j \lambda_j \frac{r_1(w_j) - \overline{r_1(z)}}{w_j - \overline{z}} = 0, \quad z\in\rho(A_1).$$

From our assumption that $r_1 = r_2$ we obtain

$$0 = \Big[\sum_{k=0}^{\Delta-1} \nu_k h_k^2 + \sum_j \lambda_j (A_2 - w_j)^{-1} h_0^2, h_l^2 \Big] = \sum_j \lambda_j r_2(w_j) w_j^l, \quad l = 0, \ldots, \Delta - 1,$$

$$\Big[\sum_{k=0}^{\Delta-1} \nu_k h_k^2 + \sum_j \lambda_j (A_2 - w_j)^{-1} h_0^2, (A_2 - z)^{-1} h_0^2 \Big] =$$
$$= \sum_{k=0}^{\Delta-1} \nu_k \overline{r_2(z)} \overline{z}^l + \sum_j \lambda_j \frac{r_2(w_j) - \overline{r_2(z)}}{w_j - \overline{z}} = 0, \quad z \in \rho(A_1) \cap \rho(A_2).$$

Since A_2 is $\mathcal{A}^\circ$-minimal, this implies that

$$\sum_{k=0}^{\Delta-1} \nu_k h_k^2 + \sum_j \lambda_j (A_2 - w_j)^{-1} h_0^2 = 0.$$

The fact that Φ is isometric is immediate from (III.2) and (III.3). By $\mathcal{A}^\circ$-minimality of A_1 and A_2, Φ has dense domain and range and thus extends to an isometric isomorphism between $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$ and $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$. We denote this extension again by Φ. Since by definition $\Phi(\mathcal{A}_1^\circ) = \mathcal{A}_2^\circ$, we also also have $\Phi(\mathcal{A}_1) = \mathcal{A}_2$.

Fix $w \in \rho(A_1) \cap \rho(A_2)$, and compute

$$(A_2 - w)^{-1} \Phi (A_1 - z)^{-1} h_0^1 = (A_2 - w)^{-1} (A_2 - z)^{-1} h_0^2 =$$
$$= \frac{(A_2 - w)^{-1} h_0^2 - (A_2 - z)^{-1} h_0^2}{w - z} = \Phi\Big(\frac{(A_1 - w)^{-1} h_0^1 - (A_1 - z)^{-1} h_0^1}{w - z} \Big) =$$
$$= \Phi\Big((A_1 - w)^{-1} (A_1 - z)^{-1} h_0^1 \Big), \quad z \in \rho(A_1) \cap \rho(A_2), z \neq w,$$

$$(A_2 - w)^{-1} \Phi (h_l^1 - w h_{l-1}^1) = (A_2 - w)^{-1} (h_l^2 - w h_{l-1}^2) = h_{l-1}^2 =$$
$$= \Phi(h_{l-1}^1) = \Phi\Big((A_1 - w)^{-1} (h_l^1 - w h_{l-1}^1) \Big), \quad l = 1, \ldots, \Delta - 1,$$

$$(A_2 - w)^{-1} \Phi h_0^1 = (A_2 - w)^{-1} h_0^2 = 0 = \Phi\Big((A_1 - w)^{-1} h_0^1 \Big).$$

It follows that $\Phi \circ (A_1 - w)^{-1} = (A_2 - w)^{-1} \circ \Phi$, and hence $\Phi \circ A_1 = A_2 \circ \Phi$ and $\rho(A_1) = \rho(A_2)$. Since

$$S_j := \big\{ (x, y) \in A_j \cap \mathcal{A}_j^2 : \ \forall z \in \rho(A_j) : y - \overline{z} x \perp (A_j - z)^{-1} h_0^j \big\},$$

we also obtain $\Phi \circ S_1 = S_2 \circ \Phi$. ❑

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H.S.V. de Snoo
Johann Bernoulli Institute for Mathematics and Computer Science
University of Groningen
PO Box 407, 9700 AK Groningen
NEDERLAND
email: hsvdesnoo@gmail.com

H. Woracek
Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10/101
1040 Wien
AUSTRIA
email: harald.woracek@tuwien.ac.at