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De Branges Spaces and Growth Aspects

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Abstract

The subject of this survey is to review the basics of Louis de Branges' theory of Hilbert spaces of entire functions, and to present results bringing together the notions of de Branges spaces on one hand and growth functions (proximate orders) on the other hand.

After a few introductory words, the paper starts off with a short companion on de Branges theory (section "A Short Companion on Hilbert Spaces of Entire Functions") where much of the terminology and cornerstones of the theory are presented. Then growth functions are very briefly introduced (section "Growth Functions"). The following two sections of the survey are devoted to growth properties. First (section "General Theorems Relating De Branges Spaces and Growth"), some general theorems, where the growth of elements of a de Branges space is discussed in relation with generating Hermite–Biehler functions and associated canonical systems, and results on growth of subspaces of a given space are presented. Second (section "Some Examples"), some more concrete examples which appear "in nature," and where growth of different rates is exhibited.

It should be said explicitly that this survey is of course far from being exhaustive. For example, since the main purpose is to study growth properties of spaces of entire functions, all what relates to spectral measures (inclusion in L^2-spaces, etc.) is omitted from the presentation.

Introduction

In the late 1950s Louis de Branges founded a theory of *Hilbert spaces of entire functions*, which was thought of as a generalization of Fourier analysis, cf. [8, 9]. In the following years he further developed his theory in the series of papers [10–13]. Comprehensive information can be found in the book [14].

This deep theory has proven to be of relevance in various contexts. As prominent examples, let us mention the spectral theory of canonical systems, Schrödinger operators, and Krein strings, where direct and inverse spectral problems can be solved using de Branges' spaces of entire functions. Other, equally intriguing applications are found in interpolation and sampling, Beurling–Malliavin type theorems, or approximation problems. Due to this variety of aspects in which de Branges' theory can be applied successfully, it has stayed an active area of research up to the day.

The elements of a de Branges space are entire functions, and hence growth properties like order and type are intrinsically connected with the notion of a de Branges space. The interplay of de Branges space structure and the classical theory of growth gives rise to some intriguing questions – quite some of them being open problems – and to beautiful theorems.

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A Short Companion on Hilbert Spaces of Entire Functions

Axiomatics of De Branges Spaces

Definition 1. A *de Branges space* is a Hilbert space $\mathcal{H}$ which satisfies the following axioms.

(dB1) The elements of $\mathcal{H}$ are entire functions and for each $w \in \mathbb{C} \setminus \mathbb{R}$ the point evaluation functional $F \mapsto F(w)$, $F \in \mathcal{H}$, is continuous in the norm $\|.\|_{\mathcal{H}}$ of $\mathcal{H}$.

(dB2) For each $F \in \mathcal{H}$, also the function $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{H}$ and $\|F^{\#}\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}$.

(dB3) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$ with $F(w) = 0$, then

$$\frac{z-\overline{w}}{z-w}F(z) \in \mathcal{H} \quad \text{and} \quad \left\|\frac{z-\overline{w}}{z-w}F(z)\right\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}.$$

Throughout this paper the notion of a de Branges space shall additionally include the following requirement.

(dB4) For each $t \in \mathbb{R}$ there exists $F \in \mathcal{H}$ with $F(t) \neq 0$.

◇

It follows from these axioms that the properties stated in (dB1) holds also for all real points w. Moreover, using (dB4), one gets that for $w \in \mathbb{R}$ and $F \in \mathcal{H}$ with $F(w) = 0$ also $\frac{1}{z-w}F(z) \in \mathcal{H}$.

Proof. This requires an analyticity argument using that all elements of $\mathcal{H}$ are analytic across $\mathbb{R}$; see [14, Theorem 20], [14, Problem 45].

From a viewpoint more oriented toward operator theory, the axiom (dB1) says that $\mathcal{H}$ is a reproducing kernel Hilbert space of entire functions, and (dB2) says that the map $.^{\#}$ induces an isometric involution on $\mathcal{H}$. The axiom (dB3) also has a neat operator theoretic interpretation; this will be presented in Remark 2. More on the operator theoretic side will be presented in the next section. Much of the operator theoretic aspects of the theory can also be developed in a similar fashion when (dB2) is dropped. For details see, e.g., [19, 20, 46, 51].

Remark 1. Notation in the literature is not uniform. Sometimes the axiom (dB4) is weakened to requiring that $\mathcal{H}$ contains a function which does not vanish identically, sometimes it is dropped completely. In most respects neither of these modifications is a gain in generality. First, admitting the trivial space $\mathcal{H} = \{0\}$ is a matter of taste; sometimes this leads to formal simplifications, sometimes it does not. Second, each space $\mathcal{H} \neq \{0\}$ satisfying (dB1)–(dB3) can be identified isometrically with a space satisfying (dB1)–(dB4). ◇

Proof. For details see [14, Problem 44], or the slightly more general [30, Lemma 2.4].

Prominent examples of de Branges spaces are the *Paley–Wiener spaces*. Citing de Branges himself, these spaces were the starting point and motivating example for developing the theory.

Example 1. For $a > 0$ denote by $\mathcal{PW}_a$ the set of all entire functions of exponential type at most a whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. If $\mathcal{PW}_a$ is endowed with the norm $\|F\|_{\mathcal{PW}_a} := \|F|_{\mathbb{R}}\|_{L^2(\mathbb{R})}$, then it becomes a de Branges space.

Terminology is explained by the fact that, by the Paley–Wiener theorem, $\mathcal{PW}_a$ is the set of all Fourier transforms of L^2-functions supported on the interval $[-a, a]$. ◇

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Another prominent class of examples appears in the context of *power moment problems*. Namely, the class of finite-dimensional spaces whose elements are polynomials.

Example 2. Let $n \in \mathbb{N}$, and let μ be a positive Borel measure on the real line which possesses power moments (at least) up to order $2n$. Denote by $\mathbb{C}[z]_n$ the space of all polynomials with complex coefficients whose degree does not exceed n. If $\mathbb{C}[z]_n$ is endowed with the inner product $(f, g) = \int_{\mathbb{R}} f(t)\overline{g(t)}\, d\mu(t)$, then it becomes a de Branges space. ◇

The Multiplication Operator

Let $\mathcal{H}$ be a de Branges space. Defined by its graph (throughout the interpretation of $S(\mathcal{H})$ as graph or as operator is interchangeably used), the *multiplication operator in* $\mathcal{H}$ is

$$S(\mathcal{H}) := \big\{(F(z); zF(z)) : F(z), zF(z) \in \mathcal{H}\big\}.$$

This operator is inextricably linked with the very basics of de Branges spaces. The following statement collects essential properties of $S(\mathcal{H})$.

Theorem 1. *Let $\mathcal{H}$ be a de Branges space. Then $S(\mathcal{H})$ is closed, symmetric, completely non-selfadjoint, and real with respect to the involution $.^{\#}$. Its set of points of regular type equals $\mathbb{C}$, and its deficiency index equals $(1, 1)$. The domain of $S(\mathcal{H})$ is not necessarily dense in $\mathcal{H}$, but always satisfies*

$$\dim \Big(\mathcal{H}\Big/\overline{\operatorname{dom} S(\mathcal{H})}\Big) \leq 1.$$

If $\overline{\operatorname{dom} S(\mathcal{H})}$ is endowed with the inner product inherited from $\mathcal{H}$, then it becomes a de Branges space.

Proof. This is, partially implicitly, in [14]; an explicit formulation (within the more general Pontryagin space setting) can be found in [29, Proposition 4.2, Corollaries 4.3, 4.7].

Remark 2. The axiom (dB3) in Definition 1 can be substituted by the following pair of requirements.

(dB3') For each $w \in \mathbb{C} \setminus \mathbb{R}$ there exists $F \in \mathcal{H}$ with $F(w) \neq 0$.
(dB3") The operator $S(\mathcal{H})$ is closed, symmetric, and has deficiency index $(1, 1)$.

◇

Proof. This is, e.g., in [46].

The extensions of $S(\mathcal{H})$ can be described in terms of their resolvents by entire functions which are in a simple way associated with the space $\mathcal{H}(E)$. Namely, for a de Branges space the set $\operatorname{Assoc}\mathcal{H}$ of *associated functions* is defined as

$$\operatorname{Assoc}\mathcal{H} := \mathcal{H} + z\mathcal{H} = \big\{F(z) + zG(z) : F, G \in \mathcal{H}\big\}.$$

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Proposition 1. *Let $\mathcal{H}$ be a de Branges space. Then the set of all functions S associated with $\mathcal{H}$ corresponds bijectively to the set of all those linear relations T in $\mathcal{H}$ which have nonempty resolvent set and extend $S(\mathcal{H})$. This correspondence is established by the formula*

$$(T-w)^{-1}F(z) = \frac{F(z) - \frac{S(z)}{S(w)}F(w)}{z-w}, \quad F \in \mathcal{H}. \tag{1}$$

Thereby, the spectrum of T coincides with the zeroes of the function S and T has nontrivial multivalued part if and only if $S \in \mathcal{H}$.

Proof. This is implicit in [14]; an explicit formulation (within the more general Pontryagin space setting) can be found in [29, Proposition 4.6].

Example 3. It is a consequence of the theorem of Paley–Wiener and the properties of the Fourier transform, that

$$\overline{\operatorname{dom} S(\mathcal{P}W_a)} = \mathcal{P}W_a.$$

Concerning the finite dimensional space $\mathbb{C}[z]_n$, it obviously holds that

$$\overline{\operatorname{dom} S(\mathbb{C}[z]_n)} = \operatorname{dom} S(\mathbb{C}[z]_n) = \mathbb{C}[z]_{n-1},$$

and hence

$$\dim\left(\mathbb{C}[z]_n \Big/ \overline{\operatorname{dom} S(\mathbb{C}[z]_n)}\right) = 1.$$

◇

The Hermite–Biehler Class

An approach alternative to the one taken in Definition 1 proceeds via the fact that each de Branges space can be generated by a single entire function. The class of entire functions appearing as generators of de Branges spaces is the following.

Definition 2. A *Hermite–Biehler function* is an entire function E which satisfies the following axioms.

(HB1) The function E has no zeroes in the open upper half-plane $\mathbb{C}^+$ ($:= \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$).
(HB2) It holds that

$$|E(\overline{z})| < |E(z)|, \quad z \in \mathbb{C}^+. \tag{2}$$

Throughout this paper the notion of a Hermite–Biehler function shall additionally include the following requirements.

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(HB3) E has no real zeroes.
(HB4) $E(0) = 1$.

The totality of all Hermite–Biehler functions is denoted as $\mathcal{H}B$. ◇

Remark 3. Again, notation in the literature is not uniform.

First, sometimes equality in (2) is permitted. Requiring strict inequality only rules out the case that E is a scalar multiple of some real entire function (an entire function F is called *real*, if $F^\# = F$. Equivalently, F could be required to assume real values along the real axis).

Second, sometimes E is allowed to have real zeroes and often the normalization condition (HB4) is not included. Requiring (HB3) and (HB4) are usually no restriction of generality. If E is an entire function subject to (HB1) and (HB2), denote by C a canonical product having the same real zeroes as E (including multiplicities) and no zeroes otherwise, and denote by $\gamma \in \mathbb{C} \setminus \{0\}$ the value of the quotient $\frac{E}{C}$ at 0. Then the function $\tilde{E} := \frac{1}{\gamma}\frac{E}{C}$ satisfies (HB1)–(HB4).

Finally, concerning terminology, sometimes one speaks of *de Branges functions* instead of Hermite–Biehler functions. ◇

Especially in connection with growth properties, it is important to observe that the Weierstraß or Hadamard product representation of a Hermite–Biehler function can be rewritten in a particular way.

Theorem 2. *Let $E \in \mathcal{H}B$ and denote by $(w_n)_n$ the (finite or infinite) sequence of its zeros listed according to their multiplicities. Then the Blaschke condition*

$$\sum_n \left| \operatorname{Im} \frac{1}{w_n} \right| < \infty \tag{3}$$

holds true. The function E admits a locally uniformly convergent product representation of the form

$$E(z) = \gamma e^{C(z)} e^{-iaz} \prod_n \left(1 - \frac{z}{w_n}\right) \exp\left(\sum_{k=1}^{p_n} \frac{z^k}{k} \operatorname{Re} \frac{1}{w_n^k}\right), \tag{4}$$

where

$$C = C^\#,\ C(0) = 0, \qquad a \geq 0 \tag{5}$$

and the sequence $(p_n)_n$ satisfies

$$\sum_n \frac{1}{|w_n|^{p_n+1}} < \infty. \tag{6}$$

Conversely, if $(w_n)_n$, $(p_n)_n$, C, and a are such that (3), (5), and (6) hold, then the product (4) converges locally uniformly and represents a function of Hermite–Biehler class.

Proof. This result is also known as *Krein's factorization theorem*. It can be found in [36] or in [45, Theorem VII.3.6].

It is common to write E as linear combination of real functions A, B. Namely,

$$E = A - iB \quad \text{with} \quad A := \frac{1}{2}(E + E^{\#}), B := \frac{i}{2}(E - E^{\#}).$$

The functions A, B also have intrinsic meaning; this will be seen in Proposition 2, (iii). In this context it is interesting to observe the following fact.

Remark 4. Let A be a real entire function all of whose zeroes are real and simple. Then there exists a function $E \in \mathcal{HB}$, such that $A = \frac{1}{2}(E + E^{\#})$. ◇

Proof. This can be regarded as common knowledge and follows, e.g., using [45, Theorems VII.1.1, VII.2.3]. An explicit reference is, e.g., [25, Lemma 5.1].

De Branges Spaces via Hermite–Biehler Functions

The relation between de Branges spaces and Hermite–Biehler functions is established by the following result.

Theorem 3. *For $E \in \mathcal{HB}$ set (denote by $H^2(\mathbb{C}^+)$ the Hardy space in the upper half plane)*

$$\mathcal{H}(E) := \left\{ F \text{ entire} : \frac{F}{E}, \frac{F^{\#}}{E} \in H^2(\mathbb{C}^+) \right\}, \tag{7}$$

$$(F, G)_E := \left(\frac{F}{E}, \frac{G}{E} \right)_{H^2(\mathbb{C}^+)} = \int_{\mathbb{R}} F(x)\overline{G(x)} \frac{dx}{|E(x)|^2}, \quad F, G \in \mathcal{H}(E).$$

Then $\mathcal{H}(E)$ endowed with the inner product $(.,.)_E$ is a de Branges space.

Conversely, if $\mathcal{H}$ is a de Branges space, then there exists a function $E \in \mathcal{HB}$ such that

$$\mathcal{H} = \mathcal{H}(E), \qquad (F, G)_{\mathcal{H}} = (F, G)_E, \ F, G \in \mathcal{H}.$$

Proof. This is obtained by combining [14, Problem 50, Theorem 23] with [49, Lemma 5.21, Theorem 6.13].

The function E representing a given de Branges space in this way is not unique. In fact, two functions $E_1, E_2 \in \mathcal{HB}$ generate the same de Branges spaces (including equality of inner products), if and only if

$$(A_2, B_2) = (A_1, B_1) \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \tag{8}$$

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with some $\alpha \in \mathbb{R}$. Here

$$A_j := \frac{1}{2}\big(E_j + E_j^\#\big),\ B_j := \frac{i}{2}\big(E_j - E_j^\#\big), \quad j = 1, 2.$$

Proof. This is [10, Theorem 1] (remembering (HB4)).

Remark 5. Briefly compare Remark 1 with Remark 3: Requiring strict inequality in (2) corresponds on the level of de Branges spaces to requiring that there exists a function in the space which does not vanish identically. Dropping (HB3) corresponds to dropping (dB4) but retaining the requirement that there exists a function in the space which does not vanish identically. Notation being as in Remark 3, one can show that $\mathcal{H}(E)$ and $\mathcal{H}(\tilde{E})$ are isometrically isomorphic.

Finally, dropping (HB4) has no influence on the de Branges space side, besides that (8) should be altered slightly. Namely to

$$(A_2, B_2) = (A_1, B_1)\begin{pmatrix} u_1 & u_2 \\ v_2 & v_2 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in \mathbb{R}$, $u_1 v_2 - u_2 v_1 = 1$. ◇

Naturally, all properties of a de Branges space $\mathcal{H}$ translate to properties of the function(s) $E \in \mathcal{H}B$ which realize $\mathcal{H}$ as $\mathcal{H}(E)$. Some examples which illustrate this principle are the following. Here $\varphi_E : \mathbb{R} \to \mathbb{R}$ denotes the unique continuous function with

$$\varphi_E(0) = 0 \quad \text{and} \quad E(x)e^{i\varphi_E(x)} \in \mathbb{R}.$$

This function is referred to as the *phase function* of E; it is just (the negative of) a continuous branch of the argument of E.

Proposition 2. *Let $\mathcal{H}$ be a de Branges space, and let $E \in \mathcal{H}B$ be such that $\mathcal{H} = \mathcal{H}(E)$.*

(i) For each $w \in \mathbb{C}$ the reproducing kernel $K(w, .)$ of $\mathcal{H}(E)$ is given as (for $z = \overline{w}$ this formula has to be interpreted appropriately as a derivative, which is possible by analyticity)

$$K(w, z) = \frac{E(z)E^\#(\overline{w}) - E(\overline{w})E^\#(z)}{2\pi i(\overline{w} - z)}, \quad z \in \mathbb{C}. \tag{9}$$

(ii) The norm $\nabla_{\mathcal{H}}(w)$ of the point evaluation functional at w is given as

$$\nabla_{\mathcal{H}}(w) = \begin{cases} \left(\frac{|E(w)|^2 - |E(\overline{w})|^2}{2\pi i(\overline{w} - w)}\right)^{\frac{1}{2}}, & w \in \mathbb{C} \setminus \mathbb{R} \\ \frac{1}{\pi}|E(w)|\sqrt{\varphi_E'(w)}, & w \in \mathbb{R} \end{cases}$$

(iii) Denote

$$S_\varphi(z) := \frac{1}{2i}\big(e^{i\varphi}E(z) - e^{-i\varphi}E^\#(z)\big), \quad \varphi \in \mathbb{R}. \tag{10}$$

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Then each function S_φ belongs to $\operatorname{Assoc}\mathcal{H}(E)$. *Via the correspondence* (1), *the family* $\{S_\varphi : \varphi \in [0,\pi)\}$ *parameterizes the set of all selfadjoint extensions of* $S(\mathcal{H}(E))$.

The finite spectrum of the extension corresponding to S_φ *equals the set of all points* $x \in \mathbb{R}$ *for which* $\varphi_E(x) \equiv \varphi$.

(iv) There exists $\varphi \in [0,\pi)$ *such that* $S_\varphi \in \mathcal{H}(E)$ *if and only if* $\overline{\operatorname{dom} S(\mathcal{H}(E))} \neq \mathcal{H}(E)$. *If* $\overline{\operatorname{dom} S(\mathcal{H}(E))} \neq \mathcal{H}(E)$, *then there exists exactly one* $\varphi_0 \in [0,\pi)$ *with* $S_{\varphi_0} \in \mathcal{H}(E)$ *and* $\mathcal{H}(E) \ominus \overline{\operatorname{dom} S(\mathcal{H}(E))} = \operatorname{span}\{S_{\varphi_0}\}$.

Proof. This is, mainly explicitly, in the work of de Branges. For (i) and (ii) see [14, Theorem 19, Problem 48], for (iii) [14, Theorem 29], and an explicit reference for (iv) is [29, Proposition 6.1].

The elements of a space $\mathcal{H}(E)$ can also be characterized by a pointwise estimate.

Proposition 3. *Let* $E \in \mathcal{HB}$. *Then an entire function* F *belongs to the space* $\mathcal{H}(E)$, *if and only if* $\int_{\mathbb{R}} |F(x)|^2 \frac{dx}{|E(x)|^2} < \infty$ *and*

$$|F(z)|^2 \le \Big(\int_{\mathbb{R}} |F(x)|^2 \frac{dx}{|E(x)|^2}\Big) \frac{|E(z)|^2 - |E(\overline{z})|^2}{2\pi i(\overline{z}-z)}, \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

Proof. This is [14, Theorem 20] and continuity.

In [14] the slightly different form of the definition of the space $\mathcal{H}(E)$ is used. Namely,

$$\mathcal{H}(E) := \Big\{F \text{ entire} : \int_{\mathbb{R}} |F(x)|^2 \frac{dx}{|E(x)|^2} < \infty, \\ \frac{F}{E}, \frac{F^\#}{E} \text{ bounded type and nonpositive mean type in } \mathbb{C}^+\Big\}.$$

We prefer to use the formula (7), since it emphasizes the following important connection with the classical theory of Hardy spaces. In fact, each de Branges space can be identified with a *model subspace* $\mathcal{K}_\Theta$ (in the sense of [47, Section 3.1]) with meromorphic Θ.

Proposition 4. *Let* $E \in \mathcal{HB}$. *Then the map* $F \mapsto \frac{F}{E}$ *induces an isometric isomorphism of the de Branges space* $\mathcal{H}(E)$ *onto the shift-coinvariant subspace*

$$\mathcal{K}_{\frac{E^\#}{E}} := H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+).$$

Proof. This is common knowledge; obviously the above map is an isomorphism.

Note that the function $\frac{E^\#}{E}$ is an inner function of a particular form. Namely,

$$\frac{E^\#(z)}{E(z)} = e^{-i\alpha z} B^\#(z),$$

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where B is the Blaschke product formed with the zeroes of E and the number α is given as $\alpha := \limsup_{y\to+\infty} \frac{1}{y} \log \left|\frac{E(-iy)}{E(iy)}\right|$. It is worth to mention that, conversely, each meromorphic inner function appears in this way, cf. [21, Section 1.2].

By the spectral theorem selfadjoint extensions of $S(\mathcal{H}(E))$ give rise to orthonormal bases.

Proposition 5. *Let $E \in \mathcal{H}B$, and let $\varphi \in [0, \pi)$. Provided the function S_φ does not belong to the space $\mathcal{H}(E)$, the set*

$$\left\{\frac{1}{\sqrt{\pi}}\left|\frac{S_{\varphi-\frac{\pi}{2}}(w)}{S'_\varphi(w)}\right|^{\frac{1}{2}} \cdot \frac{S_\varphi(z)}{z-w} : w \in \mathbb{R}, S_\varphi(w) = 0\right\} \tag{11}$$

is an orthonormal basis of $\mathcal{H}(E)$.

If $S_\varphi \in \mathcal{H}(E)$, the set (11) is an orthonormal basis of $\overline{\operatorname{dom} S(\mathcal{H}(E))}$. Hence, the family (11) together with the function $\frac{1}{\|S_\varphi\|_E} \cdot S_\varphi$ is an orthonormal basis of $\mathcal{H}(E)$.

Proof. This is [14, Theorem 22] and its proof.

Example 4. For each $a > 0$ the Paley–Wiener space $\mathcal{P}W_a$ is generated by the Hermite–Biehler function e^{-iaz}. The reproducing kernel of $\mathcal{P}W_a$ is given as

$$K_{\mathcal{P}W_a}(w, z) = \frac{\sin[a(z-\overline{w})]}{\pi(z-\overline{w})}.$$

An orthonormal basis of $\mathcal{P}W_a$ is given as (for illustration taking $\varphi = 0$)

$$\left\{\frac{1}{\sqrt{\pi}} \cdot \frac{\sin(az)}{z-w} : w \in \frac{\pi}{a}\mathbb{Z}\right\}$$

◇

Proof. This is a classical fact which is known since the early 1900s. Using the setup of de Branges theory, it follows from [14, Theorem 16].

Example 5. Let μ be a positive Borel measure which possesses moments (at least) up to order $2n$, and consider the space $\mathbb{C}[z]_n$ endowed with the $L^2(\mu)$-inner product. Set

$$s_k := \int_{\mathbb{R}} x^k \, d\mu(x), \; k = 0, \ldots, n \quad \text{and} \quad D_n := \det\left(s_{i+j}\right)_{i,j=0}^n.$$

Then the reproducing kernel of $\mathbb{C}[z]_n$ is given as

$$K(w, z) = -\frac{1}{D_n} \det \begin{pmatrix} 0 & 1 & z & \cdots & z^n \\ 1 & s_0 & s_1 & \cdots & s_n \\ \overline{w} & s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \overline{w}^n & s_n & s_{n+1} & \cdots & s_{2n} \end{pmatrix}$$

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The Hermite–Biehler function which generates the space $\mathbb{C}[z]_n$ can be given in terms of the orthogonal polynomials of the first and second kinds associated with the moment sequence $(s_k)_{k=0}^{2n}$. ◇

Proof. This is, e.g., [1, p. 9]. See also (within the more general Pontryagin space setting) [40, Proposition 3.1], [41, Section 4.4].

The Chain of Subspaces

Throughout the theory subspaces play a crucial role.

Definition 3. Let $\mathcal{H}$ be a de Branges space. A linear subspace $\mathcal{L}$ of $\mathcal{H}$ is called a *de Branges subspace of* $\mathcal{H}$, if it is itself, with the inner product inherited from $\mathcal{H}$, a de Branges space.

The set of all de Branges subspaces of $\mathcal{H}$ is denoted as $\operatorname{Sub}\mathcal{H}$. ◇

Revisiting the axioms (dB1)–(dB4), it is easy to see that $\mathcal{L}$ is a de Branges subspace of $\mathcal{H}$ if and only if it is closed in the topology of $\mathcal{H}$, invariant under the involution $.^{\#}$, and invariant under dividing zeroes.

The first true cornerstone of de Branges' theory which is encountered is that for each de Branges space $\mathcal{H}$ the set $\operatorname{Sub}\mathcal{H}$ is totally ordered (with respect to inclusion) and in some sense dense in itself. The following theorem comprehensively states these facts.

Theorem 4. *Let $\mathcal{H}$ be a de Branges space. Then the following statements hold.*

(i) $\operatorname{Sub}\mathcal{H}$ *is totally ordered with respect to inclusion.*
(ii) For each $\mathcal{L}_0 \in \operatorname{Sub}\mathcal{H}$ it holds that (for $\mathcal{L}_0 = \mathcal{H}$ the second formula is of course immaterial)

$$\dim\Big(\mathcal{L}_0\Big/\operatorname{Clos}\bigcup\{\mathcal{L}\in\operatorname{Sub}\mathcal{H}\cup\{0\}\colon \mathcal{L}\subsetneq\mathcal{L}_0\}\Big)\le 1,$$
$$\dim\Big(\bigcap\{\mathcal{L}\in\operatorname{Sub}\mathcal{H}\colon \mathcal{L}\supsetneq\mathcal{L}_0\}\Big/\mathcal{L}_0\Big)\le 1.$$

(iii) Either $\operatorname{Sub}\mathcal{H}$ *contains a one-dimensional element, or*

$$\inf_{\mathcal{L}\in\operatorname{Sub}\mathcal{H}}\nabla_{\mathcal{L}}(w)=0,\quad w\in\mathbb{C},$$

where, again, $\nabla_{\mathcal{L}}(w)$ denotes the norm of the point evaluation functional at w in the space $\mathcal{L}$.

Proof. Item (i) is [13, Theorem 1], item (ii) follows by combining [14, Problems 148–150] with [12, Theorem 1], and item (iii) is [14, Theorem 40(5)]. The proof of (ii) depends on the connection with canonical systems reviewed in the next section.

Remarkably, the original proof of de Branges of item (i) contained a minor gap which remained unnoticed for quite some time (but was closed eventually).

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Example 6. It holds that

$$\operatorname{Sub}\mathcal{P}W_a = \{\mathcal{P}W_b : 0 < b \le a\} \quad \text{and} \quad \operatorname{Sub}\mathbb{C}[z]_n = \{\mathbb{C}[z]_k : 0 \le k \le n\}.$$

◇

What here becomes apparent is that the Paley–Wiener spaces on one hand and the spaces $\mathbb{C}[z]_n$ on the other hand represent two opposite extremal cases. The first are perfectly continuous, whereas the second are discrete.

The Structure Hamiltonian of a De Branges Space

We understand by a *Hamiltonian* a function H defined on an (possibly unbounded) interval $I = (a, b)$, which takes real and non-negative 2×2-matrices as values, is locally integrable, and does not vanish on any set of positive measure.

The *canonical system* associated with H is the differential equation for a 2-vector valued function y given as

$$y'(x) = zJH(x)y(x), \quad x \in I,$$

where z is a complex parameter (the eigenvalue parameter), and J is the signature matrix $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The Hamiltonian H is said to be in *limit circle case at the endpoint* a (*lc at* a, for short), if for one (and hence for all) $x_0 \in I$

$$\int_a^{x_0} \operatorname{tr} H(x)\, dx < \infty,$$

and in *limit point case at* a (*lp at* a, for short) otherwise. The analogous notation is applied to distinguish the cases whether or not H remains integrable at the endpoint b. Notice that, since $H(x)$ is positive semidefinite, integrability of $\operatorname{tr} H$ is equivalent to integrability of all entries of H.

Two Hamiltonians H_1 and H_2 are called *reparameterizations* of each other, if there exists an increasing bijection γ between their domains, such that γ and γ^{-1} are both absolutely continuous and

$$H_2(x) = H_1(\gamma(x))\gamma'(x).$$

As a rule of thumb, Hamiltonians which are reparameterizations of each other share all their important properties.

A notion which may seem technical on first sight, but actually is of intrinsic importance, is the following. Let H be a Hamiltonian defined on $I = (a, b)$. A nonempty interval $(a', b') \subseteq I$ is called *indivisible* for H, if for some scalar function $h(x)$ and some fixed angle $\alpha \in \mathbb{R}$ (denoting $\xi_\alpha := (\cos\alpha, \sin\alpha)^T$),

$$H(x) = h(x)\xi_\alpha\xi_\alpha^T, \quad x \in (a', b') \text{ a.e.}$$

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The angle α is called the *type* of the indivisible interval (a', b') and is determined up to multiples of π. The number $\int_{a'}^{b'} \operatorname{tr} H(x)\, dx$ is called its the *length*. A point $x \in I$ is called *regular for* H, if it is not inner point of an indivisible interval. The set of all regular points for H is denoted by I_{reg}.

It is another cornerstone of de Branges' theory that the chain $\operatorname{Sub}\mathcal{H}$ can be described by a canonical system.

Theorem 5. *Let $E \in \mathcal{HB}$. Then there exists a Hamiltonian H, defined on some interval $I = (a, b)$, such that the following statements hold.*

(i) The Hamiltonian H is lc at b, for no $x_0 \in I$ the interval (a, x_0) is indivisible of type $\frac{\pi}{2}$, and for one (and hence for all) $x_0 \in I$

$$\int_a^{x_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* H(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt < \infty.$$

(ii) Let $(A_t(z), B_t(z))^T$, $z \in \mathbb{C}$, be the unique solution of the initial value problem at b

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} A_t(z) \\ B_t(z) \end{pmatrix} = zJH(t)\begin{pmatrix} A_t(z) \\ B_t(z) \end{pmatrix}, \quad t \in I, \\ \begin{pmatrix} A_b(z) \\ B_b(z) \end{pmatrix} = \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}. \end{cases}$$

Here write again $E = A - iB$ with A, B real entire functions.
Then $E_t := A_t - iB_t \in \mathcal{HB}$, $t \in I$.

(iii) We have

$$\operatorname{Sub}\mathcal{H}(E) = \left\{ \mathcal{H}(E_t) : t \in I_{\text{reg}} \right\}.$$

The Hamiltonian H is uniquely determined up to reparameterization.

Proof. This is contained in [14, Theorems 40, 35, Problems 152, 153].

We refer to the Hamiltonian H as in the above theorem (to be precise, rather to the equivalence class modulo reparameterization of one such Hamiltonian) as the *structure Hamiltonian of* E.

Remark 6. The property (i) in Theorem 5, that the left upper entry of H remains integrable toward a, is related to the normalization that $E(0) = 1$. The message is that the Hamiltonian H behaves well in *one* direction. ◇

Again citing de Branges (cf. [14, p. 140]), it is a fundamental problem to determine the class of all Hamiltonians which appear as the structure Hamiltonian of some de Branges space. However, this natural question remained unsolved up to the day.

Characterisations can be given implicitly in terms of the Weyl coefficient of H. Namely, the Weyl coefficient should be meromorphic in the whole plane. Speaking in equivalent operator theoretic terms, the selfadjoint realizations of the differential operator associated with H should have compact resolvents.

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Proof. This can be regarded as common knowledge. An explicit reference is, e.g., [25, Section 5.3, Main Theorem].

The next theorem contains some presently known partial results giving conditions on H itself in order that H is the structure Hamiltonian of some function $E \in \mathcal{H}B$.

Theorem 6. *Let $H = (h_{ij})_{i,j=1}^2$ be a Hamiltonian defined on some interval $I = (a,b)$, and assume that H is lc at b.*

(i) Assume that H satisfies

$$\int_a^b h_{11}(t)\,dt < \infty \quad \text{and} \int_a^b \Big(\int_a^t h_{11}(s)\,ds\Big) h_{22}(t)\,dt < \infty. \tag{12}$$

Set $\beta(t) := \int_b^t h_{12}(x)\,dx$. Then there exists a unique solution of

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} A_t(z) \\ B_t(z) \end{pmatrix} = zJH(t)\begin{pmatrix} A_t(z) \\ B_t(z) \end{pmatrix}, & t \in I \\ \lim\limits_{t \downarrow a} e^{\beta(t)z}\begin{pmatrix} A_t(z) \\ B_t(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases}$$

The limit $(A_b, B_b)^T := \lim_{t\uparrow b}(A_t, B_t)$ exists, and the function $E := A_b - iB_b$ belongs to $\mathcal{H}B$. The Hamiltonian H is the structure Hamiltonian of $\mathcal{H}(E)$.

(ii) Assume that H is of diagonal form, i.e., writing $H(t) = (h_{ij}(t))_{i,j=1,2}$, that $h_{12} = h_{21} = 0$. Then H is the structure Hamiltonian of a de Branges space if and only if

$$\int_a^b h_{11}(x)\,dx < \infty \quad \text{and} \quad \lim_{x\downarrow a}\Big(\int_a^x h_{11}(t)\,dt \cdot \int_x^b h_{22}(t)\,dt\Big) = 0.$$

(iii) Assume that there exists a point $x_0 \in (a,b)$ and a monotone and bounded function $\phi : (a,x_0) \to \mathbb{R}$, such that $H(x) = \operatorname{tr} H(x) \cdot \xi_{\phi(x)}\xi_{\phi(x)}^T$, $x \in (a,x_0)$ a.e. Then H is the structure Hamiltonian of a de Branges space if and only if ($\phi(a) := \lim_{x\downarrow a}\phi(x)$)

$$\int_a^b \cos^2\phi(x)\operatorname{tr} H(x)\,dx < \infty \quad \text{and}$$

$$\lim_{x\downarrow a}\Big(\big|\phi(x) - \phi(a)\big| \cdot \int_x^b \operatorname{tr} H(t)\,dt\Big) = 0.$$

Proof. Item (i) is contained in [14, Theorem 41]. It should be pointed out that the given condition is certainly satisfied if H is lc at a.

The diagonal case in item (ii) is stated without a proof in [23, p. 209, Corollary]; the case considered in item (iii) seems to be unpublished. However, both can be deduced easily from the result [26, p. 138, 2°] on strings.

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It should be noted that the best possible known (necessary or sufficient, respectively) conditions for the general case are those stated in [23, Theorem 1]. However, a proof of these results seems to be not available in the literature (and thus they are not included here).

Taking an operator theoretic viewpoint, the conditions "(12)" and "lc at its left endpoint" on the structure Hamiltonian of a Hermite–Biehler function E have a neat characterization.

Theorem 7. *Let $E \in \mathcal{HB}$ and let H be the structure Hamiltonian of E.*

(i) The Hamiltonian H satisfies (12) if and only if the selfadjoint extensions of $S(\mathcal{H}(E))$ have resolvents of Hilbert-Schmidt class.
(ii) The Hamiltonian H is lc at its left endpoint if and only if $\operatorname{Assoc}\mathcal{H}(E)$ *contains a real and zerofree function.*

Proof. An explicit reference for item (i) is [32, Theorem 2.4] (where the trace normed case is considered; the general case is easily deduced, cf. [53, Theorem 2.12]. Item (ii) is [11, Theorems Vi,VII].

It is an interesting fact that these properties of H can also be characterized in terms of E itself.

Theorem 8. *Let H be a Hamiltonian defined on some interval $I = (a, b)$, and assume that H is lc at b.*

(i) The Hamiltonian H satisfies (12) if and only if H is the structure Hamiltonian of a function $E \in \mathcal{HB}$ which is of Polya class (an entire function F is said to be of Polya class, *if $F \in \mathcal{HB}$ and for each $x \in \mathbb{R}$ the function $y \mapsto |F(x+iy)|$, $y \in (0,\infty)$, is nondecreasing; see, e.g., [14, Section 7]. If real zeroes are admitted in the definition of $\mathcal{HB}$, then this monotonicity condition describes precisely the locally uniform closure of $\mathcal{HB} \cap \mathbb{C}[z]$).*
(ii) The Hamiltonian H is lc at a if and only if H is the structure Hamiltonian of a function $E \in \mathcal{HB}$ satisfying (as usual write $E = A - iB$ with A, B real entire functions)

- *A and B have no common zeroes, and all zeroes of A and B are real and simple.*
- $\operatorname{Im} \frac{B(z)}{A(z)} \geq 0$ *for all $z \in \mathbb{C}^+$.*
- *The nonzero zeroes α_n of A and β_n of B satisfy*

$$\sum_n \frac{1}{|A'(\alpha_n)B(\alpha_n)|\alpha_n^2} < \infty, \quad \sum_n \frac{1}{|A(\beta_n)B'(\beta_n)|\beta_n^2} < \infty .$$

- *The function $\frac{1}{AB}$ has an expansion*

$$\frac{1}{A(z)B(z)} = \frac{c_{-1}}{z} + c_0 + \sum_n \frac{1}{A'(\alpha_n)B(\alpha_n)}\Big[\frac{1}{z-\alpha_n} + \frac{1}{\alpha_n}\Big] + \\ + \sum_n \frac{1}{A(\beta_n)B'(\beta_n)}\Big[\frac{1}{z-\beta_n} + \frac{1}{\beta_n}\Big],$$

with some $c_{-1}, c_0 \in \mathbb{R}$.

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Proof. The first is contained in [14, Theorem 41], the second follows from [39, Section 3, Theorem A].

Example 7. Concerning the Paley–Wiener spaces, for each $a > 0$ the structure Hamiltonian of the function $E(z) := e^{-iaz}$ is given as

$$H(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x \in (0, a).$$

Concerning the space $\mathbb{C}[z]_n$ endowed with an $L^2(\mu)$-inner product, the formulas for H are more complicated. Assume that μ is a probability measure which has moments $s_k := \int_{\mathbb{R}} x^k \, d\mu(x)$ up to order $2n + 1$, and set (by convention $s_{-1} := 0$ and $\arctan(\infty) := \frac{\pi}{2}$)

$$C_n := \det \left(s_{i+j-1}\right)_{i,j=0}^{k}, \quad D_k := \det \left(s_{i+j}\right)_{i,j=0}^{k}, \quad E_n := \det \left(s_{i+j+1}\right)_{i,j=0}^{n},$$

$$l_k := \frac{C_n^2 + E_n^2}{D_{n-1} D_n}, \qquad \phi_k := \begin{cases} \frac{\pi}{2} & , \quad k = 0 \\ \arctan\left(-\frac{E_k}{C_k}\right), & k = 1, \dots, n \end{cases}$$

Then the structure Hamiltonian of the Hermite–Biehler function associated with the space $\mathbb{C}[z]_n$ is composed of the sequence of $n + 1$ indivisible intervals of lengths $l_0, \dots, l_n$ and types $\phi_0, \dots, \phi_n$. ◇

Proof. The case of Paley–Wiener spaces is common knowledge and can be checked by simple computation. The case of $\mathbb{C}[z]_n$ is known from the theory of the Hamburger power moment problem. An explicit reference is [24]; see also [41] (within the more general Pontryagin space setting).

Inclusion of the String Equation

A string is a pair $(L, \mathfrak{m})$ where $L \in [0, +\infty]$ and $\mathfrak{m}$ is a positive (possibly unbounded) Borel measure on $\mathbb{R} \cup \{+\infty\}$ with

$$\operatorname{supp} \mathfrak{m} \subseteq [0, L], \quad \mathfrak{m}([0, x]) < \infty, \ x \in [0, L),$$
$$\mathfrak{m}(\{L\}) < \infty, \quad \mathfrak{m}(\{L\}) = 0 \text{ if } L + \mathfrak{m}([0, L]) = +\infty.$$

We refer to L as the length of the string, to the function $m(x) := \mathfrak{m}((-\infty, x))$, $x \in \mathbb{R}$, as its mass distribution function, to the number $\mathfrak{m}([0, L])$ as its total mass, and denote the string given by L and $\mathfrak{m}$ as $\mathrm{S}[L, \mathfrak{m}]$. Throughout this paper assume in addition that

$$\inf \operatorname{supp} \mathfrak{m} = 0, \quad \sup \operatorname{supp} \mathfrak{m} = L,$$

meaning that the string has two heavy endpoints, i.e., cannot start or end with an interval free of mass.

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Given a string $S[L, \mathfrak{m}]$, consider the eigenvalue equation of the Krein–Feller differential operator $-D_{\mathfrak{m}} D_x$. Written in the form of an integral boundary value problem this is

$$\begin{cases} f(x) - f(0) + z \int_{[0,x]} (x - y) f(y) \, d\mathfrak{m}(y) = 0, & x \in \mathbb{R}, \\ f'(0-) = 0 \end{cases} \tag{13}$$

see, e.g., [27, Section 1] or [22]. Thereby, $z \in \mathbb{C}$ is the eigenvalue parameter.

The Krein–Feller differential operator arises when Fourier's method is applied to the partial differential equation

$$\frac{\partial}{\partial m(s)} \left(\frac{\partial v(s,t)}{\partial s} \right) - \frac{\partial^2}{\partial t^2} v(s,t) = 0,$$

which describes the vibrations of a string with a free left endpoint, which is stretched with unit tension on the interval $[0, L)$, and whose total mass on the interval $[0, x]$ equals $\mathfrak{m}([0, x])$. If the distribution function is sufficiently smooth, the boundary value problem (13) can be rewritten as a Sturm–Liouville equation. Conversely, for most potentials, the one-dimensional Schrödinger operator on a finite interval or on the half-line can be rewritten as a string equation using an appropriate Liouville-transformation.

Strings can be considered as Hamiltonian systems in several ways.

Proposition 6. *Let* $\mathsf{S}[L, \mathfrak{m}]$ *be a string, and define a Hamiltonian on the interval*

$$I_0 := \begin{cases} (0, L) \ , & L + \int_0^L m(x)^2 \, dx = \infty \\ (0, \infty) \ , & L + \int_0^L m(x)^2 \, dx < \infty \end{cases}$$

as

$$H_0(x) := \begin{cases} \begin{pmatrix} 1 & -m(x) \\ -m(x) & m(x)^2 \end{pmatrix} , & x \in (0, L) \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} , & x > L \quad \text{if } L + \int_0^L m(x)^2 \, dx < \infty \end{cases}$$

Then H_0 *is lc at* 0 *and lp at* $\sup I_0$. *The Weyl coefficient (the* Weyl coefficient *of a canonical system is a certain analytic function having nonnegative imaginary part throughout the upper half-plane. It can be used to build a functional model for the canonical differential equation. The measure in its Herglotz integral representation is a spectral measure of one selfadjoint realization. For details see, e.g., [12, Theorem III])* q_{H_0} *of* H_0 *and the Titchmarsh–Weyl coefficient (again, the* Titchmarsh–Weyl coefficient *of a canonical system is a certain analytic function which describes the Krein–Feller operator associated with the string. Sometimes, e.g., in the classical reference [27], it is called the* coefficient of dynamic compliance*)* q *of* $\mathsf{S}[L, m]$ *are related as*

$$q_{H_0}(z) = z q(z).$$

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Proposition 7. *Let a string* $\mathsf{S}[L, \mathfrak{m}]$ *be given. Set* $\mu := \lambda + \mathfrak{m}$ *where* λ *denotes the Lebesgue measure (then* λ *and* $\mathfrak{m}$ *are both absolutely continuous with respect to* μ*). Moreover, denote by* $M(x)$ *the function* $M(x) := x + m(x)$ *defined on the interval* $[0, L)$*. Then* $M(x) = \mu([0, x))$*,* $x \in [0, L)$*.*

With this notation, define a Hamiltonian H_d *on the interval* $I_d := [0, \infty)$ *as*

$$H_d(x) := \begin{cases} \begin{pmatrix} \frac{d\lambda}{d\mu}(x) & 0 \\ 0 & \frac{dm}{d\mu}(x) \end{pmatrix}, & x \in \operatorname{ran} M \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & x \in (0, \infty) \setminus \operatorname{ran} M \end{cases}$$

Then H_d *is lc at* 0 *and lp at* ∞*. The Weyl coefficient* q_{H_d} *of* H_d *and the principal Titchmarsh–Weyl coefficient* q *of* $\mathsf{S}[L, m]$ *are related as*

$$q_{H_d}(z) = zq(z^2)\,.$$

Proof. These two propositions can be regarded as common knowledge. A reference containing a detailed and explicit account emphasizing the operator theoretic perspective is [34].

The relevance in the present context – paying special attention to growth properties – is that a lot of knowledge about strings is readily available, and Propositions 6 and 7 can be used to transfer this knowledge to particular classes of canonical systems and de Branges spaces. Some instances of this principle will be met in the section "Growth from Spectral Properties of the String Equation" below; also remember the mentioned approach to Theorem 6(ii), (iii).

Growth Functions

In complex analysis the notion of growth is most classical and plays a central role. Comparing the maximum modulus of an entire function on a disk with radius r with exponentials $\exp\big(r^\rho\big)$ leads to the common notion of order and type. Very early in history also comparison with other functions appeared. This is usually attributed to Lindelöf, who used comparison functions $\exp\big(\lambda(r)\big)$ where

$$\lambda(r) = r^a \cdot \big(\log_{(m_1)} r\big)^{b_1} \cdot \ldots \cdot \big(\log_{(m_n)} r\big)^{b_n} \tag{14}$$

for large enough r. Here $a \geq 0$, $m_i \in \mathbb{N}$, $m_1 < \ldots < m_n$, $b_1, \ldots, b_n \in \mathbb{R}$, where $m_1 = 1$ and $b_1 > 1$ if $a = 0$, and $\log_{(n)}$ is defined by

$$\log_{(1)} r := \log r, \qquad \log_{(k+1)} r := \log\big(\log_{(k)} r\big), \quad k \in \mathbb{N},$$

for large enough r.

A general theory of growth is established by Valiron's theory of *proximate orders*, cf. [45, Section I.12]. We follow the approach taken in [44, Section I.6] or [50], and use the terminology of growth functions (in essence being the exponentials of proximate orders).

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Definition 4. A function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *growth function* if it satisfies the following axioms.

(gf1) The limit $\rho_\lambda := \lim_{r\to\infty} \frac{\log\lambda(r)}{\log r}$ exists and is finite and non-negative.

(gf2) For all sufficiently large values of r, the function λ is differentiable and

$$\lim_{r\to\infty} \left(r\frac{\lambda'(r)}{\lambda(r)} \Big/ \frac{\log\lambda(r)}{\log r} \right) = 1.$$

(gf3) We have $\log r = \mathrm{o}\big(\lambda(r)\big)$ (all 'o'- and 'O'-relations are, unless otherwise specified, understood for $r \to \infty$).

◇

Remark 7. Notation in the literature is again not uniform.

First, instead of (gf2) often the condition

$$\lim_{r\to\infty} \frac{r\lambda'(r)}{\lambda(r)} = \rho_\lambda \tag{15}$$

is required. If $\rho_\lambda > 0$, clearly, this is equivalent to (gf2). However, if $\rho_\lambda = 0$, (gf2) is stronger. In order to capture growth of functions of zero order, and in the context of de Branges spaces such examples do appear naturally, it is advisable to use (gf2) rather than (15).

Second, the condition (gf3) is imposed here to rule out some (usually trivial) particular cases. For several results this is not needed, sometimes it is. ◇

Similar to the standard notion of type with respect to an order ρ, the type of an entire function with respect to some growth function is defined. In the following $M(F, r)$ denotes the maximum modulus of the entire function F on the disk with radius r centered at the origin, i.e.,

$$M(F, r) := \max_{|z|\le r} |F(z)| = \max_{|z|=r} |F(z)|.$$

Definition 5. Let F be an entire function and let λ a growth function. The λ*-type of* F is defined to be the number (denote $\log^+ x := \max\{\log x, 0\}$, $x > 0$)

$$\sigma_\lambda(F) := \limsup_{r\to\infty} \frac{\log^+ M(F, r)}{\lambda(r)} \in [0, \infty].$$

◇

Notice that the λ-type of a function F is finite if and only if $\log^+ M(F, r) = \mathrm{O}\big(\lambda(r)\big)$. Moreover, the usual notion of type with respect to an order ρ is reobtained for the growth function $\lambda(r) := r^\rho$.

The growth of an entire function relates to the distribution of its zeros. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of (nonzero) complex numbers which has no finite accumulation point. Set

$$N((z_n)_{n\in\mathbb{N}}, r) := \sum_{|z_n|\le r} \log\frac{r}{|z_n|}, \quad S((z_n)_{n\in\mathbb{N}}; r_1, r_2; k) := \frac{1}{k} \sum_{r_1<|z_n|\le r_2} \left(\frac{1}{z_n}\right)^k.$$

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The sequence $(z_n)_{n\in\mathbb{N}}$ is said to have *finite λ-density* if

$$N((z_n)_{n\in\mathbb{N}}, r) = \mathrm{O}\big(\lambda(r)\big),$$

and to be *λ-balanced* if, uniformly in k,

$$|S((z_n)_{n\in\mathbb{N}}; r_1, r_2; k)| = \mathrm{O}\Big(\frac{\lambda(r_1)}{r_1^k} + \frac{\lambda(r_2)}{r_2^k}\Big), \quad r_1, r_2 \to \infty.$$

A sequence $(z_n)_{n\in\mathbb{N}}$ which has finite λ-density and is λ-balanced is called *λ-admissible*.

Theorem 9. *Let λ be a growth function, and $(z_n)_{n\in\mathbb{N}}$ be a sequence of (nonzero) complex numbers which has no finite accumulation point. Then there exists an entire function F with $\sigma_\lambda(F) < \infty$ such that $(z_n)_{n\in\mathbb{N}}$ is the precise sequence of zeros of F (taking into account multiplicities), if and only if $(z_n)_{n\in\mathbb{N}}$ is λ-admissible.*

Proof. This is [50, Theorem 13.5.2]. See also [45, Theorems I.13.17, I.13.18].

Remark 8. Consider the case that $\lambda(r) = r^\rho$. If ρ is not an integer, then a sequence $(z_n)_{n\in\mathbb{N}}$ is λ-admissible if and only if

$$\limsup_{r\to\infty} \frac{1}{r^\rho} N((z_n)_{n\in\mathbb{N}}, r) < \infty,$$

whereas, for integer ρ,

$$\left| \sum_{|z_n|\le r} \Big(\frac{1}{z_n}\Big)^\rho \right| = O(1)$$

should be valid in addition. ◇

Proof. This is shown in [50, Proposition 13.3.3].

Remark 8 shows that Theorem 9 reduces for $\lambda(r) = r^\rho$ to the classical results of Lindelöf on the distribution of the zeros of an entire function of order ρ, finite type (for Lindelöf's Theorems see, e.g., [45, Theorems I.10.14, I.11.15]).

General Theorems Relating De Branges Spaces and Growth

Bringing Together the Concepts

The elements of a de Branges space are entire functions, and this naturally leads to the following definition.

Definition 6. Let $\mathcal{H}$ be a de Branges space and λ a growth function. The *λ-type of $\mathcal{H}$* is defined to be the number

$$\sigma_\lambda(\mathcal{H}) := \sup_{F\in\mathcal{H}} \sigma_\lambda(F) \in [0, \infty].$$

◇

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A particular case which is noteworthy in many respects is that $\lambda(r) = r$; then one deals with usual exponential type.

Example 8. The Paley–Wiener space $\mathcal{P}W_a$ is of r-type (usual exponential type) equal to a. The space $\mathbb{C}[z]_n$ is of zero λ-type for every growth function λ. ◇

Growth via Hermite–Biehler Functions

The following result says that the growth of a de Branges space $\mathcal{H}$ can be computed from the Hermite–Biehler function generating it.

Theorem 10. *Let $E \in \mathcal{H}B$ and let λ be a growth function. Then*

$$\sigma_\lambda(\mathcal{H}(E)) = \max_{F \in \mathcal{H}} \sigma_\lambda(F) = \max_{F \in \operatorname{Assoc} \mathcal{H}} \sigma_\lambda(F) = \sigma_\lambda(E).$$

Proof. This is [30, Theorem 3.4]. The proof employs some standard complex analysis arguments and depends on the representation (9) of the reproducing kernel of $\mathcal{H}$.

It is a more involved fact that the growth of $\mathcal{H}$ can also be recovered from the functions S_φ parameterizing the selfadjoint extensions of the multiplication operator.

Theorem 11. *Let $E \in \mathcal{H}B$ and let λ be a growth function. Then, with S_φ as in (10),*

$$\sigma_\lambda(\mathcal{H}(E)) = \sigma_\lambda(S_\varphi), \quad \varphi \in [0, \pi).$$

Proof. This is [2, Corollary 2.5]. For particular cases, see also [3] and [30, Corollary 3.18].

The proof depends on the following complex analysis lemma, which is of independent interest.

Lemma. *Let A, B be entire functions and let λ be a growth function. If* $\operatorname{Im} \frac{B(z)}{A(z)} \geq 0$, $z \in \mathbb{C}^+$, *then* $\sigma_A^\lambda = \sigma_B^\lambda$.

The proof of this lemma uses the Herglotz integral representation of functions with nonnegative imaginary part and subharmonicity. It is shown in [2, Proposition 2.3], a particular case can be found in [4, Lemma 2.1].

Returning to the operator theoretic interpretation of the functions S_φ, Theorem 11 shows that growth of functions in $\mathcal{H}$ reflects in spectral properties of the selfadjoint extensions of the multiplication operator. The following statement illustrates this principle.

Corollary 1. *Let $\mathcal{H}$ be a de Branges space and let $\rho > 0$. If $\sigma_{r^\rho}(\mathcal{H}) < \infty$, then the operators $(A_\varphi - z)^{-1}$ belong to each Neumann–von Schatten class $\mathfrak{S}_p$, $p > \rho$.*

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Exponential Growth via Structure Hamiltonians

Let E be a function of Hermite–Biehler class, and assume that its structure Hamiltonian H is lc at its left endpoint. Then there exists a real entire function C such that $e^C E$ is of bounded type in the upper half-plane, and hence of finite exponential type. The type $\sigma_r(e^C E)$ can be computed explicitly in terms of H. This goes back to [37] or [11, Theorem X]. The next theorem provides a slightly stronger formulation.

Theorem 12. *Let E be a Hermite–Biehler function which is of bounded type in the upper half-plane, and let H be its structure Hamiltonian (say, defined on $I = (a,b)$). Assume that H satisfies (12) and the following condition (Δ).*

(Δ) Fix $x_0 \in (a,b)$, and define functions $X_k : (a, x_0) \to \mathbb{C}^2$ recursively by

$$X_0(x) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad X_k(x) := \int_{x_0}^{x} JH(y)X_{k-1}(y)\,dy, \quad k \in \mathbb{N}.$$

There exists a number $N \in \mathbb{N}_0$ such that (the space $L^2(H|_{(a,x_0)})$ is the set of all 2-vector valued measurable functions f with $\int_a^{x_0} f^*(x)H(x)f(x)\,dx < \infty$)

$$L^2\big(H|_{(a,x_0)}\big) \cap \operatorname{span}\big\{X_k : k \le N\big\} \ne \{0\}.$$

Then $\sqrt{\det H(t)} \in L^1(a,b)$ and

$$\sigma_r(e^C E) = \int_a^b \sqrt{\det H(t)}\,dt. \tag{16}$$

Proof. This result follows from [43, Theorem 4.1].

Obviously, if H is lc at a, the hypothesis of Theorem 12 are fulfilled with $N = 0$. Hence, Theorem 12 includes the classical case.

Example 9. Let $\alpha > 0$ and set $\nu_1 := \frac{\alpha-1}{2}$, $\nu_2 := \frac{\alpha+1}{2}$. Moreover, denote by J_ν the Bessel function

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)\,2^{2n+\nu}} x^{2n+\nu}, \quad \nu \in \mathbb{R} \setminus (-\mathbb{N}).$$

For each $b > 0$, the entire function

$$E_{\alpha,b}(z) := 2^{\nu_1}\Gamma(\nu_2)z^{-\nu_1}\Big(b^{-\nu_1}J_{\nu_1}(zb) - i\,b^{\nu_2}J_{\nu_2}(zb)\Big)$$

is of Hermite–Biehler class. From the known asymptotics of Bessel functions, it follows that $\sigma_r(E_{\alpha,b}) = b$.

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The structure Hamiltonian $H_{\alpha,b}$ of $E_{\alpha,b}$ is given as

$$H_{\alpha,b}(x) = \begin{pmatrix} x^{\alpha} & 0 \\ 0 & x^{-\alpha} \end{pmatrix}, \qquad x \in (0,b).$$

It satisfies the hypothesis of Theorem 12. Obviously, $\det H_{\alpha,b} = 1$, and this illustrates validity of the formula (16). $\diamond$

Proof. The fact that $E_{\alpha,b} \in \mathcal{HB}$ can be shown using the known properties of growth and zero distribution of Bessel functions. However, what lies behind is [42, Lemma 4.13]. The fact that $H_{\alpha,b}$ satisfies the hypothesis of Theorem 12 can be shown using arguments as in [53, Example 3.15].

If the structure Hamiltonian H of E is lp at its left endpoint, but not subject to further growth restrictions, it is not known whether a general relation between H and growth of $\mathcal{H}(E)$ prevails.

Growth Behavior of Subspaces

Let $\mathcal{H}$ be a de Branges space, let λ be a growth function, and consider the function

$$\Upsilon_{\lambda,\mathcal{H}} : \begin{cases} \operatorname{Sub}\mathcal{H} \to [0,\infty] \\ \qquad \mathcal{L} \mapsto \sigma_\lambda(\mathcal{L}) \end{cases}.$$

It is obvious from the definition of the λ-type of a de Branges space that $\Upsilon_{\lambda,\mathcal{H}}$ is nondecreasing.

One may say that the speed of exponential growth $\lambda(r) = r$ manifests a borderline. This intuitive statement is concretized by the next theorem which gives a neat dichotomy.

Theorem 13. *Let λ be a growth function.*

(i) Assume that $r = \mathrm{o}\big(\lambda(r)\big)$. Then, for each de Branges space $\mathcal{H}$, the function $\Upsilon_{\lambda,\mathcal{H}}$ is constant.
(ii) Assume that $\lambda(r) = \mathrm{O}(r)$. Then, for each growth function λ_1 with $\lambda_1(r) = \mathrm{o}(\lambda(r))$, there exists a pair of de Branges spaces $\mathcal{L},\mathcal{H}$ with

$$\mathcal{L} \in \operatorname{Sub}\mathcal{H} \quad \textit{and} \quad 0 < \sigma_{\lambda_1}(\mathcal{L}) < \infty,\ 0 < \sigma_\lambda(\mathcal{H}) < \infty.$$

The space $\mathcal{H}$ can be chosen such that $1 \in \operatorname{Assoc}\mathcal{H}$ (and hence also $1 \in \operatorname{Assoc}\mathcal{L}$).

Proof. Item (i) is [30, Theorem 3.10]. Its proof uses only that for each two functions F, G in a de Branges space, their quotient $\frac{F}{G}$ is a (meromorphic) function of bounded type in both, the open upper and lower half-planes.

The proof of the existence result in item (ii) is more involved; this is [2, Theorem 3.6] and its proof.

In particular, it is seen that for a growth function $\lambda(r) = \mathrm{O}(r)$, the function $\Upsilon_{\lambda,\mathcal{H}}$ may be not constant: with the notation of Theorem 13, (ii), it holds that

$$\Upsilon_{\lambda,\mathcal{H}}(\mathcal{L}) = 0 \quad \text{whereas} \quad \Upsilon_{\lambda,\mathcal{H}}(\mathcal{H}) > 0.$$

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Interestingly, the behavior of $\Upsilon_{\lambda,\mathcal{H}}$ seems to be related to the growth of the corresponding Hermite–Biehler function along the real axis. At least, the following result may be seen as a hint in this direction.

Theorem 14. *Let λ and λ_1 be growth functions with $\lambda_1(r) = o(\lambda(r))$, and let $\mathcal{H}$ be a de Branges space with $0 < \sigma_\lambda(\mathcal{H}) < \infty$. Assume that for one (and hence for each) function $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$ (here $f(x) \asymp g(x)$ means that there exist constants $0 < c < C < \infty$, such that $cf(x) \le g(x) \le Cf(x)$ for all x in the domain of definition of f and g)*

$$\log^+ |E(x)| + 1 \asymp \lambda(|x|), \quad x \in \mathbb{R}.$$

Then no infinite dimensional subspace $\mathcal{L} \in \operatorname{Sub}\mathcal{H}$ is of finite λ_1-type.

Proof. This is [2, Theorem 4.1].

For exponential growth, i.e., for $\lambda(r) = r$, the function $\Upsilon_{r,\mathcal{L}}$ is well behaved.

Remark 9. Let $E \in \mathcal{H}B$ and assume that the structure Hamiltonian H of E is subject to the conditions of Theorem 12. Then

$$\inf_{\mathcal{L}\in\operatorname{Sub}\mathcal{H}} \Upsilon_{r,\mathcal{H}} = 0,$$

and (of course, for $\mathcal{L} = \mathcal{H}$ the formula involving the infimum is immaterial. Also, if $\operatorname{Sub}\mathcal{H}$ contains a smallest element, for this element the formula involving the supremum is immaterial)

$$\sigma_r(\mathcal{L}) = \inf_{\substack{\mathcal{L}'\in\operatorname{Sub}\mathcal{H}\\ \mathcal{L}'\supsetneq\mathcal{L}}} \sigma_r(\mathcal{L}') = \sup_{\substack{\mathcal{L}'\in\operatorname{Sub}\mathcal{H}\\ \mathcal{L}'\subsetneq\mathcal{L}}} \sigma_r(\mathcal{L}'), \quad \mathcal{L} \in \operatorname{Sub}\mathcal{H}. \tag{17}$$

◇

Proof. This is immediate from (16).

The equality (17) can be seen as a continuity property of $\Upsilon_{\lambda,\mathcal{L}}$ for $\lambda(r) = r$. For growth functions λ with $r = \mathrm{o}(\lambda(r))$, continuity of $\Upsilon_{\lambda,\mathcal{L}}$ is trivial since this function is constant. In stark contrast, if $\lambda(r) = \mathrm{o}(r)$, continuity fails miserably.

Theorem 15. *Let λ be a growth function with $\lambda(r) = o(r)$. Then there exists a de Branges space $\mathcal{H}$ with*

$$0 < \sigma_\lambda(\mathcal{H}) < \infty \quad \textit{and} \quad \operatorname{Sub}\mathcal{H} = \big\{\mathbb{C}[z]_n : n \in \mathbb{N}\big\} \cup \{\mathcal{H}\}.$$

Proof. This can be obtained by putting together [31, Theorem 2.1(A)] with the knowledge on the relation between growth and distribution of zeros Theorem 9. A more explicit, but less elementary, reference is [6, Theorem D] (from which the above follows by passing from a Stieltjes to the symmetrized Hamburger moment problem).

Theorem 15 shows in particular that the function $\Upsilon_{\lambda,\mathcal{H}}$ may have a jump of maximal possible height.

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Growth from Spectral Properties of the String Equation

There is a vast literature containing information about the solutions of a string equation and about the spectrum of Krein–Feller operators. The connection between strings and canonical systems mentioned in the section "Inclusion of the String Equation" can be exploited to transfer these results (here only an instance is presented where available knowledge about strings is used to deduce knowledge about classes of Hamiltonians. Despite, it should be pointed out that this transfer works both ways). The next theorem is an example which illustrates this principle.

Theorem 16. *Let E be a Hermite–Biehler function which is of bounded type in the upper half-plane. Assume that the structure Hamiltonian H of E is of the form (denote the domain of H as $I = (a,b)$)*

$$H(t) = \operatorname{tr} H(t)\xi_{\phi(t)}\xi_\phi(t)^T, \quad t \in (a,b), \tag{18}$$

where:

- *(i) The function $\phi(t)$ is bounded and piecewise monotone (by this it is meant that there exists a* finite *partition of the domain of ϕ such that on each interval of this partition the function ϕ is either nondecreasing or nonincreasing).*
- *(ii) With $\phi(a) := \lim_{x\downarrow a} \phi(x)$ it holds that*

$$\int_a^b |\phi(x) - \phi(a)| \operatorname{tr} H(x)\, dx < \infty.$$

Then functions $\psi_n \in L^1_{\mathrm{loc}}((a,b])$ are well defined by the recurrence

$$\psi_0(x) := 1, \qquad \psi_{n+1}(x) := \int_{(a,x]} \Big(\int_\xi^b \psi_n(s) \operatorname{tr} H(s)\, ds \Big) |d\phi(\xi)|, \quad x \in (a,b].$$

Assume in addition to (i) and (ii) that:

- *(iii) For some $n \in \mathbb{N}_0$ the function ψ_n belongs to $L^2(\operatorname{tr} H(x)dx)$.*

Then

$$\sigma_{\sqrt{r}}(E) < \infty. \tag{19}$$

Proof. This result is obtained by combining [53, Theorem 5.2], the fact that (ii) characterizes trace-class (unpublished, but can be deduced using [26, p. 140]), and [33, Proposition 3.12].

Note that the hypothesis (i)–(iii) are certainly fulfilled (condition (iii) with $n = 0$), if H is lc at a. In this case, the statement is just a slightly stronger formulation of the classical result that the fundamental solutions of the eigenvalue equation associated with a string are entire functions of order $\frac{1}{2}$ finite type (which goes back to [38], see also [27, (2.27)]).

An easily accessible condition which ensures applicability of this result is the following.

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Theorem 17. *Let $E \in \mathcal{H}B$. If the phase function φ_E of E is bounded from below (bounded from above), then the structure Hamiltonian of E is of the form* (18) *with some nondecreasing (nonincreasing, respectively) and bounded function $\phi(t)$.*

Proof. This is [52, Theorems 4.1 and 4.3].

It should be pointed out that the conclusion (19) of Theorem 16 is only an upper bound. Using Theorem 17, Remark 4, and Theorem 9, it is easy to construct examples of Hermite–Biehler functions which satisfy the hypothesis of Theorem 16 and are of arbitrary growth smaller than $r^{\frac{1}{2}}$.

Some Examples

De Branges Spaces from Schrödinger Equations

Let an integrable potential on an interval $[0, L]$ be given. Then denote by y_1 and y_2 the solutions of the homogenous equation $-\frac{d^2}{dx^2} + V = 0$ with initial values

$$y_1(0) = 1,\ y_1'(0) = 0, \qquad y_1(0) = 0,\ y_1'(0) = 1\,,$$

and assign to V the Hamiltonian

$$H_V(x) := \begin{pmatrix} y_1(x)^2 & y_1(x)y_2(x) \\ y_1(x)y_2(x) & y_2(x)^2 \end{pmatrix}, \quad x \in (0, L)\,.$$

For sufficiently smooth Hamiltonians this construction can be reversed.

The canonical system with Hamiltonian H_V is closely related to the Schrödinger equation with potential V. In fact, if a function $y(x,z)$ solves the equation $-\frac{d^2}{dx^2}y(x,z)+V(x)y(x,z) = zy(x,z)$, then the function

$$u(x,z) := \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}^{-1} \begin{pmatrix} y(x,z) \\ y'(x,z) \end{pmatrix}$$

solves the canonical system.

Clearly, the Hamiltonian H_V is lc at both endpoints 0 and L. Hence, it is the structure Hamiltonian of a de Branges space $\mathcal{H}_V$.

The spaces of the chain $\operatorname{Sub}\mathcal{H}_V$ are generated by the functions $E_x(z) := y(x,z) + iy'(x,z)$ where $y(x,z)$ is the solution of $-\frac{d^2}{dx^2}y(x,z)+V(x)y(x,z) = zy(x,z)$ with $y(0,z) = 1$, $y'(0,z) = 0$. As a set, $\mathcal{H}(E_x(z))$ is given as the space of all cosine transforms with parameter $\sqrt{z}$ of square integrable functions on $[0, x]$. Its inner product can be computed via a certain integral operator.

In connection with growth properties, the following statement holds.

Proposition 8. *Let $L \in (0,\infty)$, $V \in L^1(0, L)$, and let E_x, $x \in (0, L]$, be the family of spaces constructed above. Then*

$$\sigma_{\sqrt{r}}(\mathcal{H}(E_x)) = x.$$

Proof. The proof of the stated facts and more features of this interesting connection can be found in [48].

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De Branges Spaces from Positive Definite Functions

Let $0 < a \leq \infty$. A continuous function $f : (-2a, 2a) \to \mathbb{C}$ with $f(-t) = \overline{f(t)}$ is called *positive definite*, if for each choice of $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in (-a, a)$ the quadratic form $\sum_{i,j=1}^{n} f(t_i - t_j)\xi_i\overline{\xi_j}$ is positive semidefinite.

By Bochner's theorem, a function f is positive definite on the whole real line ($a = \infty$), if and only if it is the Fourier transform of a finite positive Borel measure on $\mathbb{R}$. It can be shown that each positive definite function f on a finite interval $(-2a, 2a)$ can be extended to a positive definite function on the whole line in at least one way. In fact, either there exists a unique positive definite extension to $\mathbb{R}$, or there exist infinitely many (a proof which proceeds via an operator theoretic argument can be found in [19, Section 3.2]).

For a positive definite function f defined on $(-2a, 2a)$, consider the linear space

$$\mathcal{L}(f) := \operatorname{span}\left\{e^{ixz} : x \in (-a, a)\right\}$$

endowed with the inner product $(.,.)_f$ given by

$$(e^{ixz}, e^{iyz})_f := f(x - y), \quad x, y \in (-a, a).$$

The Hilbert space completion of $\mathcal{L}(f)$ is denoted as $\mathcal{H}(f)$.

Theorem 18. *Let $0 < a < \infty$, and let f be a positive definite function on $(-2a, 2a)$ which possess infinitely many extensions positive definite extensions to $\mathbb{R}$. Then $\mathcal{H}(f)$ is a de Branges space. Denoting $\mathcal{H}_b := \operatorname{cls}\{e^{ixz} : |x| \leq b\}$, it holds that*

$$\operatorname{Sub}\mathcal{H} \supseteq \left\{\mathcal{H}_b : 0 \leq b < a\right\}. \tag{20}$$

Proof. This has been shown (within the more general Pontryagin space setting) in [28]. An explicit reference for the Hilbert space case is not known to us, but it can be regarded as common knowledge and deduced from the already mentioned [19, Section 3.2].

Concerning growth properties, apparently, $\sigma_r(\mathcal{H}_b) = b$.

It should be noted that it is an open problem to find conditions on f which characterize when in (20) equality holds (for "most" positive definite functions equality does not hold).

Two Examples from Probability

First, one example where rational (positive) orders appear.

Example 10. Birth-and-death processes are a particular kind of stationary Markov processes whose state space is the nonnegative integers. They model the time evolution of some population. The transition probabilities are a solution of the forward Kolmogorov equation, and this yields a connection to the theory of orthogonal polynomials and in turn to canonical systems (for details see, e.g., [35]).

For several cases order and type of the corresponding monodromy matrices (and hence corresponding de Branges spaces) was computed. It depends on the asymptotic behavior on a

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small time scale of the one-step transition probabilities. It turns out that for quartic processes the monodromy matrix is of order $\frac{1}{4}$ and for cubic processes of order $\frac{1}{3}$. The type with respect to the respective order is finite and positive, and can be calculated (in fact, as the value of some elliptic integral). ◇

Proof. This is taken from [5, 18], and [17].

Again in connection with Markov processes fractal strings are studied in the literature. In this context examples of de Branges spaces are obtained where irrational orders appear.

Example 11. Let C be the classical Cantor set, and let S_1 and S_2 denote the functions $S_1(x) := \frac{1}{3}x$ and $S_2(x) := \frac{1}{3}x + \frac{2}{3}$ defined on the unit interval $[0, 1]$. Moreover, for $\rho \in (0, 1)$, let μ_ρ be the unique probability measure on $[0, 1]$ with

$$\mu_\rho(A) = \rho\mu_\rho(S_1^{-1}(A)) + (1-\rho)\mu_\rho(S_2^{-1}(A))$$

for each Borel subset A of $[0, 1]$. Then $\operatorname{supp}\mu_\rho = C$.

The distribution function $m_\rho(x) := \mu_\rho([0, x])$, $x \in [0, 1]$, is the mass function of a regular string. Provided that $\log(\frac{\rho}{3})/\log(\frac{1-\rho}{3})$ is irrational, order and type of the corresponding monodromy matrix can be computed: Denote by $n(r)$ the counting function of the spectrum of the corresponding Krein–Feller operator (i.e., the number of spectral points in the interval $[-r, r]$). Then the limit $\lim_{r\to\infty}\frac{n(r)}{r^\gamma}$ exists and is finite and positive, where $\gamma \in (0, \frac{1}{2})$ is the unique solution of the equation

$$\left(\frac{\rho}{3}\right)^\gamma + \left(\frac{1-\rho}{3}\right)^\gamma = 1.$$

◇

Proof. This is taken from [15] and [16].

Two Examples Involving Special Functions Which Move Away from Classical Order and Type

In the concrete examples presented so far, exact growth with respect to some order ρ (meaning positive and finite λ-type w.r.t. the growth function $\lambda(r) = r^\rho$) appeared. The following two examples move away from the classical scale of order.

First, some instances of very slowly growing Nevanlinna matrices.

Example 12. Using some general results, e.g., the Riesz criterion or the Krein condition, it is often possible to conclude that a concrete moment sequence is indeterminate. Contrasting this, there are rather few examples known of indeterminate moment sequences for which the corresponding monodromy matrix can be computed explicitly. One class of such sequences are indeterminate moment problems within the q-Askey scheme. These include a variety of situations featured by classical orthogonal polynomials, e.g., q-Laguerre or Stieltjes–Wigert polynomials.

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For indeterminate moment problems within the q-Askey scheme the corresponding Nevanlinna matrices can be given explicitly in terms of special functions (mostly hypergeometric functions). It turns out that these Nevanlinna matrices (and hence the corresponding de Branges spaces) are of finite and positive λ-type with respect to the growth function $\lambda(r) := (\log r)^\alpha$, where the value of α may depend on the situation under consideration (but mostly is equal to 2). In particular, these functions are of zero order. $\diamond$

Proof. This is taken from [7] and [4].

Second, an example where growth of order $\frac{1}{2}$ maximal type occurs.

Example 13. Let ξ denote the Riemann ξ-function, i.e.,

$$\xi(z) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\,\Gamma\big(\frac{1}{2}s\big)\,\zeta(s)$$

and set

$$E(z) := \xi\big(\frac{1}{2} + i\,\sqrt{iz}\big).$$

Due to the functional equation $\xi(1-s) = \xi(s)$, this formula defines an entire function. It is of Hermite–Biehler class, and hence generates a de Branges space.

From the known asymptotics of ξ it is seen that

$$0 < \sigma_\lambda(E) < \infty \quad \text{where} \quad \lambda(r) := r^{\frac{1}{2}} \log r.$$

The de Branges space generated by E contains the constant function 1, in particular the structure Hamiltonian H of E is lc at its left endpoint. It turns out that

$$\operatorname{Sub}\mathcal{H}(E) = \big\{\mathbb{C}[z]_n : n \in \mathbb{N}\big\} \cup \{\mathcal{H}(E)\},$$

i.e., H consists of a sequence of indivisible intervals. $\diamond$

Proof. This is [31, Example 3.2].

Finally, it should be said that plenty of examples of de Branges spaces generated by special functions can be found in [14, Chapter 3].

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References

1. Akhiezer, N.I.: Klassicheskaya problema momentov i nekotorye voprosy analiza, svyazannye s neyu. Gosudarstv. Izdat. Fiz.-Mat. Lit. Moscow (1961, in Russian). English translation:

The Classical Moment Problem and Some Related Questions in Analysis. Oliver & Boyd, Edinburgh (1965)
2. Baranov, A.D., Woracek, H.: Subspaces of de Branges spaces with prescribed growth. Algebra i Analiz **18**(5), 23–45 (2006). ISSN: 0234-0852
3. Berg, C., Pedersen, H.L.: Nevanlinna matrices of entire functions. Math. Nachr. **171**, 29–52 (1995). ISSN: 0025-584X
4. Berg, C., Pedersen, H.L.: Logarithmic order and type of indeterminate moment problems. In: Difference Equations, Special Functions and Orthogonal Polynomials. With an appendix by Walter Hayman, pp. 51–79. World Scientific, Hackensack (2007)
5. Berg, C., Valent, G.: The Nevanlinna parametrization for some indeterminate Stieltjes moment problems associated with birth and death processes. Methods Appl. Anal. **1**(2), 169–209 (1994), ISSN: 1073-2772
6. Borichev, A., Sodin, M.: The Hamburger moment problem and weighted polynomial approximation on discrete subsets of the real line. J. Anal. Math. **76**, 219–264 (1998). ISSN: 0021-7670
7. Christiansen, J.S.: Indeterminate Moment Problems within the Askeyscheme. Ph.D. thesis. University of Copenhagen (2004)
8. de Branges, L.: Some Hilbert spaces of entire functions. Proc. Am. Math. Soc. **10**, 840–846 (1959). ISSN: 0002-9939
9. de Branges, L.: Some mean squares of entire functions. Proc. Am. Math. Soc. **10**, 833–839 (1959). ISSN: 0002-9939
10. de Branges, L.: Some Hilbert spaces of entire functions. Trans. Am. Math. Soc. **96**, 259–295 (1960). ISSN: 0002-9947
11. de Branges, L.: Some Hilbert spaces of entire functions. II. Trans. Am. Math. Soc. 99, 118–152 (1961). ISSN: 0002-9947
12. de Branges, L.: Some Hilbert spaces of entire functions. III. Trans. Am. Math. Soc. **100**, 73–115 (1961). ISSN: 0002-9947
13. de Branges, L.: Some Hilbert spaces of entire functions. IV. Trans. Am. Math. Soc. **105**, 43–83 (1962). ISSN: 0002-9947
14. de Branges, L.: Hilbert Spaces of Entire Functions, pp. ix+326. Prentice-Hall Inc., Englewood Cliffs (1968)
15. Freiberg, U.: Maßgeometrische Laplaceoperatoren für fraktale Teilmengen der reellen Achse. German. Ph.D. thesis. Friedrich-Schiller-Universität Jena (2000)
16. Freiberg, U.: Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets. Forum Math. **17**(1), 87–104 (2005). ISSN: 0933-7741
17. Gilewicz, J., Leopold, E., Ruffing, A. et al.: Some cubic birth and death processes and their related orthogonal polynomials. Constr. Approx. **24**(1), 71–89 (2006). ISSN: 0176-4276
18. Gilewicz, J., Leopold, E., Valent, G.: New Nevanlinna matrices for orthogonal polynomials related to cubic birth and death processes. J. Comput. Appl. Math. **178**(1–2), 235–245 (2005). ISSN: 0377-0427
19. Gorbachuk, M.L., Gorbachuk, V.I.: M. G. Krein's Lectures on Entire Operators. Operator Theory: Advances and Applications, vol. 97, pp. x+220. Birkhäuser, Basel (1997). ISBN: 3-7643-5704-5
20. Haböck, U.: Reproducing Kernel Spaces of Entire Functions. German. MA thesis. Vienna University of Technology (2001)

21. Havin, V., Mashreghi, J.: Admissible majorants for model subspaces of H^2. I. Slow winding of the generating inner function. Canad. J. Math. **55**(6), 1231–1263 (2003). doi:10.4153/CJM-2003-048-8. http://dx.doi.org/10.4153/CJM-2003-048-8
22. Kac, I.S.: The spectral theory of a string. Ukraïn. Mat. Zh. **46**(3), 155–176 (1994). ISSN: 0041-6053
23. Kac, I.S.: A criterion for the discreteness of a singular canonical system. Funktsional. Anal. i Prilozhen **29**(3) (1995, in Russian). English translation: Funct. Anal. Appl. **29**(3), 75–78, 207–210 (1995/1996). ISSN: 0374-1990
24. Kac, I.S.: Inclusion of the Hamburger power moment problem in the spectral theory of canonical systems. Zap. Nauchn. Sem. S.- Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **262** (1999, in Russian). Issled. po Linein. Oper. i Teor. Funkts. 27. English translation: J. Math. Sci. (New York) **110**(5), 2991–3004, 147–171, 234 (2002). ISSN: 0373-2703
25. Kac, I.S.: On the nature of the de Branges Hamiltonian. Ukraïn. Mat. Zh. **59**(5) (2007, in Russian). English translation: Ukrainian Math. J. **59**(5), 718–743, 658–678 (2007). ISSN: 1027-3190
26. Kac, I.S., Krein, M.G.: Criteria for the discreteness of the spectrum of a singular string. Izv. Vysš. Učebn. Zaved. Matematika **2**(3), 136–153 (1958). ISSN: 0021-3446
27. Kac, I.S., Krein, M.G.: On spectral functions of a string. In: Addition II in F.V. Atkinson, Diskretnye i nepreryvnye granichnye zadachi, pp. 648–737. Izdat. "Mir", Moscow (1968, in Russian). English translation: Am. Math. Soc. Transl. **103**(2), 19–102 (1974)
28. Kaltenbäck, M.: Hermitian indefinite functions and Pontryagin spaces of entire functions. Int. Equ. Oper. Theory **35**(2), 172–197 (1999). ISSN: 0378-620X
29. Kaltenbäck, M., Woracek, H.: Pontryagin spaces of entire functions. I. Int. Equ. Oper. Theory **33**(1), 34–97 (1999). ISSN: 0378-620X
30. Kaltenbäck, M., Woracek, H.: De Branges spaces of exponential type: general theory of growth. Acta Sci. Math. (Szeged) **71**(1–2), 231–284 (2005). ISSN: 0001-6969
31. Kaltenbäck, M., Woracek, H.: Hermite-Biehler functions with zeros close to the imaginary axis. Proc. Am. Math. Soc. **133**(1), 245–255 (2005, electronic). ISSN: 0002-9939
32. Kaltenbäck, M., Woracek, H.: Canonical differential equations of Hilbert-Schmidt type. In: Operator Theory in Inner Product Spaces. Oper. Theory Adv. Appl., vol. 175, pp. 159–168. Birkhäuser, Basel (2007)
33. Kaltenbäck, M., Winkler, H., Woracek, H.: Singularities of generalized strings. In: Operator Theory and Indefinite Inner Product Spaces. Oper. Theory Adv. Appl., vol. 163, pp. 191–248. Birkhäuser, Basel (2006)
34. Kaltenbäck, M., Winkler, H., Woracek, H.: Strings, dual strings, and related canonical systems. Math. Nachr. **280**(13–14), 1518–1536 (2007). ISSN: 0025-584X
35. Karlin, S., McGregor, J.: The classification of birth and death processes. Trans. Am. Math. Soc. **86**, 366–400 (1957). ISSN: 0002-9947
36. Krein, M.G.: On a class of entire and meromorphic functions. Russian. In: Achieser, N.I., Krein, M.G. (eds.) Some Problems in the Theorie of Moments, pp. 231–252. Charkov (1938)
37. Krein, M.G.: On the theory of entire matrix functions of exponential type. Ukrain. Mat. Žurnal **3**, 164–173 (1951). ISSN: 0041-6053
38. Krein, M.G.: On a generalization of investigations of Stieltjes. Doklady Akad. Nauk SSSR (N.S.) **87**, 881–884 (1952)
39. Krein, M.G.: On the indeterminate case of the Sturm–Liouville boundary problem in the interval (0,∞). Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 293–324 (1952). ISSN: 0373-2436

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DOI 10.1007/978-3-0348-0692-3_7-1

40. Krein, M.G., Langer, H.: On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part I. Beiträge Anal. **14**, 25–40 (1979) (loose errata)
41. Krein, M.G., Langer, H.: On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part II. Beiträge Anal. **15**, 27–45 (1980/1981)
42. Langer, M., Woracek, H.: Indefinite Hamiltonian systems whose Titchmarsh–Weyl coefficients have no finite generalized poles of non-positive type. Oper. Matrices **7**(3), 477–555 (2013)
43. Langer, M., Woracek, H.: The exponential type of the fundamental solution of an indefinite Hamiltonian system. Compl. Anal. Oper. Theory **7**(1), 285–312 (2013). ISSN: 1661-8254
44. Lelong, P., Gruman, L.: Entire functions of several complex variables. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 282, pp. xii+270. Springer, Berlin (1986). ISBN: 3-540-15296-2
45. Levin, B.J.: Distribution of zeros of entire functions. Revised. vol. 5. Translations of Mathematical Monographs. Translated from the Russian by R.P. Boas, J.M. Danskin, F.M. Goodspeed, J. Korevaar, A.L. Shields and H.P. Thielman, pp. xii+523. American Mathematical Society, Providence (1980). ISBN: 0-8218-4505-5
46. Martin, R.T.W.: Representation of simple symmetric operators with deficiency indices (1, 1) in de Branges space. Compl. Anal. Oper. Theory **5**(2), 545–577 (2011). ISSN: 1661-8254
47. Nikolski, N.K.: Operators, functions, and systems: an easy reading, vol. 1. Mathematical Surveys and Monographs. Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann, vol. 92, pp. xiv+461. American Mathematical Society, Providence (2002). ISBN: 0-8218-1083-9
48. Remling, C.: Schrödinger operators and de Branges spaces. J. Funct. Anal. **196**(2), 323–394 (2002). ISSN: 0022-1236
49. Rosenblum, M., Rovnyak, J.: Topics in Hardy classes and univalent functions. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], pp. xii+250. Birkhäuser, Basel (1994) ISBN: 3-7643-5111-X
50. Rubel, L.A.: Entire and Meromorphic Functions. Universitext. With the assistance of James E. Colliander, pp. viii+187. Springer, New York (1996) ISBN: 0-387-94510-5
51. Silva, L.O., Julio H.T. Applications of Krein's theory of regular symmetric operators to sampling theory. J. Phys. A **40**(31), 9413–9426 (2007). ISSN: 1751-8113
52. Winkler, H.: Canonical systems with a semibounded spectrum. In: Contributions to Operator Theory in Spaces with an Indefinite Metric (Vienna, 1995). Oper. Theory Adv. Appl., vol. 106, pp. 397–417. Birkhäuser, Basel (1998)
53. Winkler, H., Woracek, H.: A growth condition for Hamiltonian systems related with Krein strings. Acta Sci. Math. (Szeged) **80**, 31–94 (2014). doi:10.14232/actasm-012-028-8